Global Existence and Regularity for the 3D Stochastic Primitive Equations of the Ocean and Atmosphere with Multiplicative White Noise

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Abstract

The Primitive Equations are a basic model in the study of large scale Oceanic and Atmospheric dynamics. These systems form the analytical core of the most advanced General Circulation Models. For this reason and due to their challenging nonlinear and anisotropic structure the Primitive Equations have recently received considerable attention from the mathematical community.

In view of the complex multi-scale nature of the earth’s climate system, many uncertainties appear that should be accounted for in the basic dynamical models of atmospheric and oceanic processes. In the climate community stochastic methods have come into extensive use in this connection. For this reason there has appeared a need to further develop the foundations of nonlinear stochastic partial differential equations in connection with the Primitive Equations and more generally.

In this work we study a stochastic version of the Primitive Equations. We establish the global existence of strong, pathwise solutions for these equations in dimension 3 for the case of a nonlinear multiplicative noise. The proof makes use of anisotropic estimates, $L^p_t L^q_x$ estimates on the pressure and stopping time arguments.

Keywords: Primitive Equations, Mathematical Geophysics, Well-Posedness, Nonlinear Stochastic Partial Differential Equations, Stochastic Evolution Equations, Anisotropic Estimates, Pressure Estimates.

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1 Introduction

The Primitive Equations (PEs) are widely considered to be a fundamental model in the study of large scale oceanic and atmospheric dynamics. These systems form the analytical core of the most advanced general circulation models for the atmosphere (AGCMs) or oceans (OGCMs) or the coupled oceanic-atmospheric system (GCMs). Moreover, beyond their considerable significance in applications, the PEs have generated much interest from the mathematics community due to their rich nonlinear, nonlocal character and their anisotropic structure.

The physical derivation of the Primitive Equations goes back to the early 20th century [4, 41]. It is based on a scale analysis that accounts for the relatively constant density of the ocean (in the atmosphere the density follows a linear profile) and the contrast between the vertical and horizontal scales (on the order of several kilometers vs. thousands of kilometers). We refer the reader to e.g. [38] or [41] for further physical background for the (deterministic) PEs.

Hence the Primitive Equations express very fundamental laws of physics and one may wonder what the motivations are for introducing uncertainty in “exact” model equations. The introduction of stochastic processes in weather and climate prediction is aimed at accounting for a number of uncertainties and errors:

1. These “exact” models are numerically intractable; they cannot be fully solved with present super computers (and will not be for any foreseeable future). For this reason some sort of statistical averages are needed corresponding to the parametrization of the effects of the small unresolved scales. Indeed when the full equations are averaged it is common to introduce volumic stochastic forcing terms as a partial closure. This is reflected in a growing physics literature on ‘stochastic parametrization’. See, for example, [42, 31, 35, 3, 48].

2. The physics of these “exact” models is in fact far from being fully understood. For the atmosphere the most dramatic example of physical uncertainty is due to the radiation properties of the air (clouds) which produce the source term called $F_T$ in the atmospheric analogue of the equation (1.2d) below, that is the energy balance equation. Nowadays these uncertainties due to the radiation properties of the clouds is considered to be the most severe source of uncertainly in weather and climate modeling. Note that while our presentation focuses on the equations of the ocean (which are mathematically slightly simpler than the corresponding atmospheric or coupled oceanic-atmospheric equations) our general results and methods in this work apply also to these systems.\(^1\)

Further and related physical background in this connection is the subject of [22].

With this backdrop in mind we study in this work a stochastic version of the primitive equations and establish the global existence and uniqueness of solutions for a nonlinear, multiplicative white noise. We carry out this investigation within the framework of pathwise (or probabilistically strong) solutions where the driving noise is fixed in advance. To the best of our knowledge this is the first work to establish such a result for a general nonlinear noise structure and we have thus had to address novel challenges in the work below. In particular, in order to adapt the methods of previous works (taking advantage of the anisotropic structure of the governing equations), we had to treat many new and delicate nonlinear terms that appear as a result of the stochastic elements in the system. See the end of the introduction below for further details concerning these difficulties. Of course we are drawing on a significant mathematical literature for the Primitive equations, (e.g. in [34, 33, 32, 7, 29, 39]) on the one hand and on the stochastic Navier-Stokes

\(^1\)Physical uncertainties on the boundary of the oceanic or coupled oceanic-atmospheric system are also present and are responsible for another source of stochasticity in these models. The boundary conditions (1.4) on $\Gamma_i$ (which is physically the sea surface-air interface) are a simplified version of the more realistic boundary conditions

$$\alpha_v \psi + \partial_z \psi = g_v, \quad \alpha_T \psi + \partial_z T = g_T. \tag{1.1}$$

In the form (1.1) the boundary condition expresses two fundamental aspects of oceanic-atmospheric interaction namely the driving force of the wind (the term $g_v$) and the heating cooling of the air by the ocean (the term $g_T$). These functions $g_v$ and $g_T$ are not well known and estimated by modelers through very rough averages. Thus the uncertainties of these functions lead to stochastic PDEs with white noise on the boundary a subject which we intend to pursue in future work.
equations, (for example in [2, 46, 9, 8, 15, 37, 12, 5, 1, 6, 36, 19]), on the other. We describe this background next.

To the best of our knowledge the mathematical study of the Primitive Equations started in the early 1990s with a series of articles [34, 33, 32] establishing the existence for all time of weak solutions of these equations. Subsequent articles improved these results and derived existence and uniqueness of more regular (strong) solutions, very similar to the results available for the incompressible Navier-Stokes equations (see e.g. [27, 24]). Recently, taking advantage of the fact that the pressure is essentially two-dimensional in the PEs (unlike the Navier-Stokes equations) global results for the existence of strong solutions of the full three dimensional PEs was established in [7] and independently in [28, 29]. In subsequent work, [30], a different proof was developed which allows one to treat non-rectangular domains as well as different, physically realistic, boundary conditions. As we have said, all of these works make essential use of the fact that the pressure terms appearing in the PEs can be shown to be essentially independent of the vertical variable \( z \).

In the present work we follow an approach closer to that of [30] in that we estimate the pressure directly via earlier results for the Stokes equation as in [45]. In any case the deterministic mathematical theory for the Primitive equations has now reached an advanced stage and we refer the reader to the survey articles [39, 43] for further references and background.

In contrast to the deterministic case, the theory for the stochastic Primitive Equations remains underdeveloped. A two dimensional version of the PEs has been studied in a simplified form in [14, 23] and more recently in [20, 21] in the greater generality of physically relevant boundary conditions and nonlinear multiplicative noise. While the full three dimensional system has been studied in [25] following the methods in [7], this work covers only the case of additive noise. In this case the PEs can be directly studied pathwise via a classical change of variables. In this and a companion work [13] devoted to the local existence of solutions we depart from [25] and develop different methods which allow us, in particular, to consider a nonlinear multiplicative forcing structure. The present work may therefore be seen as the continuation of [13] which takes us from the local to the global existence of solutions.

In contrast, the theory of the related stochastic Navier-Stokes equations has undergone substantial developments dating back to the 1970’s; see e.g. [2, 46, 9, 8, 16, 12, 1, 5, 6, 18, 10, 36, 37, 44, 17, 19]. In this literature, as for the wider literature on stochastic PDEs (see [11]) two principal notions have been developed, namely Martingale (or probabilistically weak) solutions and Pathwise (or probabilistically strong) solutions. In this and the companion work [13] we concentrate on the latter notion where the driving noise is fixed in advance.

Not withstanding these developments, to emphasize the notable differences between the stochastic Navier-Stokes Equations and the primitive equations we recall that the deterministic Primitive equations are known to be well posed in space dimension three. This result is not known for the Navier-Stokes Equations and is the object of the famous Clay problem. On the other hand the primitive equations are technically more involved than the Navier-Stokes equations.

The Governing Equations

The stochastic version of the Primitive Equations we will study below takes the form:

\[
\begin{align*}
\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nu \nabla \mathbf{v} + \frac{1}{\rho_0} \nabla p + f k \times \mathbf{v} - \mu_v \Delta \mathbf{v} - \nu_v \partial_z \mathbf{v} &= F_v + \sigma_v(\mathbf{v}, T, S) \dot{W}_1, \\
\partial_z p &= -\rho g, \\
\nabla \cdot \mathbf{v} + \partial_z w &= 0, \\
\partial_t T + (\mathbf{v} \cdot \nabla) T + w \partial_z T - \mu_T \Delta T - \nu_T \partial_z T &= F_T + \sigma_T(\mathbf{v}, T, S) \dot{W}_2, \\
\partial_t S + (\mathbf{v} \cdot \nabla) S + w \partial_z S - \mu_S \Delta S - \nu_S \partial_z S &= F_S + \sigma_S(\mathbf{v}, T, S) \dot{W}_3, \\
\rho &= \rho_0 (1 + \beta_T (T - T_r) + \beta_S (S - S_r)),
\end{align*}
\]

Here, \( U := (\mathbf{v}, T, S) = (u, v, T, S) \), \( p \), \( \rho \) represent the velocity, temperature, salinity, pressure and density of the fluid under consideration; \( \mu_v, \nu_v, \mu_T, \nu_T, \mu_S, \nu_S \) are (possibly anisotropic) coefficients of the eddy and
molecular viscosity and the heat and saline diffusivity respectively; \( f \) is the Coriolis parameter appearing in the antisymmetric term in (1.2a) and accounts for the earth’s rotation in the momentum equations. The third component of the velocity field, \( w \) is a ‘diagnostic variable’ in that it is determined directly from \( \mathbf{v} \), the other two components from the horizontal velocity field (see (2.1b), below). The evolution equations (1.2) occurs for \((x_1, x_2, z)\) ranging over a cylindrical domain \( M = M_0 \times (-h, 0) \). \( M_0 \) is an open bounded subset of \( \mathbb{R}^2 \) with smooth boundary \( \partial M_0 \). Note that \( \nabla = (\partial_1, \partial_2) \) where \( \partial_1, \partial_2 \) are the partial derivatives in the (horizontal) \( x_1, x_2 \) directions. \( \Delta = \partial_1^2 + \partial_2^2 \) is the horizontal Laplace operator.

The stochastic terms are driven by Gaussian white noise processes \( W_j \) which are formally delta correlated in space and time. The stochastic terms may be written formally in the expansion

\[
\begin{pmatrix}
\sigma_v(U)\dot{W}_1(t, x) \\
\sigma_T(U)\dot{W}_2(t, x) \\
\sigma_S(U)\dot{W}_3(t, x)
\end{pmatrix} = \sum_{k \geq 1} \begin{pmatrix}
\sigma_{v,k}(U)(x, t)\dot{W}_{1,k}(t) \\
\sigma_{T,k}(U)(x, t)\dot{W}_{2,k}(t) \\
\sigma_{S,k}(U)(x, t)\dot{W}_{3,k}(t)
\end{pmatrix},
\]  

where the elements \( \dot{W}_{j,k} \) are independent white noise processes in time. We understand (1.2) in the Itô sense but the classical correspondence between the Itô and Stratonovich systems would allow one to treat both situations with the analysis herein. We recall the basic mathematical definitions and give precise conditions on the operators \( \sigma_v, \sigma_T, \sigma_S \) below.

The boundary \( \partial M \) is partitioned into the top \( \Gamma_i = M_0 \times \{0\} \), the bottom \( \Gamma_b = M_0 \times \{-h\} \) and the sides \( \Gamma_l = \partial M_0 \times (-h, 0) \). We denote by \( \mathbf{n}_H \) the outward unit normal to \( \partial M_0 \). We prescribe the following boundary conditions:

\[
\partial_z \mathbf{v} = 0, \quad w = 0, \quad \partial_z T = 0, \quad \partial_z S = 0,
\]

on \( \Gamma_i \). At the bottom \( \Gamma_b \) we take

\[
\partial_z \mathbf{v} = 0, \quad w = 0, \quad \partial_z T = 0, \quad \partial_z S = 0.
\]

Finally for the lateral boundary \( \Gamma_l \)

\[
\mathbf{v} = 0, \quad \mathbf{n}_H T = 0, \quad \mathbf{n}_H S = 0.
\]

The equations and boundary conditions (1.2), (1.4), (1.5), (1.6) are supplemented by initial conditions for \( \mathbf{v}, T \) and \( S \), that is

\[
\mathbf{v} = \mathbf{v}_0, \quad T = T_0, \quad S = S_0, \quad \text{at } t = 0.
\]

The Cauchy problem (1.2)-(1.7) given above models regional oceanic flows. We note however that equations of a quite similar structure may be given that describe the atmosphere and the coupled oceanic atmospheric system. See e.g. [39]. The methods developed here could thus be extended to treat these systems.

The manuscript is organized as follows. In the initial Section 2 we set the mathematical background for the work defining precisely the notion of pathwise solutions we are studying. Section 3 recalls the results in [13] that guarantee the local existence of solutions. Crucially, these results imply a maximal time of existence \( \xi = \xi(\omega) \); on those samples \( \omega \) where \( \xi(\omega) < \infty \) certain norms of the solution \( U = (\mathbf{v}, T, S) \), in particular the \( H^1 \) norm, must blow up at \( \xi(\omega) \). Having exhibited these preliminaries we next introduce a criteria for global existence based on the uniform control in time of \( \mathbf{v} \in L^4 \) and \( \partial_z U \in L^2 \). This sets the agenda for the remainder of the paper. In Section 4 we carry out the estimates in \( L^4 \). To this end we introduce a new ‘shifted’ variable \( \hat{\mathbf{v}} \) that satisfies a random PDE that we may analyze pathwise. In this way we are able to handle the pressure via the results in [45]. On the other hand a new terms appear in the nonlinear portion of the equations for \( \hat{\mathbf{v}} \) that we must tackle. In Section 5 we turn to the estimates for \( \partial_2 U \) in \( L^2 \). In this case the pressure disappears when we exhibit the evolution for \( |\partial_2 U|^2_{L^2} \). As such we carry out the estimates in the original variable using stochastic methods: Itô calculus, the Burkholder-Davis-Gundy inequality, etc. Finally we include in an appendix further details for various technical complements used in the body of the work: the pressure estimates for the Stokes equations after [45] and a particular version of the Gronwall lemma that we use to close the \( L^4 \) estimates in Section 4.
2 Mathematical Background and Notational Conventions

In order to introduce a precise mathematical definition of solutions for the stochastic Primitive Equations we begin by rewriting (1.2) in a slightly different form. This formulation will be the basis for all that follows below:

\[
\begin{align*}
\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} &+ w(\mathbf{v})\partial_z \mathbf{v} + \frac{1}{\rho_0} \nabla p_s - g \int_z^0 (\beta_T \nabla T + \beta_S \nabla S) \, d\bar{z} \\
+ f k \times \mathbf{v} - \mu_v \Delta \mathbf{v} - \nu_v \partial_z^2 \mathbf{v} &= F_v + \sigma_v(\mathbf{v}, T, S) \bar{W}_1, \quad (2.1a)
\end{align*}
\]

\[
\begin{align*}
\partial_z T + (\mathbf{v} \cdot \nabla) T + w(\mathbf{v})\partial_z T - \mu_T \Delta T - \nu_T \partial_z^2 T &= F_T + \sigma_T(\mathbf{v}, T, S) \bar{W}_2, \quad (2.1b)
\end{align*}
\]

\[
\begin{align*}
\partial_z S + (\mathbf{v} \cdot \nabla) S + w(\mathbf{v})\partial_z S - \mu_S \Delta S - \nu_S \partial_z^2 S &= F_S + \sigma_S(\mathbf{v}, T, S) \bar{W}_3. \quad (2.1d)
\end{align*}
\]

By integrating (1.2b) and making use of the relation (1.2f) we find that the pressure \( p \) may be decomposed into a 'surface pressure' and some lower order terms that couple the momentum equations to those for the temperature and salinity. Crucially, we note that \( p_s \) does not depend on the vertical variable \( z \). Of course, this system (2.1) is supplemented with initial and boundary conditions as given in (1.7) and (1.4), (1.5), (1.6) above.

2.1 Abstract Setting for the Equations

We next recall the abstract setting for (2.1) (equivalently (1.2)). Note that our presentation and notations are useful for the coherence of physical dimensions and for (mathematical) coercivity. Since this is not needed here we take \( \kappa_T = \kappa_S = 1 \). Similar remarks also apply to the space \( V \).

\[H := \left\{ (\mathbf{v}, T, S) \in (L^2(\mathcal{M}))^4 : \nabla \cdot \int_{-h}^0 \mathbf{v} \, dz = 0 \text{ in } \mathcal{M}_0, \mathbf{n}_H \cdot \int_{-h}^0 \mathbf{v} \, dz = 0 \text{ on } \partial \mathcal{M}_0, \int_{\mathcal{M}} T \, d\mathcal{M} = \int_{\mathcal{M}} S \, d\mathcal{M} = 0 \right\}.
\]

We equip this space with the classical \( L^2 \) inner product\(^2\) which we denote by \( | \cdot | \). Define \( P_H \) to be the Leray type projection operator from \( L^2(\mathcal{M})^4 \) onto \( H \). For \( H^1(\mathcal{M})^4 \) we consider the subspace:

\[V := \left\{ (\mathbf{v}, T, S) \in (H^1(\mathcal{M}))^4 : \nabla \cdot \int_{-h}^0 \mathbf{v} \, dz = 0 \text{ in } \mathcal{M}_0, \mathbf{v} = 0 \text{ on } \Gamma_l, \int_{\mathcal{M}} T \, d\mathcal{M} = \int_{\mathcal{M}} S \, d\mathcal{M} = 0 \right\}.
\]

We equip \( V \) with the inner product

\[
\begin{align*}
((\mathbf{U}, \mathbf{U'})_2) &:= ((\mathbf{v}, \mathbf{v'}))_1 + ((T, T'))_2 + ((S, S'))_3, \\
((\mathbf{v}, \mathbf{v'}))_1 &:= \int_{\mathcal{M}} (\mu_v \nabla \mathbf{v} \cdot \nabla \mathbf{v'} + \nu_v \partial_z \mathbf{v} \cdot \partial_z \mathbf{v'}) \, d\mathcal{M}, \\
((T, T'))_2 &:= \int_{\mathcal{M}} (\mu_T \nabla T \cdot \nabla T' + \nu_T \partial_z T \cdot \partial_z T') \, d\mathcal{M}, \\
((S, S'))_3 &:= \int_{\mathcal{M}} (\mu_S \nabla S \cdot \nabla S' + \nu_S \partial_z S \cdot \partial_z S') \, d\mathcal{M},
\end{align*}
\]

and take \( \| \cdot \| = \sqrt{((\cdot, \cdot))} \). Note that under these definitions a Poincaré type inequality \( |\mathbf{U}| \leq c\|\mathbf{U}\| \) holds for all \( \mathbf{U} \in V \). We take \( V_2 \) to be the closure of \( V \) in \( (H^2(\mathcal{M}))^4 \) and equip this space with the classical \( H^2(\mathcal{M}) \) norm and inner product.

\(^2\)One sometimes also finds the more general definition \((\mathbf{U}, \mathbf{U'}) := \int_{\mathcal{M}} (\mathbf{v} \cdot \mathbf{v'} \, d + \kappa_T T T' + \kappa_S S S') \, d\mathcal{M} \) with \( \kappa_T, \kappa_S > 0 \) fixed constants. These parameters \( \kappa_T, \kappa_S \) are useful for the coherence of physical dimensions and for (mathematical) coercivity. Since this is not needed here we take \( \kappa_T = \kappa_S = 1 \). Similar remarks also apply to the space \( V \).
In the course of the analysis below we shall make estimates involving the individual components of the solution \( U = (u, v, T, S) \). As such we shall sometime abuse notation and use \(| \cdot |\) and \(| \cdot |\) in the obvious way for \( v = (u, v) \), \( T \) or \( S \). We shall also work with the \( L^p = L^p(M) \) norms of these individual components of the solution. For \( p \geq 1 \) we denote \( v^p = (u^p, v^p) \) and let \( |v|_{L^p} := (\int_M(|u|^p + |v|^p) dM)^{1/p} \). Furthermore, for \( q, p \geq 1 \), we write
\[
|v|_{L^2} := \left( \int_0^1 \left( \int_{-h}^0 (|u|^p + |v|^p) dz \right)^{q/p} dM_0 \right)^{1/q}.
\]

**Remark 2.1.** Assume that \( v \in H^1(M) \), with \( v = 0 \) on \( \Gamma_i \) or \( \int_M v dM = 0 \). As in [39], an elementary calculation that makes use of the Sobolev embedding theorem in \( \mathbb{R}^2 \), reveals that
\[
|v|_{L^2} \leq c|v|^{1-s} \|v\|^s,
\]
where \( q \geq 2 \) and \( s = 1 - 2/q \). This observation will be used on several occasions below.

The principal linear portion of the equation is defined by\(^3\)
\[
AU = P_H \begin{pmatrix}
-\mu_v \Delta v - \nu_v \partial_z v \\
-\mu_T \Delta T - \nu_T \partial_z T \\
-\mu_S \Delta S - \nu_S \partial_z S
\end{pmatrix}, \quad \text{for any } U = (v, T, S) \in D(A)
\]
where:
\[
D(A) = \{ U = (v, T) \in V(2) : \partial_z v = \partial_z T = \partial_z S = 0 \text{ on } \Gamma_i, \\
\partial_{n_i} T = \partial_{n_i} S = 0 \text{ on } \Gamma_i, \partial_z v = \partial_z T = \partial_z S = 0 \text{ on } \Gamma_b \}.
\]
We observe that \( A \) is self adjoint, positive definite and that \( \langle AU, U^\sharp \rangle = \langle (U, U^\sharp) \rangle \) for all \( U, U^\sharp \in V \). Note also that due to [47] (see also [39]) \( |AU| \cong |U|_{H^2} \).

We next turn to the quadratically nonlinear terms appearing in (2.1). Noting that there is no momentum equation for \( w \) in (2.1) and in accordance with (2.1b) we define the diagnostic function:
\[
w(U) = w(v) := \int_z^0 \nabla \cdot v dz, \quad U = (v, T, S) \in V. \tag{2.3}
\]
Take, for \( U, U^\sharp \in D(A)\):
\[
B_1(U, U^\sharp) := P_H \begin{pmatrix}
(v \cdot \nabla) v^\sharp \\
(v \cdot \nabla) T^\sharp \\
(v \cdot \nabla) S^\sharp
\end{pmatrix}, \quad B_2(U, U^\sharp) := P_H \begin{pmatrix}
w(v) \partial_z v^\sharp \\
w(v) \partial_z T^\sharp \\
w(v) \partial_z S^\sharp
\end{pmatrix}. \tag{2.4}
\]

We let \( B(U, U^\sharp) := B_1(U, U^\sharp) + B_2(U, U^\sharp) \), and often write \( B(U) = B(U, U) \). As in [39] one may show that \( B \) is well defined as an element in \( H \) for any \( U, U^\sharp \in D(A) \) or \( V(2) \). In addition to the properties of \( B \) appearing in [39] we have the following additional bounds, which are established with anisotropic estimates along the same lines (cf. Remark 2.1).

**Lemma 2.1.** Suppose that \( U, U^\sharp \in D(A) \) and that \( U^\flat \in H \). Then
\[
|\langle B(U, U^\sharp), U^\sharp \rangle| \leq c(|v|_{L^2}^4 |U^\sharp|^{1/4} |AU|^{3/4} |U^\flat| + |v|^{1/2} |v^{(2)}|^{1/2} |\partial_z U^\flat|^{1/2} |\partial_z U^\sharp|^{1/2} |U^\flat|). \tag{2.5}
\]
\(^3\)In comparison to previous works, such as [39], we do not include all of the terms due to the pressure in the definition of \( A \). Such elements destroy the symmetry of \( A \) and are therefore relegated to a lower order term \( A_p \). See (2.6).
For the second component of the pressure in (2.1a) we take
\[ A_p U = P_H \left( -g \int_0^z (\beta_T \nabla T + \beta_S \nabla S) d\tilde{z} \right), \quad U \in V. \] (2.6)

We capture the Coriolis (rotational) forcing according to
\[ EU = P_H \left( \frac{f}{k} \times v \right), \quad U \in H. \] (2.7)

Finally we set
\[ F = P_H \begin{pmatrix} F_v \\ F_T \\ F_S \end{pmatrix}. \] (2.8)

We shall assume throughout this work that:
\[ F \in L^2(\Omega; L^2_{loc}([0, \infty); L^4(M))). \] (2.9)

We finally give a precise definition for the stochastic terms appearing in (2.1) (i.e. (1.2)). For this purpose let us briefly recall some aspects of the theory of the infinite dimensional Itô integration. As we are studying pathwise solutions of (1.2) (see Definition 2.1 below) we shall fix throughout this work a single stochastic basis \( S := (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, W) \). Here \( W \) is a cylindrical brownian motion defined on an auxiliary Hilbert space \( \mathcal{H} \) and adapted to the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). By picking a complete orthonormal basis \( \{e_k\}_{k \geq 1} \) for \( \mathcal{H} \), \( W \) may be written as the formal sum \( W(t, \omega) = \sum_{k \geq 1} e_k W_k(t, \omega) \) where the elements \( W_k \) are an independent sequence of 1D standard Brownian motions.

Consider another separable Hilbert space \( X \) and let \( L^2(U, X) = \{ R \in L(U, X) : \sum_k |R e_k|^2 < \infty \} \), that is the collection of Hilbert-Schmidt operators from \( U \) into \( X \). Given an \( X \)-valued predictable\(^4\) process \( G \in L^2(\Omega; L^2_{loc}([0, \infty), L^2(U, X))) \) one may define the (Itô) stochastic integral
\[ M_t := \int_0^t G dW = \sum_k \int_0^t G_k dW_k, \]
as an element in \( \mathcal{M}_X \), that is the space of all \( X \)-valued square integrable martingales (see [11] or [40]); here \( G_k = G e_k \). The process \( \{M_t\}_{t \geq 0} \) has many desirable properties. Most notably for the analysis here, the Burkholder-Davis-Gundy inequality holds which in the present context takes the form,
\[ \mathbb{E} \left( \sup_{t \leq T} \left| \int_0^t G dW \right|^r \right) \leq c \mathbb{E} \left( \int_0^T |G|^2_{L^2(U, X)} dt \right)^{r/2}, \] (2.10)
valid for any \( r \geq 1 \). Here \( c \) is an absolute constant depending only on \( r \).

Given any pair of Banach spaces \( X \) and \( Y \) we denote by \( Bnd_u(X, Y) \), the collection of all continuous mappings \( \Psi : [0, \infty) \times X \rightarrow Y \) such that
\[ \| \Psi(x, t) \|_Y \leq c(1 + \|x\|_X), \quad x \in X, t \geq 0, \]
\(^4\)Let \( \Phi = \Omega \times [0, \infty) \) and take \( \mathcal{G} \) to be the \( \sigma \)-algebra generated by sets of the form \( \{s, t \times F \}, \quad 0 \leq s < t < \infty, F \in \mathcal{F}_s; \quad \{0\} \times F, \quad F \in \mathcal{F}_0 \).

Recall that an \( X \) valued process \( U \) is called predictable (with respect to the stochastic basis \( S \)) if it is measurable from \( (\Phi, \mathcal{G}) \) into \( (X, B(X)) \), \( B(X) \) being the family of Borel sets of \( X \).
where the numerical constant \( c \) may be chosen independently of \( t \). If, in addition,
\[
\|\Psi(x, t) - \Psi(y, t)\|_Y \leq c\|x - y\|_X, \quad x, y \in X, \ t \geq 0,
\]
we say \( \Psi \) is in \( \text{Lip}_u(X, Y) \). We define
\[
\sigma((v, T, S)) = \sigma(U) = P_H \begin{pmatrix}
\sigma_v(U) \\
\sigma_T(U) \\
\sigma_S(U)
\end{pmatrix}, \quad U \in H,
\]
and assume that \( \sigma : [0, \infty) \times H \rightarrow L_2(\Omega, H) \) with
\[
\sigma \in \text{Lip}_u(H, L_2(\Omega, H)) \cap \text{Lip}_u(V, L_2(\Omega, V)) \cap Bnd_u(V, L_2(\Omega, D(A)))).
\tag{2.12}
\]
Under the assumption (2.12) on \( \sigma \), the stochastic integral \( t \mapsto \int_0^t \sigma(U)\,dW \) may be shown to be well defined (in the Itô sense), taking values in \( H \) whenever \( U \in L^2(\Omega, L_{\text{loc}}^2([0, \infty); H)) \) and is predictable.

**Remark 2.2.** The condition (2.12) may be shown to cover a wide class of examples, including but not limited to the classical cases of additive and linear multiplicative noise, projections of the solution in any direction, and directional forcings of Lipschitz functionals of the solution. See [22] for further details and physical motivations.

We note that the final condition \( \sigma \in Bnd_u(V, L_2(\Omega, D(A))) \) in fact may be weakened slightly to \( \sigma \in Bnd_u(D(A), L_2(\Omega, D(A))) \) for the proof of local existence. However for the global existence we need this stronger condition in order to carry out the \( L^4 \) estimates which appear in Section 4. In fact the stronger condition used in this section is an artifact of a ‘pathwise’ approach which further complicates the nonlinear term. This pathwise approach is necessitated by the appearance of certain terms explicitly involving the pressure \( p_s \).

Collecting the operators defined above we reformulate (2.1) (equivalently, (1.2)) as the following abstract evolution system
\[
dU + (AU + B(U) + A_p U + EU)\,dt = F\,dt + \sigma(U)\,dW, \quad U(0) = U_0.
\tag{2.13}
\]
We recall the following pathwise (or probabilistically strong) notions of local and global existence for this system.

**Definition 2.1.** Let \( S = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W) \) be a fixed stochastic basis and suppose that \( U_0 \) is a \( V \) valued, \( \mathcal{F}_0 \) measurable random variable with \( \mathbb{E}(\|U_0\|^2) < \infty \). Assume that \( F \) satisfies (2.9) and that (2.12) holds for \( \sigma \).

(i) A pair \((U, \tau)\) is a a local pathwise solution of (2.13) (i.e. of (1.2)) if \( \tau \) is a strictly positive stopping time and \( U(\cdot \land \tau) \) is an \( \mathcal{F}_t \)-adapted process in \( V \) so that (relative to the fixed basis \( S \))
\[
U(\cdot \land \tau) \in L^2(\Omega; C([0, \infty); V)), \quad U_{t \leq \tau} \in L^2(\Omega; L_{\text{loc}}^2([0, \infty); D(A)));
\tag{2.14}
\]
and, for every \( t \geq 0 \),
\[
U(t \land \tau) + \int_0^{t \land \tau} (AU + B(U) + A_p U + EU)\,dt' = U_0 + \int_0^{t \land \tau} F\,dt' + \int_0^{t \land \tau} \sigma(U)\,dW, \tag{2.15}
\]
with equality understood in \( H \).

(ii) Pathwise solutions of (2.13) are said to be (pathwise) unique up to a stopping time \( \tau > 0 \) if given any pair of solutions \((U^1, \tau), (U^2, \tau)\) which coincide at \( t = 0 \) on the event \( \Omega = \{U^1(0) = U^2(0)\} \subseteq \Omega \), then
\[
\mathbb{P}(\mathbb{1}_\Omega(U^1(t \land \tau) = U^2(t \land \tau)) = 0; \forall t \geq 0) = 1.
\]
(iii) Suppose that \( \{\tau_n\}_{n \geq 1} \) is a strictly increasing sequence of stopping times converging to a (possibly infinite) stopping time \( \xi \) and assume that \( U \) is a predictable process in \( H \). We say that the triple \( (U, \xi, \{\tau_n\}_{n \geq 1}) \) is a maximal strong solution if \( (U, \tau_n) \) is a local strong solution for each \( n \) and

\[
\sup_{t \in [0, \xi]} \|U\|^2 + \int_0^\xi |AU|^2 ds = \infty
\]

almost surely on the set \( \{\xi < \infty\} \).

(iv) If \( (U, \xi) \) is a maximal strong solution and \( \xi = \infty \) a.s. then we say that the solution is global.

With these foundations in place we finally state, in precise terms, the main result of the work, the global well-posedness of (1.2) in the class of pathwise, strong solutions:

**Theorem 2.1.** Suppose that \( U_0 \in L^2(\Omega, V) \) and is \( \mathcal{F}_0 \) measurable. Assume that \( F \) satisfies (2.9) and that (2.12) holds for \( \sigma \). Then, there exists a unique global pathwise solution \( U \) of (2.13) with \( U(0) = U_0 \).

The proof of Theorem 2.1 is carried out below between Propositions 4.1, 5.1, which, taken together, establish the sufficient condition given by Theorem 3.2.

### 3 Local Existence and A Criteria For Global Existence

As with the deterministic case, the first step in the analysis is to establish the local existence of solutions all the way to a ‘maximal’ existence time \( \xi > 0 \). Recall that if \( \xi(\omega) < \infty \) then the \( H^1 \) norm of the solution must blow up at this maximal time. With this in mind, we next show in Theorem 3.2 that the strong norms of solutions may be controlled by a combination of the \( L^4 \) norm of the momentum \( \nu \) and by the vertical gradient of the entire solution \( U \). The results in Theorem 3.2 thus set the program for the rest of the article.

The first step, establishing the existence of a maximal pathwise solution has been carried out in [13]. This work addresses this step for (2.13) within a wider class of abstract nonlinear SPDEs.

**Theorem 3.1.** Suppose that (2.9) and (2.12) hold for \( F \) and \( \sigma \) respectively. Given any \( \mathcal{F}_0 \) measurable \( U_0 \in L^2(\Omega, V) \), there exists a unique maximal pathwise solution \( (U, \xi) \) of (2.13) with \( U(0) = U_0 \). If, for some \( p \geq 2 \), \( U(0) \in L^p(\Omega, V) \) then, for any \( t > 0 \), we have the bounds

\[
\mathbb{E} \left( \sup_{t' \in [0, t \land \xi]} |U|^p + \int_0^{t \land \xi} \|U\|^2 |U|^{p-2} dt' \right) < \infty,
\]

and

\[
\mathbb{E} \left( \int_0^{t \land \xi} \|U\|^2 dt' \right)^{p/2} < \infty.
\]

**Remark 3.1.** The local existence of solutions for multiplicative noise is challenging since the classical compactness methods break down when \( \omega \) is no longer a parameter in the problem, as in the case of additive noise. The proof of Theorem 3.1 appearing in [13] is based on an elementary but powerful characterization of convergence in probability given in [26]. With this characterization we are able to establish a Yamada-Watanabe type result namely: ‘pathwise’ solutions follow from the existence of Martingale solutions when we have pathwise uniqueness for solutions (see Definition 2.1,(ii)). A different method based on the Cauchy convergence of the Galerkin solutions associated to the basic SPDE is developed in [19] for the Navier-Stokes equations in dimensions \( D = 2, 3 \) and for the 2D Primitive equations in [20, 21].

We define the following stopping times which are used here and below:

\[
\tau^W_K = \inf_{t \geq 0} \left\{ \sup_{t' \in [0, t \land \xi]} |U|^2 + \int_0^{t \land \xi} (\|U\|^2 + |F|_F^2) dt' \geq K \right\},
\]

where \( F \) is a strongly \( \mathcal{F}_t \) measurable \( H \) valued \( \mathcal{F}_0 \) predictable process in \( \mathbb{H} \) independent of \( \xi \).
Remark 3.2. Alternatively (3.3), (3.4) express the fact that
\[ \sup_{t'} \left| U(t') \right|^2 + \int_0^{t' \wedge \xi} \left( \left| U(t') \right|^2 + \left| F_{t',x}(t') \right|^2 \right) dt' < K \quad \text{for all } 0 \leq t < \tau_K^W, \] (3.5)
\[ \sup_{t'} \left| v_{t'} \right|^4 < K \quad \text{for all } 0 \leq t < \tau_K^{(1)}, \] (3.6)
\[ \sup_{t'} \left| \partial_s U(t') \right|^2 + \int_0^{t' \wedge \xi} \left| \partial_s U(t') \right|^2 dt' < K \quad \text{for all } 0 \leq t < \tau_K^{(2)}, \] (3.7)
and \( \tau_K^W, \tau_K^{(1)}, \tau_K^{(2)} \) are respectively the largest times such that (3.5)-(3.7) hold.

Note that (3.3), (3.4) are respectively equivalent to (3.5)-(3.7); The form (3.3), (3.4) express \( \tau_K^W, \tau_K^{(1)}, \tau_K^{(2)} \) as exit times, (3.5)-(3.7) is the form conveniently used in the a priori estimates below.

Theorem 3.2. Suppose that the conditions given in Theorem 2.1 hold and consider a maximal strong solution \((U, \xi) = ((v, T, S), \xi)\) of (1.2) whose existence is guaranteed by Theorem 3.1. For any deterministic constant \(K\), consider the stopping times \(\tau_K\) defined according to (3.4). Then, for any deterministic time \(t > 0\) and any \(K > 0\),
\[ E \left( \sup_{t' \in [0, t \wedge \tau_K]} \left| U(t') \right|^2 + \int_0^{t' \wedge \tau_K} \left| AU(t') \right|^2 dt' \right) < \infty \] (3.8)
If, moreover, it is shown that \(\lim_{K \uparrow \infty} \tau_K = \infty\) then \(\xi = \infty\) i.e. the solution \(U\) is global in the sense of Definition 2.1, (iv).

We conclude this section by briefly sketching the proof of Theorem 3.2. It relies mainly on Lemma 2.1 and standard estimates on the stochastic terms using martingale inequalities.

Proof. With an application of the Itô formula we deduce the following evolution equation for \(\|U\|^2\):
\[ d\|U\|^2 + 2\|AU\|^2 dt = -2\langle B(U) + A_p U + EU + F, AU \rangle dt + \|\sigma(U)\|^2_{L_2(U, V)} dt + 2(A^{1/2} \sigma(U), A^{1/2} U) dW. \]
Fix any \(0 \leq \tau_a \leq \tau_b \leq \tau_K\). Integrating in time, taking a supremum over the interval \([\tau_a, \tau_b]\) and then the expected value of the resulting expression yields the estimate
\[
E \left( \sup_{\tau_a \leq t' \leq \tau_b} \|U(t')\|^2 + 2 \int_{\tau_a}^{\tau_b} \|AU(t')\|^2 dt' \right) \\
\leq E\|U(\tau_a)\|^2 + cE \int_{\tau_a}^{\tau_b} \langle B(U), AU \rangle dt + cE \int_{\tau_a}^{\tau_b} \left( \|A_p U + EU + F, AU\| + \|\sigma(U)\|^2_{L_2(U, V)} \right) dt \\
+ cE \sup_{\tau_a \leq t' \leq \tau_b} \int_{\tau_a}^{t'} \langle A^{1/2} \sigma(U), A^{1/2} U \rangle dW \\
:= E\|U(\tau_a)\|^2 + J_1 + J_2 + J_3.
\]
We address the terms $J_k$ in reverse order. For $J_3$ using the Burkholder-Davis-Gundy Inequality, (2.10), and the assumption in (2.12) that $\sigma \in \text{Lip}_a(V, L_2(\Omega, U))$ we infer
\[
J_3 \leq c \mathbb{E} \left( \int_{\tau_n}^{\tau_b} \langle A^{1/2} \sigma(U), A^{1/2} U \rangle^2 dt \right)^{1/2} \leq c \mathbb{E} \left( \int_{\tau_n}^{\tau_b} \|\sigma(U)\|_{L_2(\Omega, U)}^2 \|U\|^2 dt \right)^{1/2}
\]
\[
\leq \frac{1}{2} \mathbb{E} \sup_{t' \in [\tau_n, \tau_b]} \|U\|^2 + c \mathbb{E} \int_{\tau_n}^{\tau_b} (1 + \|U\|^2) dt.
\]

For $J_2$ we use again that $\sigma \in \text{Lip}_a(V, L_2(\Omega, U))$ and make direct estimates for $A_p$ and $E$; see respectively, (2.6), (2.7) above. We infer
\[
|\langle A_p U + E U + F, AU \rangle| + \|\sigma(U)\|_{L_2(\Omega, U)}^2 \leq \frac{1}{2} |AU|^2 + c(1 + \|U\|^2 + |F|^2).
\]

On the other hand, (2.5) implies that
\[
|\langle B(U), AU \rangle| \leq |v|_{L^4}|U|^{1/4}|AU|^{7/4} + \|U\|^{1/2}|\partial_2 U|^{1/2}\|\partial_2 U\|^2 \|AU\|^{3/2}
\]
\[
\leq \frac{1}{2} |AU|^2 + c(|v|_{L^4} + |\partial_2 U|^2 \|\partial_2 U\|^2) \|U\|^2.
\]

This takes care of $J_1$. Combining these three estimates, we conclude that
\[
\mathbb{E} \left( \sup_{\tau_n \leq t' \leq \tau_b} \|U\|^2 + \int_{\tau_n}^{\tau_b} |AU|^2 dt' \right) \leq \mathbb{E} \left( \|U(\tau_0)\|^2 + c \int_{\tau_n}^{\tau_b} (|v|_{L^4}^8 + |\partial_2 U|^2 \|\partial_2 U\|^2 + 1) \|U\|^2 dt + c \int_{\tau_n}^{\tau_b} |F|^2 dt \right).
\]

Noting that the constants appearing above do not depend on $\tau_a, \tau_b$, the desired result, (3.8), now follows from the stochastic Gronwall lemma as given in [19, Lemma 5.3].

4 Estimates in $L^4(\mathcal{M})$

In this section we establish that the horizontal component of the velocity field, $v$, of the solution $U = (v, T, S)$ is finite in $L^{\infty}_{t^c} L^4_{K,z}$, at least up to the maximal time of existence, $\xi$. More precisely we prove that:

**Proposition 4.1.** Suppose that the conditions given in Theorem 2.1 hold and consider the resulting maximal strong solution $(U, \xi) = ((v, T, S), \xi)$ of (1.2). Then, for any $t > 0$,
\[
\sup_{t' \in [0, t \wedge \xi]} |v(t')|_{L^4} < \infty \quad \text{a.s.}
\]

Moreover, defining, for $K > 0$ the stopping time $\tau^{(1)}_K$ as in (3.4) we have, up to a set of measure zero, that $\lim_{K \uparrow \infty} \tau^{(1)}_K = \infty$.

The remainder of this section is devoted to the proof of Proposition 4.1. This proof relies on a decomposition of the solution $U$ of (2.15) into a sum of two stochastic systems: $\hat{U}$ and $\hat{\bar{U}}$; $\hat{U}$ solves a linear stochastic system, (4.2), whereas $\hat{\bar{U}}$ is the solution of the random PDE, (4.4). The later system can be studied ‘pathwise’, that is pointwise in $\Omega$. We are therefore able to treat certain terms involving the pressure explicitly by means of Proposition 6.1 (cf. [30], [45]).

**The Decomposition**

We describe next the decomposition and recall some basic properties of the linear portion of this splitting. Consider the following linear system, an infinite dimensional Ornstein-Uhlenbeck process
\[
d\hat{U} + A\hat{U} dt = \mathbb{1}_{t \leq \xi} \sigma(U) dW, \quad \hat{U}(0) = 0.
\]

We underline here that the element $(U, \xi)$ appearing in the stochastic terms is the unique maximal solution of (1.2) corresponding to the given initial condition $U_0$. 

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Proposition 4.2. There exists a unique global, pathwise strong solution \( \hat{U} = (\mathbf{v}, \hat{T}, \hat{S}) \) of (4.2) with \( \hat{U} \in L^2(\Omega; C([0, \infty); D(A))) \). In particular, for any \( t > 0 \),

\[
\sup_{t' \in [0,t]} |\mathbf{v}(t')|_{L^2} < \infty \quad \text{a.s.} \tag{4.3}
\]

Remark 4.1. The final condition in (2.12), \( \sigma \in \text{Bnd}(V, L^2(\Omega, D(A))) \) is imposed so that \( \hat{U} \) is guaranteed to evolve continuously in \( D(A) \). This stronger conditions is needed in several estimates below (see e.g. (4.10), (4.14)). In any case linear systems of the form (4.2) are well understood. See [11]. The estimates which justify \( \hat{U} \in L^2(\Omega; C([0, \infty); D(A))) \) are straightforward and may be found for example in [21].

We next define \( \tilde{U} = U - \hat{U} \). By subtracting (4.2) from (2.13) we find that \( \tilde{U} \) satisfies

\[
\frac{d}{dt} \tilde{U} = A\tilde{U} + B(\tilde{U} + \hat{U}) + E(\tilde{U} + \hat{U}) + A_p(\tilde{U} + \hat{U}) = F, \tag{4.4}
\]

The equation for the first component \( \mathbf{v} \) of the solution \( \tilde{U} = (\mathbf{v}, \hat{T}, \hat{S}) \) is given by

\[
\partial_t \mathbf{v} + ((\mathbf{v} + \mathbf{v}) \cdot \nabla)(\mathbf{v} + \mathbf{v}) + w(\mathbf{v} + \mathbf{v})\partial_z (\mathbf{v} + \mathbf{v}) + \frac{1}{\rho_0} \nabla \rho_s + \mathbf{f} \times (\mathbf{v} + \mathbf{v}) - \mu_\nu \Delta \mathbf{v} - \nu_\nu \partial_z \mathbf{v} = F_v + g \int_0^t \left( \beta_T \nabla (\hat{T} + \hat{T}) + \beta_S \nabla (\hat{S} + \hat{S}) \right) d\bar{z} \tag{4.5}
\]

Note that the diagnostic function, \( w(\cdot) \) is defined as above in (2.3). As, with \( \mathbf{v} \), we have \( \int_{-h}^0 \nabla \cdot \mathbf{v} d\bar{z} = 0 \). Of course, (4.5) is supplemented with boundary conditions as in (1.4), (1.5), (1.6).

Multiplying this system by \( \mathbf{v}^3 = (v_1^3, v_2^3, v_3^3) \) and then integrating over the domain \( M \) leads to the following system describing the time evolution of \( |\mathbf{v}_j^3|_{L^4}^4 \):
Estimates for the nonlinear term

We first provide estimates for the inertial terms $J_1$, $J_2$ in (4.6). For $J_1$ we estimate:

$$
\left| \sum_{k,j} \int_{\mathcal{M}} \hat{v}_j \partial_j \hat{v}_k \hat{v}_k^3 d\mathcal{M} \right| \leq c \sum_{k,j} \int_{\mathcal{M}} |\partial_j \hat{v}_k||[\hat{v}_k]^4| + |\hat{v}_j|^4| d\mathcal{M} 
\leq c |\nabla \hat{v}|_{L^6} \sum_j \left( \int_{\mathcal{M}} (|\hat{v}_j|^4)^{6/5} d\mathcal{M} \right)^{5/6}
\leq c |A\hat{U}| \sum_j \left( \int_{\mathcal{M}} (\hat{v}_j^2)^{12/5} d\mathcal{M} \right)^{(5/12)2} \leq c |A\hat{U}| \left( |\hat{v}_2|^3|\nabla_3 (\hat{v}_2)|^{1/4} \right)^2
\leq c |A\hat{U}| ||\hat{v}|_{L^4}^3|\nabla_3 (\hat{v}_2)|^{1/2} \leq c |A\hat{U}|^{4/3} ||\hat{v}|_{L^4}^4 + \kappa |\nabla_3 (\hat{v}_2)|^2
$$

Here we have used the Sobolev embeddings in $\mathbb{R}^3$ of $H^1$ into $L^6$ and of $H^{1/4}$ into $L^{5/12}$. For $J_2$ we also need to estimate

$$
\left| \sum_{k,j} \int_{\mathcal{M}} \hat{v}_j \partial_j \hat{v}_k \hat{v}_k^3 d\mathcal{M} \right| \leq c |\hat{v}|_{L^\infty} |\nabla \hat{v}|_{L^4} |\hat{v}|_{L^{4/3}} \leq c |A\hat{U}|^2 (1 + |\hat{v}|_{L^4}^4).
$$

In summary,

$$
|J_1| \leq c (1 + |A\hat{U}|^2) (1 + |\hat{v}|_{L^4}^4) + \kappa |\nabla_3 (\hat{v}_2)|^2. \quad (4.9)
$$

The estimates for $J_2$ make use of a preliminary integration by parts (cf. (2.1b))

$$
- \sum_k \int_{\mathcal{M}} w(\hat{v} + \hat{v}) \partial_z \hat{v}_k \hat{v}_k^3 d\mathcal{M}
= \sum_{j,k} \int_{\mathcal{M}} \int_0^\infty (\hat{v}_j + \hat{v}_j) d\tilde{z} \partial_{zj} \hat{v}_k \hat{v}_k^3 d\mathcal{M} + 3 \sum_{j,k} \int_{\mathcal{M}} \int_0^\infty (\hat{v}_j + \hat{v}_j) d\tilde{z} \partial_{zj} \hat{v}_k \hat{v}_k \hat{v}_k^3 d\mathcal{M}
:= J_{2,1} + J_{2,2} + J_{2,3} + J_{2,4}.
$$

For $J_{2,1}$ we use the embedding of $H^{3/4}$ into $L^4$ in $\mathbb{R}^3$ and find,

$$
|J_{2,1}| \leq c |\hat{v}|_{L^8} |\nabla \partial_z \hat{v}| |\hat{v}|_{L^{8/3}} \leq c |A\hat{U}| |\hat{v}|_{L^8} \leq c |A\hat{U}| \sum_j \left( \int_{\mathcal{M}} (\hat{v}_j^2)^{4} \right)^{(1/4)2}
\leq c |A\hat{U}| \left( |\hat{v}_2|^4 |\nabla_3 (\hat{v}_2)|^{3/4} \right)^2 \leq c |A\hat{U}|^4 |\hat{v}|_{L^4}^4 + \frac{\kappa}{2} |\nabla_3 (\hat{v}_2)|^2.
$$

For $J_{2,2}$, by making use of Agmon’s inequality and that $H^{1/2}$ is embedded into $L^3$ we infer:

$$
|J_{2,2}| \leq |\hat{v}|_{L^\infty} |\nabla \partial_z \hat{v}| |\hat{v}|_{L^{3/2}} \leq c |\hat{v}|_{L^\infty} |\nabla \partial_z \hat{v}| |\hat{v}_2|^{3/2} \leq c |A\hat{U}|^2 \left( |\hat{v}_2|^1 |\nabla_3 (\hat{v}_2)|^{1/2} \right)^{3/2} \leq c |A\hat{U}|^2 |\hat{v}_2|^{3/2} |\nabla_3 (\hat{v}_2)|^{3/4}
\leq c |A\hat{U}|^{16/5} |\hat{v}_2|^{12/5} \frac{\kappa}{2} |\nabla_3 (\hat{v}_2)|^2 \leq c (1 + |A\hat{U}|^4) (1 + |\hat{v}|_{L^4}^4) + \frac{\kappa}{2} |\nabla_3 (\hat{v}_2)|^2.
$$
$J_{2,3}$ and $J_{2,4}$ seem to require ‘anisotropic’ type estimates:

$$|J_{2,3} + J_{2,4}| \leq \sum_{j,k} \int_{\mathcal{M}_0} \int_0^1 (\hat{v}_j + \hat{v}_j) \, dz \leq \sum_{j,k} \int_{\mathcal{M}_0} |\hat{v}_j + \hat{v}_j| \, dz \leq \sum_{j,k} \int_{\mathcal{M}_0} |\hat{v}_j + \hat{v}_j| \, dz \leq \sum_{j,k} |\hat{v}_j + \hat{v}_j| \, dz \leq \sum_{j,k} |\hat{v}_j + \hat{v}_j| \, dz \leq \sum_{j,k} |\hat{v}_j + \hat{v}_j| \, dz$$

Note that we made use of Remark 2.1, (2.2) with $q = 12$ in order to estimate $|\hat{v}_j + \hat{v}_j| \leq c\|U\|$. Summarizing the above estimates and taking advantage of the observation (4.8) we conclude that

$$|J_2| \leq c(1 + \|U\|^2)(1 + |A\hat{U}|^4)(1 + |\hat{v}_s|_L^4) + \mu \nu \sum_{j,k} \int_{\mathcal{M}} (\hat{v}_j \hat{v}_k)^2 \hat{v}_k^2 \, d\mathcal{M} + \nu \sum_{j,k} \int_{\mathcal{M}} (\hat{v}_j \hat{v}_k)^2 \hat{v}_k^2 \, d\mathcal{M}. \quad (4.10)$$

**Estimates for the Pressure**

We next attend to the term $J_3$. Using the fact that the pressure term, $p_s$, depends only on the horizontal variable $x$ we find

$$|J_3| = \left| \sum_{j} \frac{1}{\rho_0} \int_{\mathcal{M}_0} \partial_j \hat{p}_s \int_{-h}^0 \hat{\nu}_j^2 \, dz \, d\mathcal{M}_0 \right| \leq c|\nabla p_s|_{L_x^{4/3}} \left| \int_{-h}^0 \hat{\nu}_j^2 \, dz \right|_{L_x^{4/3}}.$$

By applying the Sobolev embedding in $D = 2$ that $W^{4/3,1}$ is embedded in $L^4(\mathcal{M}_0)$ we have

$$\left| \int_{-h}^0 \hat{\nu}_j^2 \, dz \right|_{L_x^4} \leq c \sum_{j,k} \left| \partial_j \hat{\nu}_j \right|_{L_x^{4/3}} \leq c \sum_{j,k} \left| \int_{-h}^0 (\partial_j \hat{\nu}_j \hat{v}_k) \hat{v}_k \, dz \right|_{L_x^{4/3}} \leq c \sum_{j,k} \left( \int_{\mathcal{M}_0} (\int_{-h}^0 (\partial_j \hat{v}_k)^2 \hat{v}_k^2 \, dz)^{2/3} \int_{-h}^0 \hat{v}_k^2 \, dz \, d\mathcal{M}_0 \right)^{3/4} \leq c|\hat{v}_s|_{L^4} \sum_{j,k} \left( \int_{\mathcal{M}} (\partial_j \hat{v}_k)^2 \hat{v}_k^2 \, d\mathcal{M} \right)^{1/2}.$$

Combining these estimates and using Young’s inequality we conclude that:

$$|J_3| \leq c|\nabla p_s|_{L_x^{4/3}}^2 |\hat{v}_s|_{L^4}^2 + \mu \nu \sum_{j,k} \int_{\mathcal{M}} (\partial_j \hat{v}_k)^2 \hat{v}_k^2 \, d\mathcal{M}. \quad (4.11)$$

**Remark 4.2.** Note that it is at precisely this stage in the estimates that we see the crucial role played by the two dimensional spatial dependence of the surface pressure $p_s$. This insight concerning the importance of the lower dimension pressure is the key to the recent breakthroughs for global existence in [7, 28, 29, 30]. In the present study we follow the approach in [30] which treats the pressure explicitly via bounds on the pressure for the Stokes equation found in [45] and reproduced for the convenience of the reader below in Proposition 6.1.
In order to estimate the pressure terms appearing after the final inequality above we now apply Proposition 6.1 below. To this end we average (4.5) in the vertical direction. Define the operator \( \mathfrak{A}(\dot{v}) = \frac{1}{h} \int_{h}^{3h} \dot{v} dz \) and let \( \mathfrak{q} = \rho_0 \mathfrak{A}(\dot{v}) \). By applying \( \mathfrak{A} \) to (4.5) we find that \( \mathfrak{q} \) satisfies the following Stokes system over \( \mathcal{M}_0 \subset \mathbb{R}^2 \):

\[
\begin{align*}
\partial_t \mathfrak{q} - \mu_v \Delta \mathfrak{q} + \nabla \hat{p}_s &= G(U), \\
\nabla \cdot \mathfrak{q} &= 0, \quad \mathfrak{q}_{\partial \mathcal{M}_0} = 0,
\end{align*}
\]

where, for \( \mathfrak{v} = \dot{v} + \hat{v} \),

\[
G(U) = G(\mathfrak{v}, T, S) = G_1(U) + G_2(U) \colon = -\rho_0 \mathfrak{A} ((\mathfrak{v} \cdot \nabla) \mathfrak{v} + (\nabla \cdot \mathfrak{v}) \mathfrak{v}) + \rho_0 \mathfrak{A} \left( g \int_{h}^{3h} (\beta_T \nabla T + \beta_S \nabla S) d\xi - f_k \times \mathfrak{v} + F_v \right) .
\]

(4.12)

Note that, using (1.2c) and taking into account the boundary conditions (1.4), (1.5), \( \mathfrak{A}(\mathfrak{v}) \partial_{\mathfrak{z}} \mathfrak{v} = \mathfrak{A}((\mathfrak{v} \cdot \nabla) \mathfrak{v}) \) and \( \mathfrak{A}(\partial_{\mathfrak{z}} \mathfrak{v}) = 0 \).

Making use of Proposition 6.1 below, we find, for any \( 0 < t < \infty \), and for any pair of stopping times \( \tau_a, \tau_b \), with \( 0 \leq \tau_a \leq \tau_b \leq t \wedge \xi \) that

\[
\int_{\tau_a}^{\tau_b} |\nabla \hat{p}_s|_{L^2}^2 \, dt' \leq c \left( \left\| \mathfrak{q}(\tau_a) \right\|^2 + \int_{\tau_a}^{\tau_b} |G(U)|_{L^2}^2 \, dt' \right).
\]

Observe that \( \| \mathfrak{q}(\tau_a) \| \leq c\|\hat{v}(\tau_a)\| \) and that

\[
\int_{\tau_a}^{\tau_b} |G_2(U)|_{L^2}^2 \, ds \leq c \int_{\tau_a}^{\tau_b} |G_2(U)|^2 \, ds \leq c \int_{\tau_b}^{\tau_b} (\|U\|^2 + |F|^2) \, ds.
\]

On the other hand,

\[
\int_{\tau_a}^{\tau_b} |G_1(U)|_{L^2}^2 \, dt' \leq c \int_{\tau_a}^{\tau_b} |(\mathfrak{v} \cdot \nabla) \mathfrak{v} + (\nabla \cdot \mathfrak{v}) \mathfrak{v}|_{L^3}^2 \, dt'
\]

\[
\leq c \int_{\tau_a}^{\tau_b} \left( \int_{\mathcal{M}} |\mathfrak{v}|^4/3 |\nabla \mathfrak{v}|^4/3 \, d\mathcal{M} \right)^{3/2} \, dt' \leq c \int_{\tau_a}^{\tau_b} |\mathfrak{v}|_{L^4}^2 \|\nabla\mathfrak{v}\|^2 \, dt'.
\]

In conclusion:

\[
\int_{\tau_a}^{\tau_b} |\nabla \hat{p}_s|_{L^2}^2 \, dt' \leq c \left( \|\hat{v}(\tau_a)\|^4 + \int_{\tau_a}^{\tau_b} ((1 + |\mathfrak{v}|_{L^4}^4) \|U\|^2 + |F|^2) \, dt' \right).
\]

(4.13)

Combining (4.13) and (4.11) we find, for any pair of stopping times \( 0 \leq \tau_a \leq \tau_b \leq t \wedge \xi \),

\[
\int_{\tau_a}^{\tau_b} |J_3| \, dt' \leq \frac{1}{2} \sup_{t' \in [\tau_a, \tau_b]} \|\hat{v}\|_{L^4}^4 + \mu_v \int_{\tau_a}^{\tau_b} \sum_{j,k} \int_{\mathcal{M}} (\partial_j \hat{v}_k)^2 \hat{v}_k^2 \, d\mathcal{M} \, dt'
\]

\[
+ c \left( \|\hat{v}(\tau_a)\|^4 + (1 + \sup_{t' \in [\tau_a, \tau_b]} (|\mathfrak{v}|_{L^4}^4 + |\mathfrak{v}|_{L^4}^4) \int_{\tau_a}^{\tau_b} (\|U\|^2 + |F|^2) \, dt' \right).
\]

(4.14)

Estimates for the Lower Order Terms

To estimate the term \( J_4 \) we use the embedding of \( L^3 \) into \( H^{1/2} \) in \( 2D \) and we observe that

\[
|J_4| \leq c \|U\| \sum_{k} \left( \int_{\mathcal{M}} (\hat{v}_k^2)^3 \, d\mathcal{M} \right)^{1/3} \leq c \|U\| \left( \|\hat{v}^2\|_{L^3}^{1/2} |\nabla^3 \hat{v}^2|_{L^2}^{1/2} \right)^{3/2}
\]

\[
\leq \kappa |\nabla^3 \hat{v}^2|^2 + c(1 + \|U\|^2)(1 + |\hat{v}|_{L^4}^4).
\]

(4.15)
The estimate for the Coriolis term $J_5$ is direct
\[ |J_5| \leq c|\nabla L_x| \hat{v}^4_{L^4} \leq c\|U\|(1 + |\hat{v}^4_{L^4}|). \] (4.16)

Finally for $J_6$ we find
\[ |J_6| \leq |F_x|L_x| \hat{v}^3_{L^4} \leq c(|F|^2_{L^4} + 1)\|\hat{v}^4_{L^4} + 1). \] (4.17)

Integrating (4.6) in time from $\tau_0$ to $\tau_B$ and then putting together the estimates (4.8), (4.9), (4.10), (4.14), (4.16), (4.17) we find that for any pair of stopping times $\tau_0$, $\tau_B$ with $0 \leq \tau_0 \leq \tau_B \leq T \land \xi$
\[
\sup_{t' \in [\tau_0, \tau_B]} |\hat{v}^4_{L^4} | \leq c\|\hat{v}(\tau_0)\|^4 + c \left(1 + \sup_{t' \in [\tau_0, \tau_B]} |\hat{v}^4_{L^4} | + \sup_{t' \in [0, T \land \xi]} |\hat{v}^4_{L^4} | \right) \int_{\tau_0}^{T_B} H(t') dt',
\]
where
\[ H(t') := (1 + \|U(t')\|^2)(1 + |A\hat{U}(t')|^4) + |F(t')|^2_{L^4} \]

We finally apply the version of Gronwall’s lemma in Proposition 6.2 below with $p = 2$, $X(t') = |\hat{v}(t')|^4_{L^4}$, $f(t') = c\|v(t')\|^2$, $g(t') = cH(t')$ and $h(t') = c(1 + \sup_{t' \in [0, T \land \xi]} |\hat{v}^4_{L^4} |)H(t')$. Observe that, as a consequence of Theorem 3.1, $X \in C([0, T \land \xi])$ and $f$ is continuous at $t = 0$. Also, due to Theorem 3.1, Proposition 4.2 and the standing assumption $(2.9)$, we easily infer that $f, g, h \in L^1([0, T \land \xi])$. Thus by, Proposition 6.2, we conclude that $\sup_{t' \in [0, T \land \xi]} |\hat{v}^4_{L^4} | < \infty$. Combining this with (4.3) we infer (4.1). Since it follows directly from (4.1) that $\lim_{K \uparrow \infty} \tau_K^{(1)} = \infty$, this completes the proof of Proposition 4.1.

5 Vertical Gradient Estimates

We next turn to estimate $\partial_z U$ in the $L^2(\mathcal{M})$ norm. The aim here is to establish the other requirement needed to satisfy the global existence criteria given in Theorem 3.2:

**Proposition 5.1.** Assume the conditions set out in Theorem 2.1 and suppose that $(U, \xi)$ is the corresponding strong pathwise solution of (1.2). Then, for any $t > 0$,
\[
\sup_{t' \in [\xi \land t]} |\partial_z U|^2 + \int_{\xi \land t}^t \|\partial_z U\|^2 dt' < \infty. \] (5.1)

Hence, defining $\tau_K^{(2)}$ according to (3.4), $\lim_{K \uparrow \infty} \tau_K^{(2)} = \infty$.

Proposition 5.1 follows as a direct consequence of Corollary 5.1 and Proposition 5.3 below. For the analysis we make use of several additional stopping times:
\[
\tau_K := \inf_{t \geq 0} \left\{ \sup_{t' \in [0, \xi \land t]} |\partial_z v(t')|^2 + \int_0^{t \land \xi} \|\partial_z v(t')\|^2 \geq K \right\},
\]
\[
\tau_K^T := \inf_{t' \geq 0} \left\{ \sup_{t' \in [0, \xi \land t]} (|T(t')|^4_{L^4} + |S(t')|^4_{L^4} \geq K \right\} ,
\]
\[
\tau_K^M := \tau_K^T \land \tau_K \land \tau_K^{(1)} .
\]

Here $\tau_K^{(1)}$ is defined as in (3.4). We first tackle $\partial_z v$ in $L^2$. From these estimates along with those carried out above for $v$ in $L^4(\mathcal{M})$ we infer that $\lim_{K \uparrow \infty} \tau_K^{(2)} = \infty$. We next would like to estimate $\partial_z T \partial_z S$ in $L^2(\mathcal{M})$. However, in order to be able to treat the nonlinear terms which arise (see, for example, (5.20) below) we first must estimate $T$ and $S$ in $L^4(\mathcal{M})$. This is the reason for the second collection of stopping times defined in (5.2). In this case, in contrast to the momentum equations above, there are no pressure terms which means that we can perform these $L^4$ estimates in the original variables by means of the Itô lemma. With these estimates we deduce that $\lim_{K \uparrow \infty} \tau_K^{M} = \infty$. We finally establish bounds for $\partial_z T$ and $\partial_z S$ in Proposition 5.3.
Additional Estimates for the Momentum Equations

**Proposition 5.2.** Assume the conditions set out in Theorem 2.1 and suppose that \( (U, \xi) \) is the resulting strong pathwise solution of \((1.2)\). Define \( \tau^{(1)}_K \) as in \((3.4)\). Then, for any \( t > 0 \),

\[
E \left( \sup_{\tau \in [0, \tau^{(1)}_K \wedge \xi^{\wedge t}]} |\partial_z v|^2 + \int_{0}^{\tau^{(1)}_K \wedge \xi^{\wedge t}} \|\partial_z v\|^2 dt' \right) < \infty. \tag{5.3}
\]

As an immediate consequence of this proposition and Proposition 4.1 above we infer:

**Corollary 5.1.** Assume that the conditions set out in Theorem 2.1 hold and that \( (U, \xi) \) is the resulting strong, pathwise solution of \((1.2)\). Then for, any \( t > 0 \),

\[
\sup_{\tau \in [\xi^{\wedge t}, t]} |\partial_z v|^2 + \int_{0}^{\tau^{\wedge t}} \|\partial_z v\|^2 dt' < \infty \quad a.s. \tag{5.4}
\]

Moreover, defining the stopping times \( \tau^{(1)}_K \) according to \((5.2)\) we have that \( \lim_{K \to \infty} \tau^{(1)}_K = \infty \).

**Proof of Proposition 5.2:** We apply \( \partial_z \) to \((2.1a)\) to infer an evolution equation for \( \partial_z v \):

\[
d \partial_z v + (\partial_z [(v \cdot \nabla)v + w(v)\partial_z v] + g\partial_z [\beta_T \nabla T + \beta_S \nabla S] + f\mathbf{k} \times (\partial_z v)) dt \\
- (\mu_v \Delta \partial_z v + \nu_v \partial_{zzz} v) dt = \partial_z F_v dt + \partial_z \sigma_v(v,T,S) dW. \tag{5.5}
\]

Note that, since \( \tilde{p}_v \) is independent of \( z \), the pressure term does not appear above. After an application of the Itô formula to this system we find

\[
dx |\partial_z v|^2 + 2|\partial_z v|^2 dt = -2 \int_{\mathcal{M}} (\partial_z [(v \cdot \nabla)v + w(v)\partial_z v] + g\partial_z [\beta_T \nabla T + \beta_S \nabla S]) \cdot \partial_z v d\mathcal{M} dt \\
+ 2(\partial_z F_v, \partial_z v) dt + |\partial_z \sigma_v(v,T,S)|_{L^2(\mathcal{M})}^2 dt + 2|\partial_z \sigma_v(v,T,S), \partial_z v| dW \\
= (J^1_v + J^2_v + J^3_v + J^4_v + J_v^5) dt + J^6_v dW. \tag{5.6}
\]

Here we have made use of the boundary conditions \((1.4), (1.5), (1.6)\) in order to infer \(- (\mu_v \Delta \partial_z v + \nu_v \partial_{zzz} v), \partial_z v) = \|\partial_z v\|^2\). Note also that there is a cancelation in the ‘Coriolis term’ \((k \times \partial_z v, \partial_z v) = 0\).

We begin by treating the nonlinear terms. For \( J^1_v \), an integration by parts reveals

\[
J^1_v = - \sum_{j,k} \int_{\mathcal{M}} \partial_z v_j \partial_z v_k \partial_z v_k d\mathcal{M} - \sum_{j,k} \int_{\mathcal{M}} v_j \partial_z v_j \partial_z v_k d\mathcal{M}
= \sum_{j,k} \int_{\mathcal{M}} \partial_j \partial_z v_j \partial_z v_k d\mathcal{M} + \sum_{j,k} \int_{\mathcal{M}} \partial_z v_j \partial_z v_j \partial_z v_k d\mathcal{M} - \sum_{j,k} \int_{\mathcal{M}} v_j \partial_z v_j \partial_z v_k d\mathcal{M}.
\]

Hence with direct estimates using Holder’s inequality and the Sobolev embedding of \( H^{3/4} \) into \( L^4(\mathcal{M}) \)

\[
|J^1_v| \leq c |\partial_z v| |v|_{L^4(\mathcal{M})} |\partial_z v|_{L^4} \leq c |\partial_z v|^{7/4} |v|_{L^{4/3}(\mathcal{M})} |\partial_z v|^{1/4} \leq c |v|_{L^{4/3}(\mathcal{M})} |\partial_z v|^2 + \frac{1}{3} |\partial_z v|^2. \tag{5.7}
\]

For the second portion, \( J^2_v \), of the nonlinear term we have:

\[
J^2_v = \int_{\mathcal{M}} w(v) \partial_z v \cdot \partial_{zz} v d\mathcal{M} = \frac{1}{2} \int_{\mathcal{M}} w(v) \partial_z (\partial_z v \cdot \partial_z v) d\mathcal{M}
= \frac{1}{2} \int_{\mathcal{M}} \nabla \cdot v \partial_z v \cdot \partial_z v d\mathcal{M} = - \sum_j \int_{\mathcal{M}} v_j \partial_j \partial_z v \cdot \partial_z v d\mathcal{M},
\]

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and hence, as for $J_1^2$ we may estimate

$$|J_2^2| \leq c |v|_{L^4}^2 |\partial_z v|^2 + \frac{1}{3} \|\partial_z v\|^2. \quad (5.8)$$

Direct estimates and using the assumption (2.12) lead to

$$|J_3^1| + |J_3^2| + |J_5^2| \leq c(1 + \|U\|^2 + |F|^2) + \frac{1}{3} \|\partial_z v\|^2. \quad (5.9)$$

On other hand, applying the Burkholder-Davis-Gundy inequality, (2.10), and again using (2.12) we find that for $0 \leq t_0 \leq t \wedge \xi \wedge \tau_K^{(1)},$

$$\mathbb{E} \sup_{t' \in [\tau_n, \tau_0]} \left| \int_{\tau_n}^{t'} J_6^5 dW \right| \leq c \mathbb{E} \left( \int_{\tau_n}^{\tau_0} (\partial_z \sigma_z(U, \partial_z v))^2 dt \right)^{1/2} \leq c \mathbb{E} \left( \int_{\tau_n}^{\tau_0} |\sigma_z(U)|_{L^2(U, V)}^2 |\partial_z v|^2 dt \right)^{1/2} \quad (5.10)$$

$$\leq c \mathbb{E} \left[ \sup_{t' \in [\tau_n, \tau_0]} |\partial_z v| \left( \int_{\tau_n}^{\tau_0} (1 + \|U\|^2) dt \right)^{1/2} \right] \leq \frac{1}{2} \mathbb{E} \sup_{t' \in [\tau_n, \tau_0]} |\partial_z v|^2 + c \mathbb{E} \int_{\tau_n}^{\tau_0} (1 + \|U\|^2) dt.$$

Gathering the estimates (5.7), (5.8), (5.9), (5.10) we apply the stochastic Gronwall Lemma (see [19, Lemma 5.3]) and infer (5.3). This completes the proof.

**Additional Estimates for the Temperature and Salinity Equations**

We turn to the estimates for the temperature and the salinity, beginning with the estimates in $L^4(\mathcal{M}).$

**Lemma 5.1.** Suppose the standing conditions as in Theorem 2.1 are satisfied and let $(U, \xi)$ be the maximal strong solution of (2.13). Define $\tau_{K}^{\infty}$ as in (3.3). Then, for any $t > 0,$

$$\mathbb{E} \left( \sup_{t' \in [0, t \wedge \tau_{K}^{\infty}]} (\|T\|_{L^4}^4 + \|S\|_{L^4}^4) \right) < \infty, \quad (5.11)$$

Moreover,

$$\sup_{t' \in [0, t \wedge \tau_{K}^{\infty}]} (\|T\|_{L^4}^4 + \|S\|_{L^4}^4) < \infty. \quad a.s. \quad (5.12)$$

With $\tau_{K}^{\infty}$ defined as in (5.2), we have that $\lim_{K \to \infty} \tau_{K}^{\infty} = \infty.$

**Proof.** We carry out the estimates for the temperature $T.$ Those for the salinity $S$ are identical. In order to find an evolution equation for $\|T\|_{L^4}^4$ we apply the Itô lemma to (2.1c) pointwise for almost every $(x, z) \in \mathcal{M}.$ We then integrate the resulting system over $\mathcal{M}$ and find that

$$d\|T\|_{L^4}^4 + \left( 12 \mu T \sum_{j} \int_{\mathcal{M}} (\partial_{x_j} T)^2 T^2 d\mathcal{M} + 12 \nu T \int_{\mathcal{M}} (\partial_{z} T)^2 T^2 d\mathcal{M} \right) dt$$

$$= 4 \int_{\mathcal{M}} F_T T^3 d\mathcal{M} dt + 6 \sum_{l \geq 1} \int_{\mathcal{M}} \sigma_T (U)^2 T^2 d\mathcal{M} dt + 4 \sum_{l \geq 1} \int_{\mathcal{M}} \sigma_T (U)_l T^3 d\mathcal{M} dW_l. \quad (5.13)$$

Here we have once again used the cancelation properties of the nonlinear portion of (2.1c) (cf. (4.7)).

The external body forcing term is estimated as above (cf. (4.17)):

$$\left| \int_{\mathcal{M}} F_T T^3 d\mathcal{M} \right| \leq c (|F_T|_{L^1}^2 + 1) (\|T\|_{L^4}^4 + 1). \quad (5.14)$$
For the Itô correction term, using (2.12), we find
\[
\left| \sum l \int_{\mathcal{M}} \sigma_T(U)^2 T^3 d\mathcal{M} dt \right| \leq c \sum l |\sigma_T(U)|_{L^4}^2 |T|_{L^4}^2 \\
\leq c |\sigma_T(U)|_{L^2(\mathcal{U,V})} |T|_{L^4}^2 \\
\leq c(1 + \|U\|^2)(1 + |T|_{L^4}^2) \tag{5.15}
\]

With an application of the BDG inequality, (2.10), we find
\[
\mathbb{E} \sup_{t \in [\tau_a, \tau_b]} \left| \int_{\tau_a}^{t} \sum l \int_{\mathcal{M}} \sigma_T(U)^2 T^3 d\mathcal{M} dt \right| \leq \mathbb{E} \left( \int_{\tau_a}^{\tau_b} \sum l \left( \int_{\mathcal{M}} \sigma_T(U)^2 T^3 d\mathcal{M} \right)^2 dt \right)^{1/2} \\
\leq \mathbb{E} \left( \int_{\tau_a}^{\tau_b} \sum l |\sigma_T(U)|_{L^4}^2 |T|_{L^4}^2 dt \right)^{1/2} \\
\leq \mathbb{E} \left( \int_{\tau_a}^{\tau_b} |\sigma(U)|_{L^2(\mathcal{U,V})} |T|_{L^4}^2 dt \right)^{1/2} \\
\leq \mathbb{E} \sup_{t \in [\tau_a, \tau_b]} |T|_{L^4}^2 \left( \int_{\tau_a}^{\tau_b} (1 + \|U\|^2)|T|_{L^4}^2 dt \right)^{1/2} \\
\leq \frac{1}{2} \mathbb{E} \sup_{t \in [\tau_a, \tau_b]} |T|_{L^4}^2 + \mathbb{E} \int_{\tau_a}^{\tau_b} (1 + \|U\|^2)(1 + |T|_{L^4}^2) dt. \tag{5.16}
\]

Here, similarly to (5.10) above, \(\tau_a, \tau_b\) may be any stopping time such that \(0 \leq \tau_a \leq \tau_b \leq t \land \tau_{K}^W \land \xi\). Combining the estimates (5.14), (5.15), (5.16) with (5.13) we apply the stochastic Gronwall lemma to infer (5.11). Now, due to Theorem 3.1, \(\lim_{K \uparrow \infty} \tau_{K}^W = \infty\). With this observation (5.12) and \(\lim_{K \uparrow \infty} \tau_{K}^T = \infty\) follow directly from (5.11). This completes the proof.

The final step is now to carry out the estimates for \(\partial_z T\) and \(\partial_z S\) in \(L^2(\mathcal{M})\):

**Proposition 5.3.** Assume the conditions of Theorem 2.1 consider \((U, \xi)\) the maximal strong solution of (2.13). Take \(\tau_{K}^M\) according to (5.2). Then, for any \(t > 0\),
\[
\mathbb{E} \left( \sup_{t \in [0, t \land \tau_{K}^M \land \xi]} (|\partial_z T|^2 + |\partial_z S|^2) + \int_{0}^{t \land \tau_{K}^M \land \xi} (|\partial_z T|^2 + |\partial_z S|^2) dt' \right) < \infty. \tag{5.17}
\]

Moreover, up to a set of measure zero,
\[
\sup_{t \in [0, t \land \xi]} (|\partial_z T|^2 + |\partial_z S|^2) + \int_{0}^{t \land \xi} (|\partial_z T|^2 + |\partial_z S|^2) dt' < \infty. \tag{5.18}
\]

**Proof.** As above, in Lemma 5.1 we provide the estimates for \(T\); those for \(S\) are identical. To this end we apply \(\partial_z\) to (2.1c) and find:
\[
d\partial_z T + \partial_z [(v \cdot \nabla)T + w(v)\partial_z T] dt - (\mu_T \Delta \partial_z T + \nu_T \partial_{zzz} \partial_z T) dt = \partial_z F_T dt + \partial_z \sigma_T(v, T, S) dW. \tag{5.19}
\]

An application of the Itô formula then reveals:
\[
d|\partial_z T|^2 + 2|\partial_z T|^2 = 2(\partial_z F_T, \partial_z T) dt + |\partial_z \sigma_T(v, T, S)|_{L^2(\mathcal{U,H})}^2 dt \\
- 2 \int_{\mathcal{M}} \partial_z [(v \cdot \nabla)T + w(v)\partial_z T] \partial_z T d\mathcal{M} dt + 2(\partial_z \sigma_T(v, T, S), \partial_z T) dW = (J_1^T + J_2^T + J_3^T + J_4^T) dt + J_5^T dW \tag{5.20}
\]
For the nonlinear term, $J_3^T$, an integration by parts reveals that,

$$-J_3^T = \sum_j \int_M (v_j \partial_j z T + \partial_j v_j \partial_j T) \partial_j z T \, dM = \sum_j \int_M (v_j \partial_j z T \partial_j z T - \partial_j v_j \partial_j z T - \partial_z v_j \partial_j z T) \, dM,$$

and we may therefore estimate

$$|J_3^T| \leq c(|\mathbf{v}|_{L^4} ||\partial_z T||_{L^4} + ||\partial_z \mathbf{v}||_{L^4} ||\partial_z T||_{L^4} + ||\partial_z \mathbf{v}||_{L^4} ||\partial_z T||_{L^4} + ||\partial_z \mathbf{v}||_{L^4} ||\partial_z T||_{L^4} + ||\partial_z \mathbf{v}||_{L^4} ||\partial_z T||_{L^4}) \leq c(|\mathbf{v}|_{L^4} ||\partial_z T||_{L^4}^{7/4} ||\partial_z T||_{L^4}^{1/4} + ||\partial_z \mathbf{v}|| ||\partial_z T||_{L^4}^{1/4} ||\partial_z T||_{L^4}^{3/4} ||\partial_z \mathbf{v}||_{L^4}^{1/4} ||\partial_z T||_{L^4}) \leq c(|\mathbf{v}|_{L^4} ||\partial_z T||_{L^4}^{5/2} + ||\partial_z \mathbf{v}||^{2} ||\partial_z T||_{L^4}^{1/2} + ||\partial_z \mathbf{v}||_{L^4}^{2} ||\partial_z T||_{L^4}^{5/2}) + \frac{1}{2} ||\partial_z T||_{L^4}^{2} + ||\partial_z T||_{L^4}^{2}.$$

Also,

$$J_4^T = \int_M w(\mathbf{v}) \partial_j z T \partial_j z T \, dM = -\frac{1}{2} \int_M \partial_j w(\mathbf{v}) (\partial_j z T)^2 \, dM = \frac{1}{2} \int_M \partial_j v_j (\partial_j z T)^2 \, dM = -\int_M v_j \partial_j z T \partial_j z T \, dM,$$

so that

$$|J_4^T| \leq c(|\mathbf{v}|_{L^4} ||\partial_z T||_{L^4}^{7/4} ||\partial_z T||_{L^4}^{1/4} \leq c(|\mathbf{v}|_{L^4} ||\partial_z T||_{L^4}^{5/2} + \frac{1}{2} ||\partial_z T||_{L^4}^{2}.$$

The remaining drift terms are estimated directly using assumption (2.12):

$$|J_1^T + J_2^T| \leq c(1 + ||U||^2 + ||F||^2) + ||\partial_z T||^2.$$

As above, in (5.10), we estimate the terms involving $J_6^T$ with the BDG inequality. We obtain for $\tau_a \leq \tau_b \leq t \wedge \tau^M \wedge \xi$

$$\mathbb{E} \sup_{t \in [\tau_a, \tau_b]} \left| \int_{\tau_a}^t J_6 dW \right| \leq \frac{1}{2} \mathbb{E} \sup_{t \in [\tau_a, \tau_b]} ||\partial_z T||^2 + c\mathbb{E} \int_{\tau_a}^{\tau_b} (1 + ||U||^2) dt'.$$

With the stochastic Gronwall lemma, the estimates (5.21), (5.22), (5.23), (5.24) imply (5.17). (5.18) now follow directly from (5.17), Proposition 4.1, Corollary 5.1 and Lemma 5.1. The proof is thus complete.

6 Appendix: Auxiliary Results

We collect here, for the convenience of the reader, two technical results that have been used in an essential way in the analysis above.

6.1 Pressure Estimates

We have made use of a special case of [45, Theorem 2.12] which provides $L^r_t L^r_x$ estimates for the pressure terms appearing in the linear Stokes equation.

**Proposition 6.1.** Suppose that $d \geq 2$ and that $\mathcal{M}_0 \subset \mathbb{R}^d$ is a bounded open domain with smooth boundary. Assume that $r \in (1, 2)$ and that $f \in L^2_{loc}([0, \infty); L^r(\mathcal{M}_0)).$ If $\mathbf{q}, p$ solves the Stokes equation in $\mathcal{M}_0 \times [0, \infty)$

$$\partial_t \mathbf{q} - \nu \Delta \mathbf{q} + \nabla p = f, \quad \nabla \cdot \mathbf{q} = 0, \quad \mathbf{q}|_{\partial \mathcal{M}_0} = 0,$$

then, for any $0 \leq \tau_0 \leq \tau_1 < \infty$,

$$\int_{\tau_0}^{\tau_1} \|\nabla p\|_{L^r(\mathcal{M}_0)}^2 ds \leq c \left( ||\mathbf{q}(\tau_0)||^2 + \int_{\tau_0}^{\tau_1} ||f||_{L^r(\mathcal{M}_0)}^2 ds \right),$$

where $c = c(d, r, \mathcal{M}_0)$ is an absolute constant independent of $f, \tau_0, \tau_1$. 
6.2 A Gronwall Lemma

We shall also make use of a particular version of the Gronwall Lemma to close the estimates for \( \mathbf{v} \) in \( L^4 \) for the proof of Proposition 4.1 in Section 4.

Proposition 6.2. Suppose that, for some \( t \geq 0 \), we are given \( f, g, h \in L^1([0, t]) \) and \( X \in C([0, t]) \). Assume that \( X, f, g, h \) are all positive for a.e. \( t' \in [0, t] \) and that \( f \) is continuous at \( t' = 0 \). If, for almost every \( 0 \leq \tau_a \leq \tau_b < t \), and for some fixed \( p \geq 1 \), we have

\[
\sup_{t' \in [\tau_a, \tau_b]} X(t') \leq f(\tau_a)^p + \sup_{t' \in [\tau_a, \tau_b]} X(t') \int_{\tau_a}^{\tau_b} g(t') \, dt' + \int_{\tau_a}^{\tau_b} h(t') \, dt',
\]

then there exists an absolute constant \( c = c(t, p, |f(0)|, |f|_{L^1}, |g|_{L^1}) \) such that,

\[
\sup_{t' \in [0, t]} X(t') \leq c \left( 1 + \int_{0}^{t} h(t') \, dt' \right).
\]

Proof. We begin with some preliminaries to determine the constant \( c \) in (6.3). Choose \( \epsilon > 0 \) so that \( f(\tau_a) \) \( g(t') \, dt' < 1/2 \) for any pair \( 0 \leq s' \leq s \leq t \) such that \( s - s' < \epsilon \). Now choose \( n \) in such a way that \( t/n < \epsilon/2 \). Then pick \( M > \max\{2, |f(0)|\} \) large enough such that \( \lambda(|f| \geq M) < t/(4n) \), where \( \lambda \) is the Lebesgue measure on the interval \([0, t]\). This later quantity \( M \) exists as a consequence of the Chebyshev inequality and the fact that \( f \in L^1([0, t]) \). We now show (6.3) is satisfied by taking \( c = 2M^p \).

To this end fix any \( t^* \in (t/2, t) \). Set \( t_0 = 0 \), \( t_n = t^* \). For each \( k = 1, \ldots, n - 1 \) we may pick an element \( t_k \in (t^* k/n, t^* (k + 1)/n) \) so that \( f(t_k) < M \). Note also that according to this choice \( t_k + 1 - t_k < \epsilon \). Thus, for \( k = 0, 1, \ldots, n \), by applying (6.2) with \( \tau_a = t_k \), \( \tau_b = t_{k+1} \), we infer,

\[
\sup_{t' \in [t_k, t_{k+1}]} X(t') \leq M^p + \frac{1}{2} \sup_{t' \in [t_k, t_{k+1}]} X(t') + \int_{t_k}^{t_{k+1}} h(t') \, dt'.
\]

Noting that \( X \in C([0, t]) \), we may rearrange and infer:

\[
\sup_{t' \in [t_k, t_{k+1}]} X(t') \leq 2M^p + 2 \int_{t_k}^{t_{k+1}} h(t') \, dt' \leq 2M^p \left( 1 + \int_{t_k}^{t_{k+1}} h(t') \, dt' \right).
\]

The analogue of (6.3), with \( t^* \) replacing \( t \) on the left hand side of the inequality now follows from a simple induction. Since \( M \) may be chosen independently of \( t^* \in (t/2, t) \) the proof is now complete.

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