Uniqueness of solutions for moist advection problems

Roger Temam\textsuperscript{a} and Joseph Tribbia\textsuperscript{b}

\textsuperscript{a}Institute for Scientific Computing and Applied Mathematics, Indiana University, Bloomington, IN, USA
\textsuperscript{b}National Center for Atmospheric Research, Boulder, CO, USA

\textsuperscript{*}Correspondence to: Roger Temam, Institute for Scientific Computing and Applied Mathematics, Indiana University, 831 East Third Street, Bloomington, IN 47405, USA. E-mail: temam@indiana.edu

This article presents, for physics-oriented readers, some recent mathematical results that establish the uniqueness of solutions for moist advection problems in climate models. The presentation is made for a simplified model, which is nevertheless typical and significant.

\textbf{Key Words:} uniqueness

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1. Introduction

It is known that our lack of understanding of the physics of clouds is a major cause of uncertainty in current weather predictions. The numerous related difficulties include the vast range of scales involved, from outer dimensions of 100–1000 km to the size (of the order of a micron) of the particles and droplets of water from which they are made. Another major difficulty that we address here is related to the change of phase occurring in the cloud, through e.g. evaporation and precipitation. These changes of phase are mathematically expressed by discontinuities, which render difficult a theoretical understanding of the corresponding equations. Because of this difficulty, the uniqueness of solutions of the corresponding initial and boundary value problems has not been known. In Remark 3.4, we present and elaborate on some simple classical examples that show how the discontinuity can produce non-uniqueness in some cases. In its turn, this lack of (known) uniqueness is an impediment to numerical simulations, as no solution is known that can serve as a test reference case, and if such a solution is not known to be unique then there is a lack of faith in the conclusions that one can draw when numerical simulations do not match observations.

The moist advection problem that we will consider is relatively simple from the physical viewpoint. It consists of the T–q system and its mathematical interpretation

2. The T–q system and its mathematical interpretation

The system that we consider is the basic system for moist advection, described in e.g. Haltiner (1971), Haltiner et al. (1980) and Rogers et al. (1989). The equations for the temperature \( T \) and content \( q \) of water vapour in the air satisfy the equations

\[
\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T + \frac{\partial q}{\partial p} \omega = -\frac{A T}{p} \omega = -\frac{\delta FL}{p} \omega, \tag{2.1}
\]

\[
\frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q + \frac{\partial q}{\partial p} \omega = -A q \omega = -\frac{\delta F}{p} \omega. \tag{2.2}
\]

Here \( \mathbf{v} = (v, \omega) \) is the (given) velocity of the fluid in the \( x, y, p \) coordinate system; \( \nabla = \left( \partial_x, \partial_y \right) \) is the horizontal gradient; \( c_p \) is the specific heat capacity of air at constant pressure. Equations (2.1), (2.2) include the dissipation operators \( A_T \) and \( A_q \), which do not appear in the references quoted above:

\[
A_T T = -\mu_T \Delta T - v_T \frac{\partial}{\partial p} \left[ \frac{\partial T}{RT} \right]^2, \tag{2.3}
\]

\[
A_q q = -\mu_q \Delta q - v_q \frac{\partial}{\partial p} \left[ \frac{\partial q}{RT} \right]^2. \tag{2.4}
\]

Here, \( \mu_T, v_T, \mu_q, v_q \) are turbulent positive viscosity coefficients and \( \Delta = \partial_x^2 + \partial_y^2 \) is the horizontal Laplacian; \( T = \bar{T}(p) \) is the average temperature over the isobaric surface with pressure \( p \).

Finally,

\[
F = F(T) = q_T \left( \frac{LR - c_p R_s T}{c_p R_s T^2 + q_s L^2} \right), \tag{2.5}
\]

\[
\delta = 1 \text{ for } \omega < 0 \text{ and } q > q_s, \tag{2.6}
\]

\[
\delta = 0 \text{ for } \omega \geq 0 \text{ or } q < q_s. \tag{2.7}
\]

Here \( q_s \) is the saturation specific humidity, a function of \( T \) and \( p, \bar{R} \) is the gas constant for air, \( R_s \) the gas constant for water...

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vapour and $L$ is the latent heat of condensation of water; see e.g. Gill (1982) and Pedlosky (1987). Both $q_s$ and $L$ are functions of temperature $T$ ($L$ a slightly varying function of $T$), but it is sufficient for study of the issues addressed in this article to treat these as constant.

We recall that the equations (2.1)–(2.4), which we do not study as such in this article, are derived from the hydrostatic primitive equations in the context of a small aspect ratio, which justifies the use of anisotropic diffusion operators and the pressure coordinate in the vertical. The equations (2.1)–(2.4) are studied from the points of view of mathematical formulation and existence and uniqueness of solutions in Coti Zelati and Temam (2012) and Coti Zelati et al. 2013. We realize that these equations are themselves a simplified model of moist advection. Besides the fact that they are decoupled from the dynamics (they assume that the velocity field $u$ is known), they do not account for any of the microphysics introduced in contemporary studies for e.g. water vapour, cloud water and rain water. Nevertheless, this model, which is studied in the classical references of Haltiner (1971), Haltiner and Williams (1980) and Rogers and Yau (1989), is sufficient for study of the issues addressed in this article. For a presentation of contemporary, more realistic models, the reader is referred to Grabowski and Smolarkiewicz (1990, 2002) and the references therein; these articles also discuss extensively the presentation of contemporary, more realistic models, the reader is referred to Grabowski and Smolarkiewicz (1990, 2002) and the references therein; these articles also discuss extensively the references therein.

Considering the case where $\omega > 0$, we see that $\delta$ takes any possible value between 0 and $1$ for $q < q_s$, because $\omega$ is equal to 0 for $q > q_s$. Thus, these terms cannot be defined at $x^* = \text{max}(x,0)$.

Additionally, it appears that one can prove the existence of a solution of (3.1), (3.2) with the same representation $h_q$ of $H(q - q_s)$, which means that we show the existence of $T, q$, and $h_q \in \mathcal{H}(q - q_s)$, such that

$$
(\dot{h}_q - h_q, q_1 - q_2) \geq 0,
$$

(3.4)

for any realization $h_q$ of $H(q - q_s)$. Equation (3.4) can be verified in an elementary way by considering all the possible cases $q_t - q_{t-1} < 0, > 0, > 0, = 0, i = 1, 2$. It can also be inferred from the fact that almost everywhere $H$ is the derivative of the convex function $x^* = \text{max}(x,0)$.

With this in mind, let us first examine the uniqueness of solution $\{q, h_q\}$ of (3.6) when $T$ is given. If $\{q_1, h_q \in \mathcal{H}(q_1 - q_s), i = 1, 2\}$ are two solutions of (3.6) (with the same $T$), then

$$
\frac{d}{dt}(q_1 - q_2) + \frac{F_0w}{p}\left(h_{q_1} - h_{q_2}\right) = 0,
$$

(3.7)

which entails after multiplication by $q_1 - q_2$ that

$$
\frac{d}{dt}\left[|q_1(t) - q_2(t)|^2 + 2\frac{F_0w}{p}\left[h_{q_1}(t) - h_{q_2}(t), q_1(t) - q_2(t)\right]\right] = 0,
$$

and since $F = F(T) \geq 0$ and $\omega^- \geq 0$, we infer from (3.4) that

$$
\frac{d}{dt}\left[|q_1(t) - q_2(t)|^2 \leq 0,
$$

(3.8)

and uniqueness follows.
Now, if we try to establish the uniqueness of the solution $T$ of
(3.5) when $q$ and $h_q$ are known, then
\[
\frac{d}{dt}(T_1 - T_2) = \frac{\omega - \nu}{p} L h_q [F(T_1) - F(T_2)]. \tag{3.9}
\]
Since $F$ is a Lipshitz function of $T$,
\[
|F(T_1) - F(T_2)| \leq c_1 |T_1 - T_2|, \tag{3.10}
\]
and we infer from (3.9) that
\[
\frac{d}{dt}(T_1 - T_2)^2 \leq c_1 |T_1 - T_2|^2,
\]
\[
|T_1(t) - T_2(t)|^2 \leq e^{-2c_1} |T_1(0) - T_2(0)|^2 = 0,
\]
where $c_1$ is another positive constant, then $T_1 = T_2$ at all times
for $T_1(0) = T_2(0)$.

Finally, if we were to prove the uniqueness of solutions for the
(simplified) system (3.5)–(3.6), then we would consider two
solutions $(T_1, q_1), (T_2, q_2),$ write the differences of the
equations satisfied by $T_1, T_2$ and $q_1, q_2$, and write the system that we
obtain for $T = T_1 - T_2, q = q_1 - q_2$; after multiplication by $2T$ and $2q$,
respectively, we obtain
\[
\frac{d}{dt} T^2 = \frac{2RT^2}{p} \omega - 2f(T_1, q_1) h_q T + 2f(T_2, q_2) h_q T,
\]
\[
\frac{d}{dt} q^2 = 2f(T_1, q_1) h_q q - 2f(T_2, q_2) h_q q.
\]
Alternatively,
\[
\frac{d}{dt} T^2 = \frac{2RT^2}{p} \omega - 2[f(T_1, q_1) - f(T_2, q_2)] h_q T - 2f(T_2, q_2) h_q T,
\]
\[
\frac{d}{dt} q^2 = [2f(T_1, q_1) - 2f(T_2, q_2)] h_q q
\]
\[
+ 2f(T_2, q_2)(h_q - h_q) q.
\]
All the terms on the right-hand sides of the equations above can be
handled properly, as in either (3.8) or (3.10), except for the term $-2f(T_2, q_2)(h_q - h_q) T$, for which none of the arguments above would apply, owing to lack of smoothness or positivity
($f(T_2, q_2) \leq 0$).

We can circumvent this difficulty by considering the moist
static energy,
\[
e = c_p T + L q,
\]
for which we have, by combining equations (3.5) and (3.6) and
using the fact that the same $h_q$ appears in both equations,
\[
\frac{d}{dt} e = \frac{R \omega}{p} (e - L q) = 0. \tag{3.11}
\]
The system consisting of (3.11) and (3.6) is of course
equivalent to the system consisting of (3.5) and (3.6); we add the
initial conditions $q(0) = q_0, e(0) = e_0 = c_p T_0 + L q_0$. The
uniqueness of solutions for (3.6), (3.11) can be proved as above,
using the monotonicity argument (3.4). More precisely, let
$q_1, h_{q_1}, q_2, h_{q_2}, e_1, e_2$ be two solutions of (3.6), (3.11) with
the same indicated initial data and let $q = q_1 - q_2, e = e_1 - e_2$.
Then $e(0) = q(0) = 0$ and
\[
\frac{d}{dt} e = \frac{R \omega}{p} (e - L q) = 0, \tag{3.12}
\]
\[
\frac{d}{dt} q = \frac{R \omega}{p} (h_q - h_{q_2}) = 0.
\]
We multiply the first equation above by $e$ and the second one by
$q$ and add the corresponding equations to obtain
\[
\frac{1}{2} \frac{d}{dt} (|e|^2 + |q|^2) + \frac{R \omega}{p} (h_q - h_{q_2})(q_1 - q_2)
\]
\[
= \frac{R \omega}{p c_p} (e - L q) q.
\]
We use (3.4) again and bound the right-hand side of (3.13) to find
\[
\frac{d}{dt} (|e|^2 + |q|^2) \leq c (|e|^2 + |q|^2), \tag{3.14}
\]
where $c$ is an appropriate positive constant; $e(t) = q(t) = 0$ follows from (3.14) since $e(0) = q(0) = 0$.

**Remark 3.1** When dealing with the actual system (2.8), (2.9), the
proofs are substantially more involved in view of the existence of the
nonlinear advective terms, in particular, and the fact that the
viscosity coefficients $(\mu, \nu)$, $(\eta_q, \nu_q)$ are allowed to be different
in the $T$ and $q$ equations; for details see Cottet Zelati et al. (2013).

**Remark 3.2** Concerning the uniqueness of solutions, nothing is
changed if we introduce given (known) source terms $S_T, S_q$ on the
right-hand sides of (2.8), (2.9) or (3.1), (3.2): $S_T$ has a physical
meaning related to radiation, while $S_q$ can be an evaporative
flux from the surface and can be introduced for mathematical
generality, or just for convenience as in Remark 3.3 below. For the
(physically relevant) source terms $S_T, S_q$ we may imagine a source
term depending on the solution, with possible discontinuities. We
will investigate some important examples in future work.

**Remark 3.3** Based on the analysis above, one possible way to
construct a semi-analytic reference solution of equations (2.8),
(2.9), possibly coupled with the dynamic equations and with other
equations, is as follows. We choose a function $u = (v, \omega)$ and a
function $q$ of physical relevance, $0 \leq q \leq 1$ or $0 \leq q \leq q_*$;
somehow arbitrarily we choose a function $h_q \in H(q - q_*),$ if $q$ reaches the value $q_*$. From this data we compute the source term $S_q$. Then, using that
same function $h_q$, we solve (2.8) with right-hand side
\[
\frac{F \omega}{p} L h_q
\]
(and no source term $S_T$). This equation for $T$, supplemented with
boundary conditions, is totally standard and could be solved with a
high reliable level of precision; we thus obtain a semi-analytic test
solution that can be used to test numerical methods.

**Remark 3.4** On the suggestion of one of the referees we present two
even simpler models, which display the mechanism of uniqueness
that we established above and also the mechanism of non-uniqueness
that we did not derive for equations (3.5)–(3.6). We consider the
discontinuous multi-valued differential equation
\[
u' \in (1-t)^2 + H(u), \tag{3.15}
\]
\[
u(0) = -1/3. \tag{3.16}
\]
Since $u(0) < 0, H(u) = 0$ at time 0 and $u(t) = t - t^2 + (t^3 - 1)/3$ as long as this number remains $< 0$, i.e. until $t_1 = 1,
$ at which time $u(1) = 0$. Note that $u'(1) = H(0). The evolution
of the system after time $t = 1$ will depend on the value $\alpha \in [0, 1]$ of $H(0)$ that the system 'chooses'. The system 'choosing' a value
for $H(0)$ is not a physically absurd statement. We may imagine
that this choice depends on auxiliary physical mechanisms (such as
microphysics in our case) that are not accounted for in (3.15). If the
system 'chooses' the value $H(0) = 0$, then $u$ will remain equal to $0$ as long as $H(0) = 0$. If the system chooses $H(0) = \alpha, 0 \leq \alpha \leq 1$
at time $t = 1$, then $u'(0) > 0$, $u$ will become $> 0$, $H(u) = 1$ and $u(t) = 2t - t^2 + (t^3 - 1)/3$ for $t > 1$; note that the actual value of $\alpha$ does not matter as long as $0 < \alpha \leq 1$. Hence the non-uniqueness.

This example illustrates the mechanism of non-uniqueness that is built into the $T$ equation, which we circumvent by introducing the moist static energy $e$. In contrast, the following variant of (3.15) illustrates the mechanism of uniqueness built into the $q$ equation as explained before. We thus consider

$$u' \in 2 - H(u), \quad (3.17)$$

$$u(0) = -1. \quad (3.18)$$

Then $u(t) = 2t - 1$ until $t = 1/2$. At that time, $u$ vanishes and the value ‘chosen’ by $H(0)$ does not matter, since $u'(0) > 0$ in any case and $u(t) > 0, u(t) = t - 1/2$ for $t > 1/2$. Hence the uniqueness of the solution for (3.17), (3.18). The important difference between (3.15) and (3.17) is the difference of the signs in front of $H(u)$, as in (2.8)–(2.9) or (3.1)–(3.2).

For non-uniqueness, one might more simply consider the equation

$$u' \in H(u), \quad u(0) = 0, \quad (3.19)$$

solutions of which read

$$u(t) = 0 \quad \forall \ t \geq 0 \text{ or}$$

$$u(t) = 0 \quad \text{for } 0 \leq t \leq t_1 \text{ and}$$

$$u(t) = (t - t_1) \quad \text{for } t \geq t_1, t_1 \geq 0 \text{ arbitrary.}$$

The advantage of (3.15) is that it allows a comparison of the mechanism of non-uniqueness with the mechanism of uniqueness for (3.17) and leaves the theoretical possibility of $H(0)$ being selected at time $t = 1$ by the evolution of a hypothetical auxiliary process. The advantage of (3.19), for which the solution is explicitly known, is that it shows that non-uniqueness can also generate instability.

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