Splitting the concordance group
of algebraically slice knots

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Abstract

As a corollary of work of Ozsváth and Szabó [8], it is shown that the classical concordance group of algebraically slice knots has an infinite cyclic summand and in particular is not a divisible group.

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Let $\mathcal{A}$ denote the concordance group of algebraically slice knots, the kernel of Levine’s homomorphism $\phi: \mathcal{C} \to \mathcal{G}$, where $\mathcal{C}$ is the classical knot concordance group and $\mathcal{G}$ is Levine’s algebraic concordance group [6]. Little is known about the algebraic structure of $\mathcal{A}$: it is countable and abelian, Casson and Gordon [2] proved that $\mathcal{A}$ is nontrivial, Jiang [5] showed it contains a subgroup isomorphic to $\mathbb{Z}^\infty$, and the author [7] proved that it contains a subgroup isomorphic to $\mathbb{Z}^\infty_2$. We add the following theorem, a quick corollary of recent work of Ozsváth and Szabó [8].

**Theorem 1** The group $\mathcal{A}$ contains a summand isomorphic to $\mathbb{Z}$ and in particular $\mathcal{A}$ is not divisible.

**Proof** In [8] a homomorphism $\tau: \mathcal{C} \to \mathbb{Z}$ is constructed. We prove that $\tau$ is nontrivial on $\mathcal{A}$. The theorem follows since, because $\text{Im}(\tau)$ is free, there is the induced splitting, $\mathcal{A} \cong \text{Im}(\tau) \oplus \text{Ker}(\tau)$. No element representing a generator of $\text{Im}(\tau)$ is divisible.

According to [8], $|\tau(K)| \leq g_4(K)$, where $g_4$ is the 4–ball genus of a knot, and there is the example of the $(4,5)$–torus knot $T$ for which $\tau(T) = 6$. We will show that there is a knot $T^*$ algebraically concordant to $T$ with $g_4(T^*) < 6$. Hence, $T\# - T^*$ is an algebraically slice knot with nontrivial $\tau$, as desired.

Recall that $T$ is a fibered knot with fiber $F$ of genus $(4-1)(5-1)/2 = 6$. Let $V$ be the $12 \times 12$ Seifert matrix for $T$ with respect to some basis for $H_1(F)$. The quadratic form $q(x) = x V x^t$ on $\mathbb{Z}^{12}$ is equal to the form given by $(V + V^t)/2$. Using [3] the signature of this symmetric bilinear form can be computed to be 8, so $q$ is indefinite, and thus by Meyer’s theorem [4] there is a nontrivial primitive element $z$ with $q(z) = 0$. Since $z$ is primitive, it is a member of a symplectic basis for $H_1(F)$. Let $V^*$ be the Seifert matrix for $T$ with respect to that basis. The canonical construction of a Seifert surface with Seifert matrix $V^*$ ([9], or see [1]) yields a surface $F^*$ such that $z$ is represented by a simple closed curve on $F^*$ that is unknotted in $S^3$. Hence, $F^*$ can be surgered in the 4–ball to show that its boundary $T^*$ satisfies $g_4(T^*) < 6$. Since $T^*$ and $T$ have the same Seifert form, they are algebraically concordant.

**Addendum** An alternative proof of Theorem 1 follows from the construction of knots with trivial Alexander polynomial for which $\tau$ is nontrivial, to appear in a forthcoming paper.
References


