Abstract. In this article, we develop a general foliated version of the entropy rigidity theorem and “Real Schwarz Lemma” of Besson, Courtois and Gallot.

1. Introduction

The volume growth entropy, \( h(g) \), of a Riemannian manifold \((X, g)\) is most commonly defined as

\[
    h(g) = \limsup_{R \to \infty} \frac{\log \Vol B(x, R)}{R},
\]

where \( B(x, R) \) is the sphere of radius \( R \) centered at \( x \in X \). This quantity is independent of \( x \in X \). In the case that \((X, g)\) is compact we make the convention that \( h(g) \) be the volume growth entropy \( h(\tilde{g}) \) of the universal cover with its lifted metric \((\tilde{X}, \tilde{g})\). In this case, by [Man79] the lim sup can be replaced by the ordinary limit and moreover \( h(g) \) equals the value of the topological entropy of the geodesic flow when \((X, g)\) is nonpositively curved.

Let \((M, g)\) be any compact, connected and oriented \( n \)-dimensional Riemannian manifold without boundary, and let \((N, g_0)\) be a compact connected negatively curved locally symmetric space of dimension \( n \). Suppose there exists a continuous map \( f: M \to N \) of non-zero degree. Besson, Courtois and Gallot proved the following two remarkable theorems.

**Theorem 1.1** (Volume Entropy Rigidity Theorem [BCG95]). With the above notations one has

\[
    h^n(g) \Vol(M, g) \geq |\deg f| h^n(g_0) \Vol(N, g_0).
\]

Moreover equality occurs if and only if \( f \) is homotopic to a Riemannian covering (i.e. a locally isometric covering).

**Theorem 1.2** (Real Schwarz Lemma [BCG99, BCG99]). Now suppose \((N, g_0)\) is an arbitrary negatively curved closed manifold and we scale the metrics \( g \) and \( g_0 \) so that \( \Ric(g) \geq - (n - 1) g \) and \( K(g_0) \leq -1 \), then

\[
    \Vol(M, g) \geq |\deg(f)| \Vol(N, g_0),
\]

and equality occurs if and only if \((M, g)\) and \((N, g_0)\) both have constant curvature \(-1\) and \( f \) is homotopic to a Riemannian cover.

The proofs of these theorems are highly dependent on the compactness of these spaces. In [BCS05] (see also [CF03]) we extended these theorems to the finite volume noncompact case. (An additional assumption was required for the first theorem to hold.)
which was later removed in \cite{Sto06}. While finite volume manifolds typically have infinite topological type and are very far from being compact, they nevertheless have an imposed control on the geometric growth of their ends. In order to even hope of extending the above theorems to a wider category of noncompact manifolds, one must first find some context equipped with a finite measure which is still intricately connected to the geometry of the manifolds. Such a context arises for leaves of a compact foliated space. In \cite{BC02} we extended a special case of the Volume Entropy Rigidity Theorem to leaves of a compact foliated manifold. However, there we required a number of technical assumptions on the domain leaves.

The purpose of this article is to prove natural foliated versions of the above two Theorems in (almost) complete generality, apart from some simply stated necessary assumptions.

We let \((M, \mathcal{F}_M)\) be a foliated connected compact topological space with oriented Riemannian leaves of dimension \(n > 2\). Examples of compact foliated spaces include any foliation of a compact manifold or any compact lamination, which is a subfoliation of a saturated compact set in an foliated manifold. However other examples exist which also naturally arise in dynamics, topology and geometry (see \cite{CC03} for a plethora of natural examples and also definitions.)

Similarly let \((N, \mathcal{F}_N)\) be a compact connected space foliated by negatively curved locally (rank one) symmetric spaces. We assume for both \(\mathcal{F}_M\) and \(\mathcal{F}_N\) that the leaves are smooth and that the metrics on the leaves, along with their first and second derivatives, vary continuously on any transversal. In particular, this implies that the universal covers of the leaves of \(\mathcal{F}_N\) are all homothetic to a fixed symmetric space \((X, g_0)\). However the homothetic factor may vary continuously in transverse directions.

We refer the reader to \cite{CC03}, especially Chapter 11 Sections 1-5, for the basic notions of foliated spaces and attendant constructions. We will always assume that we have chosen a fixed regular foliated atlas of a finite number of charts for \(M\) and \(N\). At some points we may choose new atlas with extra properties which we may assume to be subordinate to these.

Now we suppose that the foliated space \((M, \mathcal{F}_M)\) is equipped with any finite transverse holonomy invariant measure \(\nu\). Not every foliation carries such a measure (see \cite{Pla75} for conditions). However, if \(f\) is a homeomorphism we can obtain our results using only a quasi-invariant measure, which always exists (see \cite{Hur94} or \cite{Zim82} for definitions and existence).

The measure \(\nu\) provides us with a global finite measure \(\mu_M\) on \(M\) which is locally the product of \(\nu\) on transversals with the leafwise Riemannian volumes \(dg_L\) (see \cite{CC03}). In the case that \(\nu\) is only quasi-invariant, then the measure \(\mu_M\) may depend on the choice of partition of unity subordinate to the foliation atlas by which the local measures are synthesized into the global one.

Similarly, we form a measure \(\mu_N\) on \(M\) using the Riemannian volumes on the leaves of \(\mathcal{F}_M\) and a transversal measure \(\nu'\) defined in Section 3 (Definition 3.3). This measure \(\nu'\) is somewhat canonical in that it only depends on \(\nu\) and the map on the space of leaves induced by \(f\). Its key properties are also necessary in order for our results to strictly generalize the existing results in the case of a trivial foliation.

The definition of volume growth entropy given earlier is not well adapted to leaves of foliations. For example, it does not clearly behave in a predictable way on the closure of a set of leaves. The following notion at least partially rectifies this.
For any Riemannian manifold \((X, g)\) we define the absolute volume growth entropy to be the quantity

\[
\overline{h}(g) = \limsup_{R \to \infty} \sup_{x \in X} \frac{\log \text{Vol } B(x, R)}{R}
\]

where \(B(x, R)\) denotes the ball of Radius \(R\) around \(x \in X\). We have \(\overline{h}(g_o) \geq h(g_o)\) and \(\overline{h}(g_o) = h(g_o)\) when \(X\) is the universal cover of a compact manifold (see Lemma 2.1). By convention, we will always compute \(\overline{h}\) on the universal cover of leaves. Hence we write \(\overline{h}(g_L)\) for \(\overline{h}(\tilde{g}_L)\) where \((L, \tilde{g}_L)\) is the universal cover of \((L, g)\) with the lifted metric.

We can take the average leafwise absolute volume growth entropy to obtain the foliated volume growth entropy of \(\mathcal{F}_M\), defined by

\[
h_{\mathcal{F}_M} = \frac{1}{\mu_M(M)} \int_M \overline{h}(g_{L_t}) d\mu_M(x).
\]

Note that \(h_{\mathcal{F}_M}\) is not to be confused with the “foliated entropy” as defined by Ghys, Langevin and Walczak \((\text{GLW88})\) and \(\text{LW91}\) which generalizes the topological entropy of a flow. The latter entropy captures information about the orbit growth of the holonomy groupoid.

The final ingredient we need is the foliated degree \(\deg_{\mathcal{F}}(f)\) of a leafwise proper foliated map \(f\), defined to be the following average of the leafwise degrees,

\[
\deg_{\mathcal{F}}(f) = \frac{1}{\mu_N(N)} \int_N \int_{T \cap f^{-1}(L_t)} \deg(f_{\mid L_t}) d\nu_x(t) d\mu_N(x),
\]

where \(\nu_x\) is the conditional probability measure on \(T \cap f^{-1}(L_x)\).

Note that since \(f\) is a proper map, \(\deg(f_{\mid L_t})\) is still defined for noncompact leaves \(L_t\) by a number of equivalent notions. For instance, \(\deg(f_{\mid L_t})\) can be defined as the number \(k \in \mathbb{Z}\) such that \(f_\ast([L_t]) = k[L_{f(t)}]\) where

\[
f_\ast : H^1_{\text{lf}}(L_t, \mathbb{Z}) \to H^1_{\text{lf}}(L_{f(t)}, \mathbb{Z})
\]

is the induced map on locally finite homology (or one could use Borel-Moore homology) and \([L_t]\) and \([L_{f(t)}]\) represent the respective fundamental classes generating the top dimensional groups. The degree of a map is an invariant of proper homotopies, and so foliated degree is invariant under leafwise proper homotopies. However, one difference is that \(\deg_{\mathcal{F}}(f)\) need not be an integer (see Example 2.2).

We can now state the first of our main results.

**Theorem 1.3.** Suppose \(\mathcal{F}_M\) and \(\mathcal{F}_N\) are as above and \(f : (M, \mathcal{F}_M) \to (N, \mathcal{F}_N)\) is a continuous leafwise proper foliated map. If \(\overline{h}(g_L) > 0\) whenever \(\deg f_{\mid L_t} \neq 0\), then

\[
h_{\mathcal{F}_M} \mu_M(M) \vol_{\mu_M}(M) \geq |\deg_{\mathcal{F}}(f)| h_{\mathcal{F}_N} \mu_N(N) \vol_{\mu_N}(N).
\]

Moreover, equality holds if and only if \(\nu\)-almost every leaf \(L_t\) is homothetic to a degree \(|\deg f_{\mid L_t}|\) Riemannian cover of \(f(L_t)\).

**Remarks 1.4.**

- In the equality case, we have \(\overline{h}(g_L) = h(g_L)\).
- Following the ideas of Sambussetti in [Sam99], we see no obstruction to improving the above result to the case where the volume growth entropy of the universal covers of each leaf, \(\overline{h}(g_{L_t})\), is replaced by the potentially
smaller volume growth entropy, $\hat{h}(g_L)$, of the cover corresponding to the subgroup $\ker(f_L) < \pi_1(L)$. However, we did not check this carefully.

- If $\deg f_L \neq 0$, then $f : L \to L_o$ is surjective.
- In the case when the dimension of leaves is 2 (the first case when $\hat{h}(g)$ can be positive), then the inequality still holds. However, the equality case does not hold even in the case of a trivial foliation as is seen by the nontriviality of the moduli space of compact surfaces of constant curvature $-1$.
- When $f$ is assumed to be a homeomorphism we can allow $\nu$ to be an arbitrary quasi-invariant measure instead. However, then we need to take transversal measure $\nu'$ on $N$ to be the pushforward under the special map $F$ we will construct in section 2.
- The necessity of the assumption that $\hat{h}(g_L) > 0$ whenever $\deg f_L \neq 0$ is demonstrated by the following example.

**Example 1.5.** Consider the foliation of $M = T^4$ by isometric copies of $\mathbb{R}^3$ with irrational slope in the torus $M$. We let $\nu$ be the transverse measure on arising as the (conditional) decomposition measure of the volume form on $M$ with respect to the leafwise Euclidean volumes.

Similarly let $N = T^4$ with the same topological foliation, but whose leaves are equipped with metrics of constant curvature $-1$ so that they are each individually isometric to $\mathbb{H}^3$. Finally we let $f : M \to N$ be the identity map which is clearly leafwise proper.

A straightforward computation shows that $\hat{h}(\mathbb{H}^3) = h(\mathbb{H}^3) = 2$ so the right hand side of the inequality from Theorem 1.3 will be the positive number $8 \ast \text{Vol}_{\mu_N}(N)$. However $\hat{h}(\mathbb{R}^3) = 0$ since the flat metrics have polynomial growth. Hence, the left hand side of the inequality is necessarily 0, contradicting the inequality.

However, as soon as the entropy of the leaves of $M$ are positive, there is a tradeoff between the entropy and the volume form so that each side of the main inequality is invariant under leafwise metric rescaling.

We also have the following generalization of Theorem 1.2 to leaves of compact foliated spaces.

**Theorem 1.6 (Foliated Real Schwarz Lemma).** Suppose $\mathcal{F}_M$ and $\mathcal{F}_N$ are as in the Theorem 1.3 except that the metrics on the leaves of $\mathcal{F}_M$ satisfy $\text{Ric}(g_L) \geq -(n-1)g_L$ and the metrics $g_L_o$ on the leaves of $\mathcal{F}_N$ are only assumed to be negatively curved with $K_{g_L_o} \leq -1$.

Then for any continuous leafwise proper foliated map $f : (M, \mathcal{F}_M) \to (N, \mathcal{F}_N)$, we have

$$\text{Vol}_{\mu_M}(M) \geq |\deg f| \text{Vol}_{\mu_N}(N)$$

Moreover, equality occurs if and only if $\nu$—almost every leaf $L$ is homothetic to a degree $|\deg f_L|$ Riemannian cover of $f(L)$.

The next section explains the natural barycenter map. In Section 3 we prove a general foliated coarea formula and develop a foliated version of degree theory. We prove Theorems 1.3 and 1.6 in Section 4.

2. **The natural map**

We will begin by establishing some useful relations between different notions of entropy.
Lemma 2.1. The absolute volume growth entropy enjoys the following properties:

1. We always have $\overline{h}(g) \geq h(g)$.
2. We have $\overline{h}(g) = h(g)$ whenever $(X, g)$ admits a compact quotient, or more generally admits a uniform group of quasi-isometries such that any point can be carried into a fixed compact set.
3. The quantity $\overline{h}(g)$ is a lower-semicontinuous function on a foliated space.
4. If $(X, g)$ is a leaf of $\mathcal{F}_M$ then we have

   $$h(g) = \inf \left\{ s > 0 \left| \int_0^\infty e^{-st} \text{Vol}(S(x, t)) dt < \infty \right\},$$

   where $S(x, t)$ is the geodesic sphere of radius $t$ about $x \in X$.

Proof. The first statement is immediate from the definitions. For the second statement note that under the hypothesis, there is a quasi-isometry that carries $B(x, R)$ into $B(y, R + D)$ for some constant $D > 0$ independent of $x$ and $y$ in $X$. It follows that there is a universal constant $C > 1$ independent of $x$ and $y$ such that $\frac{1}{C} \leq \frac{\text{Vol}(B(x, R))}{\text{Vol}(B(y, R))} \leq C$. The result then follows immediately from the definitions.

For the third statement we first recall that the closure of a set of leaves is again a union of leaves. This can be seen easily by noting that the limits of leaves contain entire plaques, and so these can be extended indefinitely to plaques in neighboring foliated charts. Now let $L'$ be a limit leaf in the closure of a sequence of leaves $\{L_i\}$. If $B(x_j, R_j)$ is a sequence of balls in $L'$ with nearly supremal volume for each radius $R_j$, then because of the $C^2$ continuity of the metrics on the leaves, which carry the quotient metrics, that there is a diagonal sequence of balls $B(x_{ij}, R_{ij})$ in $L_i$ such that $\lim_{j \to \infty} \frac{\text{Vol}(B(x_{ij}, R_{ij}))}{\text{Vol}(B(x_{ij}, R_{ij}))} = 1$. It follows immediately from the definition of the absolute volume growth entropy that $\liminf_i \overline{h}(g_{L_i}) \geq \overline{h}(g')$.

For the final statement we set $\delta = \inf \left\{ s > 0 \left| \int_0^\infty e^{-st} \text{Vol}(S(x, t)) dt < \infty \right\}$ and first observe that from the definition of $h(g)$ there is a subexponential function $f : \mathbb{R} \to \mathbb{R}$ such that $\text{Vol}(S(x, t)) \leq f(t)e^{h(g)t}$. Estimating this in the integral in the definition of $\delta$ immediately implies $\delta \leq h(g)$ since if $\delta - h(g) > 0$ then for any $0 < \epsilon < \delta - h(g)$ we have $\int_0^\infty e^{-\epsilon t} f(t) dt < \infty$ contradicting the definition of $\delta$.

On the other hand, if $h(g) > \delta$ then for any positive $\epsilon < h(g) - \delta$ there is a sequence $t_i \to \infty$ such that $\text{Vol}(S(x, t_i)) > e^{(h(g)-\epsilon)t_i}$. Since the derivatives of the metric are universally bounded from the assumptions of it being a leaf in the compact foliated space $\mathcal{F}_M$, we also have for any positive $\epsilon' < h(g) - \delta - \epsilon$ that $\text{Vol}(S(x, t)) > e^{(h(g)-\epsilon'-\epsilon)t}$ for $t$ in a compact neighborhood $[t_i - c, t_i + c]$ for some $c > 0$ independent of $i$. In particular, for any positive $\epsilon'' < h(g) - \delta - \epsilon - \epsilon'$, we have

$$\int_0^\infty e^{-(\delta+\epsilon+t)} \text{Vol}(S(x, t)) > 2\epsilon' \sum_{i=1}^\infty 1 = \infty.$$

This contradicts the definition of $\delta$, and finishes the proof. \hfill \Box

Example 2.2. Consider the compact laminated space $M$ shown in Figure 1 consisting of two circles, say $A$ and $B$, and a copy of $\mathbb{R}$ with each end limiting onto one of them. This is a compact foliated space with exactly three one dimensional leaves and can be covered by four foliated charts, two covering the neighborhood
of each circle. A transversal for each of the four foliated charts is homeomorphic to the set \( \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subset \mathbb{R} \).

We can consider the extension of the map \( f(x) = 2x \) on \( \mathbb{R} \) to a self map of the whole space. This carries the odd multiples of \( \pi \) and the even multiples of \( \pi \) to the even multiples of \( \pi \) and so forth. Therefore \( f \) extends continuously to the map \( e^{2\pi i t} \mapsto e^{4\pi i t} \) on each of the two circles. It is easy to check that it is leafwise proper and hence \( f \) has degree 2 on each of the circles, but degree one on the copy of \( \mathbb{R} \).

There is no finite transverse invariant measure which is nonzero on the copy of \( \mathbb{R} \). However, we can choose a finite transverse quasi-invariant measure \( \nu \) on \( M \) consisting of a countable number of atoms with weights \( \nu(\{0\}) = 1 \) and \( \nu(\left\{ \frac{1}{n} \right\}) = 2^{-n} \) on each of the four transversals. Provided that the metric on \( \mathbb{R} \) and the \( S^1 \)'s are normalized so that each of the plaques have the same length, with this measure \( \nu \) we may check that the foliated degree of \( f \) is \( \deg_{\mathscr{F}}(f) = \frac{3}{2} \).

To obtain a similar example using a transverse invariant measure, we alter the map by taking \( f(x) = \begin{cases} 3x & x > 0 \\ 2x & x \leq 0 \end{cases} \) on \( \mathbb{R} \) instead. The unique continuous extension of this map to \( M \) will have degree three on one circle and degree two on the other. In this case we can use the transverse invariant measure \( \nu(\{0\}) = 1 \) and \( \nu(\left\{ \frac{1}{n} \right\}) = 0 \) on each transversal to obtain a foliated degree of \( \deg_{\mathscr{F}}(f) = \frac{5}{2} \).

We now turn our attention to the definition of the natural map by the barycenter construction that was first carried out for closed manifolds in [BCG95].

Suppose \( L \) is a leaf of \( \mathscr{F}_M \) in theorem 1, then \( f(L) \) is contained in some leaf (suppose \( \tilde{X} \)) in \( \mathscr{F}_N \). We denote its lift to the universal cover as \( \tilde{f} : \tilde{L} \to \tilde{X} \).

**Definition 2.3.** For \( s > h(g) \), we consider the following family of measures on \( \tilde{L} \):

\[
\mu_{s,y} = \frac{e^{-sd(y,z)}d\tilde{g}}{\int_{\tilde{L}} e^{-sd(y,z)}d\tilde{g}}
\]

where \( d \) is the distance on \( \tilde{L} \), induced by the metric \( \tilde{g} \) on \( \tilde{L} \).

The assignment \( y \mapsto \mu_{s,y} \) defines an embedding of \( \tilde{L} \) into the space of probability measures, \( \mathcal{P}(\tilde{L}) \), on \( \tilde{L} \).

The significance of this family of probability measures is that they are smooth analogs of the Patterson-Sullivan measures. In fact, when \( (L,g) \) is the cover of a compact negatively curved manifold \( \hat{L} \), the weak limit \( \lim_{s \searrow b(\hat{g})} \mu_{s,y} \) converges to the Patterson-Sullivan measure \( \mu_y \) associated to \( \pi_1(\hat{L}) \) on \( \partial L \) (see [Kni94]).

By pushing forward the measure \( \mu_{s,y} \) on \( \tilde{L} \) we obtain a new measure \( \tilde{f}_* \mu_{s,y} \) on \( \tilde{X} \). And then making the convolution with the harmonic measure on \( (\tilde{X}, \tilde{g}_0) \) in order to regularize it, we obtain the family of probability measures \( \sigma_{s,y} \) on \( \partial \tilde{X} \).
More explicitly, the probability measure \( \sigma_{s,y} \) on \( \partial \tilde{X} \) can be expressed as:

\[
\sigma_{s,y} = \left( \frac{\int_{\tilde{L}} e^{-s d(y,z)} e^{-h(g_o) B_o(f(z), \theta)} dv_0(z)}{\int_{\tilde{L}} e^{-s d(y,z)} dv_0(z)} \right) d\theta
\]

where \( B_o(f(z), \theta) \) is the Busemann function of \( f(z) \in \tilde{X} \) at \( \theta \in \partial \tilde{X} \) with respect to the basepoint \( \tilde{f}(0) \in \tilde{X} \).

Now we define for each \( s > h(g_L) \) an equivariant map \( \tilde{F}_s : \tilde{L} \to \tilde{X} \) by

\[
\tilde{F}_s(y) = \text{bar}(\sigma_{s,y}) = \text{the unique critical point of } B(x, \cdot)
\]

where

\[
B(x, \cdot) := \int_{\partial \tilde{X}} B_o(\cdot, \theta) d\sigma_{s,y}(\theta)
\]

We can summarize the construction of \( \tilde{F}_s \) by the following diagram.

\[
\begin{array}{ccc}
\mathcal{M}_1(\tilde{M}) & \xrightarrow{\tilde{f}_s} & \mathcal{M}_1(\tilde{N}) \\
\mu_s & \searrow & \text{bar} \\
\tilde{M} & \rightarrow & \tilde{N}
\end{array}
\]

Each stage (see [BCG96]) is equivariant under the intertwined action of \( \pi_1(M) \) or \( f_s \pi_1(M) < \pi_1(N) \). Hence \( \tilde{F}_s \) descends to the natural map \( F_s : M \to N \).

The proof of Theorem 1.3 relies on the following lemma.

**Lemma 2.4.** For \( s > h(g_L) \), the map \( \tilde{F}_s \) is proper.

To prove Lemma 2.4, we need Lemma 3.2 of [BCS05]:

**Lemma 2.5.** There is \( D > 0 \) with the following property: let \( \lambda \) be a finite measure on \( \tilde{M}_o \), \( \sigma_\lambda \) the convolution with the family of visual measures and \( K \subset \tilde{M}_o \) a convex set with \( \lambda(K) \geq \frac{9}{10} ||\lambda|| \), then the barycenter \( \text{bar}(\sigma_\lambda) \) of \( \sigma_\lambda \) is within distance \( D \) of \( K \).

In the context of the above lemma, \( M_o \) is an oriented complete finite volume Riemannian manifold with pinched negative curvature less than \(-1\). However, the proof does not use the finite volume condition. Moreover, in our case, we may scale the metric \( g_o \) such that the curvature is bounded above by \(-1\), apply the theorem, and then scale back. (Recall that since \( L \) is a leaf of a compact foliation, the curvatures are bounded away from \(-1\).) The constant \( D \) then changes by a factor of the square root of the required scaling constant.

**Proof of Lemma 2.4.** If \( \tilde{F}_s \) is not proper, then there would exist a sequence of points \( y_n \) in \( \tilde{L} \) which has no convergent subsequences such that \( \tilde{F}_s(y_n) \) tends to a point \( z \in \tilde{X} \).

First, we prove that there exists an \( R > 0 \), depending only on \( s \) and not \( y_n \), such that \( \mu_{s,y_n} \) is \( \frac{9}{10} \) supported on the ball \( B(y_n, R) \).

For \( s > h \), we define \( R_s^* \) to be the largest \( R \) such that \( \frac{\log \Vol(B(x,R))}{R} \geq s \), and if there is no such \( R \), then \( R_s^* = 0 \).

**Claim:** For all \( x \in \tilde{L} \), \( R_s^* \) is bounded.
Proof of Claim: If $R^*_x \to \infty$ as $i \to \infty$, i.e., there exists a sequence $x_i$ such that \( \frac{\log \Vol(B(x_i, R^*_x))}{R^*_x} \geq s \), then

\[
\bar{\kappa} = \limsup_{R \to \infty} \sup_{x_i \in L} \frac{\log \Vol(B(x_i, R))}{R} \geq s,
\]

which contradicts the assumption that $s > \bar{\kappa}$.

Now we are going to use $R^*_x$ to prove the existence of $R$.

For $y_n$,

\[
\mu_{s,y_n}(B(y_n, R_{y_n})) = \frac{\int_{B(y_n, R_{y_n})} e^{-sd(y_n, z)}d\nu(z)}{\int_L e^{-sd(y_n, z)}d\nu(z)} = \frac{\int_{R_{y_n}} e^{-st} \Vol'(B(y_n, t))dt}{\int_0^\infty e^{-st} \Vol'(B(y_n, t))dt} = \frac{\int_{R_{y_n}} e^{-st} \Vol'(B(y_n, t))dt}{\int_{R_{y_n}} e^{-st} \Vol'(B(y_n, t))dt + \int_{R_{y_n}} e^{-st} \Vol'(B(y_n, t))dt}
\]

For $s - \epsilon$, there is some constant $C$ such that for all $x \in \bar{L}$, $R^*_x < C$, then if $R \geq C$, $\frac{\log \Vol(B(x, R))}{R} < s - \epsilon$. Let $R_{y_n} \geq C$, we have

\[
\int_{R_{y_n}} e^{-st} \Vol'(B(y_n, t))dt < s - \epsilon \int_{R_{y_n}} e^{-st} dt \leq C (s - \epsilon) e^{-\epsilon t} dt = \frac{(s - \epsilon)}{\epsilon} e^{-\epsilon C}.
\]

The latter expression tends to 0 as $C$ tends to $\infty$.

Moreover $\int_{R_{y_n}} e^{-st} \Vol'(B(y_n, t))dt \geq \int_{R_{y_n}} e^{-st} \Vol'(B(y_n, t))dt$. Hence, we can take $C$ large enough such that

\[
\int_{R_{y_n}} e^{-st} \Vol'(B(y_n, t))dt < \frac{1}{9} \int_{R_{y_n}} e^{-st} \Vol'(B(y_n, t))dt
\]

Summarizing, we have so far proved that there exists an $R > 0$ such that $\mu_{s,y_n}$ is supported on the ball $B(y_n, R)$.

Second, we prove diam($\bar{\sigma}(B(y, R))$) is bounded.

Claim: $f : L \to f(L)$ is Lipschitz in the large, i.e., there exist constant numbers $C$ and $K$ such that $d(f(x), f(y)) \leq Cd(x, y) + K$.

Proof of claim: For $x, y \in \bar{L}$, there exists a finite plaque chain connecting the two points. Because $M$ is compact, we know there exists some constant $C$ such that diam($f(P)$) < $C$ for all plaques $P \subset L$. Since $f(x)$ and $f(y)$ are connected by the images of those plaques, we know there exists $K$ such that $d(f(x), f(y)) \leq Cd(x, y) + K$.

Hence, by the claim, $\bar{\sigma}$ is also Lipschitz in the large. Therefore, diam($\bar{\sigma}(B(y, R))$) is bounded. Third, by Lemma 3.2 of [BCS05], bar($\sigma_{s,y_n}$) = $\bar{F}_s(y_n)$ is within distance
$D_{y_n}$ of $B(\bar{f}(y_n), \bar{R})$. Since the sequence $\bar{F}_s(y_n)$ tends to a point $z \in \bar{X}$, then $z$ is within distance $D$ of $B(\bar{f}(y_n), \bar{R})$. Hence $d(z, \bar{f}(y_n)) \leq D + \bar{R}$ and consequently
\[
d(\bar{f}(y_0), \bar{f}(y_n)) \leq d(\bar{f}(y_0), z) + d(z, \bar{f}(y_n)) \leq d(\bar{f}(y_0), z) + D + \bar{R}
\]
However, $d(\bar{f}(y_0), \bar{f}(y_n)) \to \infty$ as $n \to \infty$ because $\bar{f}$ is leafwise proper and $d(y_0, y_n) \to \infty$ as $n \to \infty$. This produces a contradiction. Hence, $\bar{F}_s$ is proper. \[\square\]

Remark 2.6. Actually, in the above proof, we don’t have to use Lemma 3.2 of [BCS05]. Since $\bar{F}_s(y_n) = \text{bar}(\sigma_{s,y_n})$, $\bar{F}_s(y_n)$ satisfies
\[
\min_{y \in \bar{X}} B(y_n, y) = \min_{y \in \bar{X}} \int_{\partial \bar{X}} B_o(y, \theta) d\sigma_{s,y_n}(\theta) = \int_{\partial \bar{X}} B_o(\bar{F}_s(y_n), \theta) d\sigma_{s,y_n}(\theta) = B(y_n, \bar{F}_s(y_n)).
\]
Hence,
\[
\int_{\partial \bar{X}} B_o(\bar{F}_s(y_n), \theta) d\sigma_{s,y_n}(\theta) \leq \int_{\partial \bar{X}} B_o(y, \theta) d\sigma_{s,y_n}(\theta) \quad \text{for all } y \in \bar{X};
\]
(\hspace{1cm} i.e. \hspace{1cm})
\[
\int_{\partial \bar{X}} (B_o(\bar{F}_s(y_n), \theta) - B_o(y, \theta)) d\sigma_{s,y_n}(\theta) \leq 0 \quad \text{for all } y \in \bar{X};
\]
\[\text{i.e. \hspace{1cm}} \int_{\partial \bar{X}} B(y, \bar{F}_s(y_n), \theta) d\sigma_{s,y_n}(\theta) \leq 0 \quad \text{for all } y \in \bar{X}.
\]

Because there exists $\bar{R}$ such that $\bar{F}_s$, $\mu_{s,y_n}$ is $\frac{2}{10}$ supported on the ball $B(\bar{f}(y_n), \bar{R})$, so $\sigma_{s,y_n}$ is also $\frac{\rho}{10}$ supported on the ball $B(f(y_n), \bar{R})$.

Since $\frac{\rho}{d(f(y_0), f(y_n))} \to 0$ as $n \to \infty$, the cone formed by the point $\bar{F}_s(y_n)$ and the base $B(\bar{f}(y_n), \bar{R})$ has very small angle. Then (2.1) cannot be true for for all $y \in \bar{X}$.

In fact, the above proof shows that $\bar{F}_s$ is actually a finite distance to the map $f$.

In other words, there is a constant $C_L > 0$ such that $\sup_{x \in L} d(F_s(x), f(x)) < C_L$. Note that there may not be a universal upper bound since $C_L$ depends on how close $s$ is to $\bar{h}(g_L)$, which itself may also be discontinuous.

The following is immediate from the properness of $\bar{F}_s$ on $L$.

Corollary 2.7. $F_s$ is proper and homotopic to $f$ on $L$ with $\deg F_s = \deg f$.

Now we combine these leafwise natural maps into a global map from $M$ to $N$, which in abuse of notation, we still denote by $F_s$. This combined map is a measurable foliation-preserving map which is $C^1$ when restricted to leaves (see [BC02]). Note that $F_s$ may not be globally continuous on $M$. For even though it is locally leafwise defined for a given $s > \bar{h}(g_L)$ parameter, it is not evident that the function $\bar{h}(g_L)$ need be any more regular than lower-semicontinuous on $M$, at least in the typical case when the leaf space is non-Hausdorff.

3. Foliated coarea formula

Recall that the foliation $(M, \mathcal{F}_M)$ possesses an holonomy invariant measure $\nu$ and the measure $\mu_M$ is the globally defined and finite measure given locally by $d\mu_M = dq_{gL} \times d\nu$. This measure is independent of the covering by foliated charts.

Since we are considering measures in this section, it will be both more convenient, and more transparent for computations, to choose foliated charts $\{U_i\}$ which are pairwise disjoint as open sets of $M$ and satisfy $M = \cup_i U_i$ and $\mu_M(M - \cup_i U_i) = 0$. We will call such an “almost” foliated atlas, a foliated partition atlas. One way
to construct such an atlas to start with an ordinary foliated atlas \( \{ Y_i \} \) equipped with a foliated partition of unity \( \{ \Psi_i \} \), and then begin with the super-level set of \( \frac{1}{2}, U_i^2 = \Psi_i^{-1}(\{\frac{1}{2}, 1\}) \). Continue by adding the super-level set of \( \frac{1}{3}, U_i^3 = U_i^2 \cup \Psi_{i|M-U_i^2}^{-1}(\{1/3, 1\}) \), and so on until we arrive at \( U_i^m \) where \( m \) is the Lebesgue number of the original cover \( \{ Y_i \} \). We then take \( U_i \) to be the interior of \( U_i^m \). If \( \nu \) has (at most countably many) atoms, then a little care must be taken in the choice of the original cover \( \{ Y_i \} \), but it is evident that this can be done.

On \( N \) we need the natural analogue, \( \nu' \), of \( \nu \) in order to construct a similar global measure \( \mu_N \) locally defined by \( d\mu_N = ddg_o \times d\nu' \). The following example explains the problem with trying to take \( \nu' \) to be a naive representative version of “\( f_*\nu \).” As we shall see the problem is not merely a matter of fixing the map to take transversals to transversals.

**Example 3.1.** Consider a locally isometric covering map \( f : M \to N \) of degree \( d = \ell m \) between two compact Riemannian manifolds \((M,g)\) and \((N,g_o)\). Treating these as trivial foliations, suppose we can lift a covering \( \{ O_i \}_{i=1}^k \) of \( N \) by disks to a covering by disks \( \{ U_{i,j} \} \) where \( i \in \{ 1, \ldots, k \} \) and \( j \in \{ 1, \ldots, \ell \} \) and \( f \) restricted to each \( U_{i,j} \) is a local isometry to \( O_i \) of degree \( m \). Each \( O_i \) can be interpreted as a foliation chart for the trivial foliation on \( N \) and similarly the \( U_{i,j} \) are foliation charts for \( M \). If \( t_{i,j} \in U_{i,j} \) and \( t'_i = f(t_{i,j}) \) is a choice of local transversals (points in this case), then we can take take \( \nu \) to be the transverse invariant probability measure on \( T = \bigcup_{i,j} \{ t_{i,j} \} \) which is just the atom of weight \( \frac{1}{k\ell} \) on each point. In this case, we can set \( \nu' \) to be the ordinary push-forward under \( f \), namely \( \nu' = f_*\nu \), to obtain a (holonomy invariant) transverse measure on \( N \). Since push-forwards preserve mass, \( \nu' \) has weights \( \frac{1}{k} \) on each \( t'_i \).

Using a partition of unity \( \Psi_{i,j} \) for \( \{ U_{i,j} \} \) and \( \Phi_i \) for \( \{ O_i \} \), we may compute,

\[
\text{Vol}_{\mu_N}(N) = \sum_i \nu'(t'_i) \int_{O_i} \Phi_i(x) d\nu(x) = \frac{1}{k} \text{Vol}_{g_o}(N)
\]

and

\[
\text{Vol}_{\mu_M}(M) = \sum_{i,j} \nu(t_{i,j}) \int_{U_{i,j}} \Psi_{i,j}(x) d\nu(x) = \frac{1}{k\ell} \text{Vol}_{g}(M) = \frac{d}{k\ell} \text{Vol}_{g_o}(N) = \frac{d}{\ell} \text{Vol}_{\mu_M}(N).
\]

In other words, these foliated notions of volumes do not in general behave according to the expected relation of the ordinary volumes, namely

\[
\text{Vol}_{g}(M) = |\deg(f)| \text{Vol}_{g_o}(N) = d \text{Vol}_{g_o}(N).
\]

Moreover, if we vary the factors \( \ell \) and \( m \) for a fixed \( d \), we see that this discrepancy is an intrinsic problem arising from how we count pre-images (plaque-wise versus globally).

The problem demonstrated by the above example is naturally fixed by renormalizing the measure \( \nu' \) by dividing by \( \ell \), the average number of plaque pre-images.

Another problem that needs to be addressed is that for general maps the image of a transversal is not a transversal. In particular, one can not immediately make sense of the push-forward of \( \nu \) as a transverse measure.

One might try to simply project the pre-image of transversals to the transversal of the box each point happens to belong to. However, in the following example we will see that such a procedure does not even lead to a measure when plaques map onto multiple plaques within the same chart.
We define a measure \( U \cup \) then we may compute the following:

**Example 3.2.** Consider a trivial foliation of a flat 2-torus \((M, g)\) with \(k\) isometric charts each consisting of one plaque \(P\) so that each transversal \(T_i\) is a single point. Let \(N = M\) with the same metric, but we let each of the \(k\) charts on \(N\) contain two identical plaques, say \(P_o\), by cutting the chart in half and assigning two transversal points to each \(T'_i\). Finally, let \(f : M \to N\) be a double cover which stretches each chart of \(M\) onto two of \(N\) so that \(\text{Jac}(f) = 2\).

Thus we may compute that \(\int_M \text{Jac}(f) d\mu_M = 2k \text{Vol}_g(P)\nu_o\) where \(\nu_o = \nu(T_i)\) which is assumed to be the same for each \(i\).

By contrast, if we use the prescription for \(\nu'\) suggested after the previous example, then we may compute the following:

\[
\int_N \deg(f) d\mu_N = 2k \text{Vol}_g(P_o)\nu'(T'_i).
\]

Here \(\nu'(T'_i) = \nu_o\) since it is the mass, \(2\nu_o\), of the preimage of \(T_i\) divided by the average number of plaque pre-images per plaque, which is also 2.

Finally, since \(\text{Vol}_g(P) = 2\text{Vol}_g(P_o)\) we arrive at the following discrepancy \(\text{Jac}(f)\text{Vol}_g(M) \neq \deg(f)\text{Vol}_g(N)\) that contradicts the usual degree formula for covering maps which are not locally isometric but with constant Jacobian.

What is really going on in the above example is that \(\nu'\) thus defined is not an honest measure since it is not even finitely additive. For instance, the measure of the two point set \(\nu'(T'_i)\) is the same as that of each of the disjoint points of \(T'_i\) taken separately. To fix this, we need to also multiply by the average number of plaques of the target intersected by each plaque in the domain.

For the remainder of the section, we assume we have chosen foliated partition atlases \(\{U_i\}\) for \(\mathcal{F}_M\) and \(\{O_i\}\) for \(\mathcal{F}_N\) with corresponding global transversals \(T = \bigcup_i T_i\) and \(T' = \bigcup_i T'_i\) respectively. Let \(r_i : U_i \to T_i\) be the local transversal projection maps, and \(r : \bigcup_i U_i \to T\) be the map that restricts to \(r_i\) on each of the disjoint sets \(U_i\). Lastly, denote plaques by \(P_t \subset U_i\) for \(t \in T_i\) and \(P_{t'} \subset O_j\) for \(t' \in T'_j\).

The above two examples motivate the following definition.

**Definition 3.3.** For a proper foliated map \(F\), let \(\lambda_F\) be the measure defined on subsets \(A \subset T'\) by

\[
\lambda_F(A) = \sum_i \int_{r_i(F_i^{-1}(A))} \# \{t' \in A : F(P_t) \cap P_{t'} \neq \emptyset\} d\nu(t).
\]

We define a measure \(\nu'_F\) on \(T'\) by

\[
\nu'_F(A) = \int_A \int_{T \cap F_i^{-1}(P_{t'})} \frac{1}{\# \{t \in T \cap L_s : F(P_t) \cap P_{t'} \neq \emptyset\}} d\nu_{t'}(s) \ d\lambda_F(t'),
\]

where \(\nu_{t'}\) is the conditional probability measure for \(\nu\) on the subset \(T \cap F^{-1}(P_{t'})\).

In words, \(\nu'_F(A)\) is the \(\lambda_F\) measure of the function which is the \(\nu\)-average value over leaves of the reciprocal of the plaque pre-image counting function. Note that the inner integral above is only necessary when \(F\) carries multiple leaves to one.

Finally, \(\nu'\) in the main theorems signifies \(\nu'_F\).

In the above definition properness is only used to guarantee that the integrand of \(\nu'_F\) is well-defined. I.e., if a leaf \(L_{t'}\) is in the image of \(F\), then so is every plaque \(P_{t'}\).

The following shows that the measure \(\nu'_F\) induced from a map \(F\) and measure \(\nu\) are in some sense canonical.
Lemma 3.4. The measure $\nu'_F$ is well defined and only depends on $\nu$ and the induced map of $F$ between the corresponding leaf spaces.

Proof. First we note that the $\mathbb{Q}$-valued function $t' \mapsto \frac{1}{\#\{s \in T \cap F^{-1}(L_{t'}) : F(P_t) \cap P_{t'} \neq \emptyset\}}$ for $s \in T \cap F^{-1}(L_{t'})$ is bounded above by 1 and is Borel provided $F$ and $\nu$ are. Hence $\nu'_F$ is a well defined measure provided that $\lambda_F$ is. We note that, apart from the effect of the lack of injectivity of $r_i$, $\lambda_F$ is a pull-back measure of $\nu$. However, the multiplicity $\#\{t' \in A : F(P_t) \cap P_{t'} \neq \emptyset\}$ exactly takes care of this lack of injectivity to make $\lambda_F$ $\sigma$-additive.

For the invariance, we note that if $G$ is any other map which induces the same action on the leaf space as $F$, then the corresponding measures $\lambda^G_F$ and $\lambda^G$ may differ in each case because, for a given set $A \subset T_j$, the pre-images $F^{-1}(r_j^{-1}(A))$ and $G^{-1}(r_j^{-1}(A))$ may intersect disjoint subsets of plaques of each chart $U_i$. However, in the case of each map, we normalize by dividing by the average number of preimage plaques in $M$ per image plaque in $A$. (This is precisely the combined effect of each of the integrands in the expression for $\lambda_F$ and $\nu'_F$.)

Since the measure $\nu$ is holonomy invariant, the Radon-Nikodym derivative satisfies $\frac{d\lambda^G}{d\lambda^F}(t) = 1$ for any $t \in T$ where $h$ is any holonomy map carrying any other $s \in T$, belonging to the same leaf, to $t \in T$. Now $G$ can pointwise be expressed solely in terms of $F$ and pre-compositions with holonomy maps. Consequently, after the normalization the leafwise contribution to $\nu'_p(A)$ is precisely the same as that of $\nu'_G(A)$. In particular the two measures coincide. \qed

For the following we let $F$ be an arbitrary transversely $\nu$-measurable leafwise $C^1$ and proper foliated map between two foliated spaces $M$ and $N$. Let $\text{Jac} F$ denote the leafwise Jacobian of $F$ and we define the foliated pre-image counting function $p_F : N \rightarrow \mathbb{N} \cup \{\infty\}$ by

$$p_F(y) = \int_{T \cap F^{-1}(L_y)} \# \{x \in L_t : F(x) = y\} \, d\nu_y(t)$$

where as before $\nu_y$ is the conditional (probability) measure for $\nu$ on $T \cap F^{-1}(L_y)$. In other words, $p_F$ is the leafwise average of the ordinary pre-image counting function.

For the purposes of the next Proposition we will define $\mu_N$ as the measure which is locally $g_o \times \nu'_F$, and we will not require the metrics $g_o$ on the target leaves to be locally symmetric or even the same (we will still denote the metric on the target leaf $F(L)$ by $g_o$ instead of $g_{F(L)}$).

Proposition 3.5 (Foliated coarea formula). With these notations we have,

$$\int_M |\text{Jac} F(x)|d\mu_M(x) = \int_N p_F(y)d\mu_N(y)$$

Proof. First we show that we may suppose $F$ is surjective. Observe that

$$\int_N p_F(y)d\mu_N(y) = \int_{F(M)} p_F(y)d\mu_N(y) + \int_{N - F(M)} p_F(y)d\mu_N(y)$$

and for $y \in N - F(M)$, $p_F(y) = 0$. Hence, we have

$$\int_N p_F(y)d\mu_N(y) = \int_{F(M)} p_F(y)d\mu_N(y).$$
Thus provided that we have
\[ \int_M |\text{Jac } F(x)|d\mu_M(x) = \int_{F(M)} p_F(y)d\mu_N(y), \]
then we still obtain the conclusion of the proposition. We will therefore assume
without loss of generality that \( N = F(M) \).

Let \( \{U_i\}_{i=1}^m \) (resp. \( \{O_j\}_{j=1}^l \)) be the given foliated partition atlas of \( M \) (resp. \( N \)) with global transversals \( T = \cup_i T_i \) (resp. \( T' = \cup_j T'_j \)). As before, when \( t \in T_i \), we denote by \( P_t \) the plaque passing through \( t \), and for \( t' \in T'_j \), \( P_{t'} \) denotes the plaque of \( O_j \) through \( t' \). We prove the analogue of the coarea formula first on a single foliated chart \( U_i \) in \( (N, F_N) \). By first applying the usual coarea formula to the plaques, we obtain,
\[
\int_{T_i} \int_{P_t} |\text{Jac } F(x)|d\mu(t) = \int_{T_i} \int_{F(P_t)} \# \{ x \in P_t : F(x) = y \} d\nu(y)d\nu(t)
\]
In other words, this is the average number of pre-images per plaque. Re-expressing this in terms of the measure \( \nu'_{F} \), requires incorporating the average number of pre-image plaques per plaque. Since we use this average over all flow boxes, we first need to sum over all \( U_i \).

The contribution of the above integral, after summing over \( i \), coming from the flow box \( O_j \) is therefore

\[
= \int_{T_j} \int_{T \cap F^{-1}(L_{t'})} \sum_{t \in T \cap L_s : F(P_t) \cap P_{t'} \neq \emptyset} \# \{ x \in P_t : F(x) = y \} d\nu_{o}(y) \ d\nu'(t')
\]
where we recall that \( \nu_{t'} \) is the conditional probability measure on \( T \cap F^{-1}(L_{t'}) \) and \( \nu'_{F} \) is the measure defined in 3.3.

After switching the inner-most integrals, the previous integral can be re-expressed in terms of \( d\mu_N \) as

\[
= \int_{O_j} \int_{T \cap F^{-1}(L_{t'})} \# \{ x \in L_s : F(x) = y \} d\nu_{o}(s) d\mu_N(y)
= \int_{O_j} p_F(y) d\mu_N(y).
\]

Now we sum over all foliated charts \( \{O_j\}_{j=1}^l \) in \( N \) to obtain the original statement,

\[ \int_M |\text{Jac } F(x)|d\mu_M(x) = \sum_{i=1}^m \int_{O_j} p_F(y)d\mu_N(y) = \int_N p_F(y)d\mu_N(y). \]

\( \square \)

**Remark 3.6.** If instead of an invariant measure \( \nu \), we had only a quasi-invariant measure for \( \nu \), then we could still express the co-area formula in terms of an integral of a sum over \( N \), but it would not be the function \( p_F(y) \). Instead it would be a version of this pre-image sum weighted in terms of the different radon-nikodym derivatives of the holonomy translates of each \( \nu \) on different plaques. This would lead to a rather ugly form for the foliated degree, which would not be expressible...
in terms of a leafwise homotopy invariant. However, as remarked, the proof could still go through provided that $F$ is leafwise a homeomorphism.

4. Proofs of Theorems 1.3 and 1.6

Before proving the main theorems, the main external technical tool that we need is the following result.

**Proposition 4.1** (Proposition 6.1 of [BCG96]). Fix a leaf $L$, for each $s > \tilde{h}(g_L)$ the map $F_s$ is a $C^1$ map and satisfies $|\text{Jac } F_s(x)| \leq \left(\frac{s}{\tilde{h}(g_0)}\right)^n$ for every $x \in L$

Before proving Theorem 1.3, we need to recall the analytic characterization of degree. On each leaf $L_t$, and for Lebesgue almost any $y \in L_f(t)$ we have

$$\deg(f_{\mid L_t}) = \deg(F_{\mid L_t}) = \sum_{x \in L_t: F_s(x) = y} \text{sgn}(\text{det } dF_s(x)).$$

The main point is that the absolute value of this quantity is at most the number of pre-images of $y$ on the leaf. Moreover, this quantity remains constant for any $s > \tilde{h}(g_L)$. We also recall here the definition of the foliated degree given in the introduction,

$$\deg_{\mathcal{F}}(f) = \frac{1}{\mu_N(N)} \int_N \int_{T \cap f^{-1}(L_y)} \deg(f_{\mid L_y}) \, d\nu_y(t) \, d\mu_N(y).$$

**Proof of the inequality in Theorem 1.3.** By Lemma 3.4 we have $\nu'_f = \nu'_{F_s}$, and for each $y \in N$, we have

$$\int_{T \cap f^{-1}(L_y)} \deg(f_{\mid L_y}) \, d\nu_y(t) = \int_{T \cap f^{-1}(L_y)} \deg(F_{\mid L_y}) \, d\nu_y(t) \leq p_{F_s}(y).$$

Hence we may apply Propositions 3.5 and 4.1 to the foliations to obtain the following equalities and inequalities:

$$\deg_{\mathcal{F}}(f) \int_N d\mu_N \leq \int_N p_{F_s}(y) \, d\mu_N = \int_M |\text{Jac } F_s(x)| \, d\mu_M \leq \int_M \left(\frac{s(g_L)}{\tilde{h}(g_0)}\right)^n \, d\mu_M$$

where $d\mu_M$ is locally $d(g \times d\nu)$ and $d\mu_N$ is locally $d(g \times d\nu')$. Thus

$$\deg_{\mathcal{F}}(f) \int_N h(g_0)^n \, d\mu_N \leq \int_M s(g_L)^n \, d\mu_M$$

for all $s(g_L) \geq \tilde{h}(g_L)$

Let $s(g_L) \to \tilde{h}(g_L)$, we get

$$\deg_{\mathcal{F}}(f) \int_N h(g_0)^n \, d\mu_N \leq \int_M \tilde{h}(g_L)^n \, d\mu_M.$$
Applying Propositions 3.5 and 4.1 to the foliations of the ergodic components, we obtain the inequalities,

\[ \deg f|_{M_\alpha} \int_{N_\alpha} h(g_0)^n d\mu_{N_\alpha} \leq h(g_0)^n \int_{M_\alpha} \frac{|\text{Jac} F_s|}{h(g_0)} d\mu_{M_\alpha} \]

\[ = \int_{M_\alpha} (s|_{M_\alpha})^n d\mu_{M_\alpha}. \]

Here,

\[ \deg f|_{M_\alpha} = \int_{\mathcal{T}_\alpha} \deg f|_{M_\alpha} \frac{dv(t)}{\nu(T_\alpha)} = \deg f|_{L^\alpha}. \]

Hence, we have

\[ \deg f|_{M_\alpha} \int_{N_\alpha} h(g_0)^n d\mu_{N_\alpha} \leq \int_{M_\alpha} (s|_{M_\alpha})^n d\mu_{M_\alpha}. \]

Hence, let \( s \to \overline{h}(g_L) \), we have

\[ (2) \quad \deg f|_{M_\alpha} \int_{N_\alpha} h(g_0)^n d\mu_{N_\alpha} \leq \int_{M_\alpha} (\overline{h}(g_L)|_{M_\alpha})^n d\mu_{M_\alpha}. \]

Because the equality holds, i.e.

\[ \int_{M_\alpha} \overline{h}^n(g_L) d\mu_M = \deg f \int_N h^n(g_0) d\mu_N \]

i.e.

\[ \int_{\alpha \in A} \int_{M_\alpha} \overline{h}^n(g_L) d\mu_{M_\alpha} d\alpha = \int_{\alpha \in A} \deg f \int_{N_\alpha} h^n(g_0) d\mu_{N_\alpha} d\alpha \]

Because

\[ \int_{\alpha \in A} \deg f|_{M_\alpha} \int_{N_\alpha} h^n(g_0) d\mu_{N_\alpha} d\alpha \]

\[ = h^n(g_0) \int_{\alpha \in A} \deg f|_{M_\alpha} \text{Vol}(N_\alpha) d\alpha \]

\[ = h^n(g_0) \int_{\alpha \in A} (\deg f|_{L^\alpha}) \frac{\text{Vol}(N_\alpha)}{\text{Vol}(N)} d\alpha \]

\[ = h^n(g_0) \frac{\text{Vol}(N)}{\text{Vol}(N)} \int_{\alpha \in A} (\deg f|_{L^\alpha}) \frac{\text{Vol}(N_\alpha)}{\text{Vol}(N)} d\alpha \]

\[ = h^n(g_0) \text{Vol}(N) \int_{\alpha \in A} (\deg f|_{L^\alpha}) \frac{\text{Vol}(N_\alpha)}{\text{Vol}(N)} d\alpha \]

\[ = h^n(g_0) \text{Vol}(N) \deg f \int_{\alpha \in A} \int_{N_\alpha} d\mu_{N_\alpha} d\alpha \]

\[ = \int_{\alpha \in A} \deg f \int_{N_\alpha} h^n(g_0) d\mu_{N_\alpha} d\alpha \]

So

\[ \int_{\alpha \in A} \int_{M_\alpha} \overline{h}^n(g_L) d\mu_{M_\alpha} d\alpha = \int_{\alpha \in A} \deg f|_{M_\alpha} \int_{N_\alpha} h^n(g_0) d\mu_{N_\alpha} d\alpha \]
and by the above (2), we get for all \( \alpha \in A \), we have

\[
\deg_{\mathcal{F}}(f)|_{M_\alpha} \int_{N_\alpha} h(g_\alpha)^n d\mu_{N_\alpha} = \int_{M_\alpha} \overline{h}(g_\alpha)^n d\mu_{M_\alpha}
\]

If we rescale the metric on \( N_\alpha \) such that \( \deg_{\mathcal{F}}(f)|_{M_\alpha} \int_{N_\alpha} d\mu_{N_\alpha} = \int_{M_\alpha} d\mu_{M_\alpha} \), and since \( \overline{h}(g_\alpha) \) is constant on \( M_\alpha \), we could get \( \overline{h}(g_\alpha) = h(g_\alpha) \).

Therefore, after rescaling, the equality holds, if and only if \( \overline{h}(g_\alpha) = h(g_\alpha) \) and \( \deg_{\mathcal{F}}(f)|_{M_\alpha} \int_{N_\alpha} d\mu_{N_\alpha} = \int_{M_\alpha} d\mu_{M_\alpha} \) hold.

Therefore, if the equality holds, the functions \( x \mapsto |\text{Jac} F_s(x)| \) converges to the constant function \( x \mapsto 1 \) in \( L^1(L) \) as \( s \to \overline{h}(g_\alpha) \).

**Remark 4.2.** We have here used the fact that on each ergodic component \( M_\alpha \), we have \( \deg_{\mathcal{F}}(f)|_{M_\alpha} = \deg_{\mathcal{F}}(f)|_L = \deg(f)_L \).

Below we are going to prove that there exists a sequence \( s_i \to \overline{h}(g_\alpha) \), such that the natural maps \( F_{s_i} : L \to L_o \) converge uniformly on compact sets to a 1-Lipschitz map \( F \) with

\[
\text{Vol}(F^{-1}(A)) = |f_L| \text{Vol}(A)
\]

for all measurable \( A \subset L_o \).

Once this has been proved, the arguments in Section 5 of \[Bes98\] show that \( F \) is a local isometry. Since \( L \) and \( L_o \) are complete we deduce that \( F \) is a \( \deg(f)_L \)-Riemannian covering.

The remainder of the proof is devoted to showing that the maps \( F_{s_i} \) converge to a 1-Lipschitz map \( F \) for which (4.1) holds. Choose a point \( p \in L \), \( R \gg 1 \) and consider the sequence of maps

\[
F_{s_i}^R := F_{s_i}|_{B_p(R, L)} : B_p(R, L) \to L_o.
\]

Then following from \( |\text{Jac } F_{s_i}(x)| \to 1 \) in \( L^1(L) \) as \( s_i \to \overline{h}(g_\alpha) \) and the fourth point in Theorem 2.1 of \[BCS05\], there is a \( C > 0 \) such that, for \( i \) large enough, the map \( F_{s_i}^R \) is \( C \)-Lipschitz. We want to apply the Arzela-Ascoli theorem and prove that, after passing to a subsequence, the maps \( F_{s_i}^R \) converge when \( i \) tends to \( \infty \) to an \( C \)-Lipschitz map \( F^R : B_p(R, L) \to L_o \), thus, we have to prove that for all \( i \), \( F_{s_i} \) is bounded.

**Lemma 4.3.** \( F_{s_i}(p) \) is bounded, for \( s_i \in (\overline{h}(g_\alpha), \overline{h}(g_\alpha) + 1] \).

**Proof.** Since \( F_{s_i}(p) \) is a continuous map, for \( s_i \in [\overline{h}(g_\alpha) + \epsilon, \overline{h}(g_\alpha) + 1] \), \( 0 < \epsilon < 1 \), \( F_{s_i}(p) \) is bounded. Suppose \( F_{s_i} \to \infty \) as \( s_i \to \overline{h}(g_\alpha) \), then we have

**Claim 1:** The measure on \( f(L) \), \( \sigma_{s_i, p} \to \mu \), a dirac measure of weight \( \geq 1/2 \) as \( s_i \to \overline{h}(g_\alpha) \).

**Proof of claim 1.**

Claim 1 tells us that \( \sigma_{s_i, p} \to \mu \), a dirac measure of weight \( \geq 1/2 \) as \( s_i \to \overline{h}(g_\alpha) \), then we know on \( f(L) \), there exists a sequence of balls \( B_{s_i} \), s.t. \( \sigma_{s_i, p}(B_{s_i}) \to \mu \) as \( s_i \to \overline{h}(g_\alpha) \).

By the proof of Lemma 2.4, we know for any \( s > \overline{h}(g_\alpha) \), we have a bound \( R^s \) such that \( \mu_{s, p} \) is \( \frac{n}{10} \) supported on the ball \( B(p, R^s) \) and \( \text{diam}(f(B(p, R^s))) \) is bounded. Therefore, \( \sigma_{s, p} \) is \( \frac{n}{10} \) supported on the finite diameter set \( f(B(p, R^s)) \).

Hence, for \( i \) large enough, \( f(B(p, R^s)) \) is disjoint with \( B_{s_i} \), but \( \frac{n}{10} + \frac{1}{2} > 1 \), which contradicts \( \sigma_{s, p} \) is a probability measure. Hence, we proved the lemma. \( \square \)
Lemma 4.3 and the Arzela-Ascoli theorem imply that, after passing to a subsequence, the maps $F_{s_i}^R$ converge to a $C$-Lipschitz map $F^R : B_p(R, L) \to L_o$. Since $R$ was arbitrarily chosen this implies that the maps $F_{s_i}$ converge uniformly on compact sets, when $i$ tends to $\infty$, to an $C$-Lipschitz $F : L \to L_o$. As remarked above, the functions $x \mapsto |\text{Jac} F_s(x)|$ converges to the constant function $x \mapsto 1$ in $L^1(L)$ as $s \to \hat{h}(g_L)$. The last point in Theorem 2.1 of [BCG95, Lemma 7.8] shows that the functions

$$x \mapsto \max_{\nu \in T_s M} \frac{\|dF_{s_i} \nu\|_L}{\|\nu\|_L}$$

converge (pointwise) on a set of full measure to the constant function $x \mapsto 1$. This implies that $F : L \to L_o$ is actually 1-Lipschitz [BCG95, Lemma 7.8].

We are going to show that $F$ satisfies (4.1).

**Lemma 4.4.** For $\nu'$-a.e. leaf $L_0$ and any $A \subset L_0$ measurable, we have $\text{Vol}(F^{-1}(A)) = |\deg(f_{s_i})| \text{Vol}(A)$.

**Proof.** Because $F : L \to L_o$ is 1-Lipschitz, we have $\text{Vol}(F^{-1}(A)) \geq |\deg f_{s_i}| \text{Vol}(A)$ for all $A \subset L_0$. Hence, it suffices to show $\text{Vol}(F^{-1}(A)) \leq |\deg f_{s_i}| \text{Vol}(A)$ for all $A \subset L_0$. Assume that this fails to be true for some measurable set $A$ and choose $\varepsilon > 0$ with

$$\text{Vol}(F^{-1}(A)) > |\deg f_{s_i}| \text{Vol}(A) + (|\deg f_{s_i}| + 2)\varepsilon$$

There is an open set $U \subset L_0$ containing $A$ and a compact set $K \subset F^{-1}(A)$ with

$$\text{Vol}(U) \leq \text{Vol}(A) + \varepsilon \quad \text{and} \quad \text{Vol}(K) \geq \text{Vol}(F^{-1}(A)) - \varepsilon.$$

Since the maps $F_{s_i}$ converge uniformly on compact sets to the map $F$, we obtain $F_{s_i}(K) \subset U$ for all $i$ large enough. We then deduce that

$$\text{Vol}(F^{-1}(U)) \geq \text{Vol}(K) \geq \text{Vol}(F^{-1}(A)) - \varepsilon > |\deg f_{s_i}| \text{Vol}(A) + (|\deg f_{s_i}| + 1)\varepsilon$$

for all but finitely many $i$. Without loss of generality, we may assume $U \subset P_r$, where $P_r$ is some plaque in $L_o$. Otherwise, we can use the partition of unity to split up $U$. The calculation in the proof of the inequalities shows that for all $i$ large enough,

$$|\deg f_{s_i}| \text{Vol}(P_r, g_o) = |\deg f_{s_i}| \text{Vol}(P_r \setminus U) + |\deg f_{s_i}| \text{Vol}(U)$$

$$\leq |\deg f_{s_i}| \int_{P_r \setminus U} dg_o + \text{Vol}(F_{s_i}^{-1}(U)) - \varepsilon$$

$$\leq \int_{P_r \setminus U} p(F_{s_i}, y)dg_o + \text{Vol}(F_{s_i}^{-1}(U)) - \varepsilon$$

$$= \int_{L \cap F_{s_i}^{-1}(P_r) \setminus F_{s_i}^{-1}(U)} |\text{Jac} F_{s_i}(x)|dg_L(x) + \text{Vol}(F_{s_i}^{-1}(U)) - \varepsilon$$

$$\leq \int_{L \cap F_{s_i}^{-1}(P_r) \setminus F_{s_i}^{-1}(U)} \left( \frac{s_i}{\hat{h}(g_o)} \right)^n dg_L(x) + \text{Vol}(F_{s_i}^{-1}(U)) - \varepsilon$$

$$= \left( \frac{s_i}{\hat{h}(g_o)} \right)^n \text{Vol}(L \cap F_{s_i}^{-1}(P_r) \setminus F_{s_i}^{-1}(U)) + \text{Vol}(F_{s_i}^{-1}(U)) - \varepsilon$$
Since $s_i$ tends to $h(g_L) = h(g_o)$ we obtain passing to the limit that

$$|\deg f_{L_i}| \Vol(P_r, g_o) \leq \Vol(F^{-1}_{s_i}(P_r)) - \epsilon$$

Hence, if there is any such $\epsilon > 0$ for each plaque in an $\nu'$-positive measure set of plaques, then the quasi-invariance of $F_*, \nu$ implies that

$$|\deg f_{L_i}| \Vol(N_{\alpha}) \leq \Vol(M_{\alpha}) - \epsilon'$$

for some $\epsilon' > 0$, which contradicts our assumption.

\[\square\]

End of the proof of the equality case and Theorem 1.3. By Lemma 4.4 for any $A \subset L_o$ measurable we have $\Vol(F^{-1}(A)) = |\deg f_{L_i}| \Vol(A)$. Hence $F$ is a 1-Lipschitz map satisfying (4.1). As remarked above, this concludes the proof of Theorem 1.3.

\[\square\]

We can now easily extend the above proof to the proof of the Foliated Real Schwarz Lemma.

\textbf{Proof of Theorem 1.6.} We begin by noting that under the hypotheses, the Bishop-Gromov comparison theorem applied to universal covers implies that $h(g_L) \leq n - 1$ for any leaf $L \subset M$, and ordinary sectional curvature estimates imply that $h(g_L) \geq n - 1$ for any leaf $L \subset N$.

The analogue for Proposition 4.1 was proved for the barycenter map between two leaves under these assumptions in [BCG98]. This gives the local estimate

$$|\Jac F_s(x)| \leq \left(\frac{s}{n-1}\right)^n,$$

where again $n$ is the dimension of the leaf. Replacing $h(g_o)$ by $n - 1$ everywhere in the proof of Theorem 1.3 yields a verbatim proof of Theorem 1.6. For the very last statement, we note that since we obtain leafwise isometries for $\nu$-almost every leaf, we are in the equality case of the Bishop-Gromov theorem and so both leaf metrics are locally hyperbolic.

\[\square\]

\textbf{REFERENCES}


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