1.1) An automobile manufacturer has four colors available for automobile exteriors and three for interiors. How many different color combinations can he produce?

\[ 4 \cdot 3 = 12 \]

1.2) What is the probability that at least 2 of the presidents of the United States have died on the same day of the year? If you bet this has happened, would you win your bet?

Assume, for simplicity’s sake, that each year has exactly 365 days in it and everyone has an equal probability of dying on any day of the year. There are 39 presidents who have died, so this is basically the Birthday Problem with 39 people. Thus the probability that at least 2 of the presidents of the United States have died on the same day of the year is

\[ 1 - \left( \frac{364}{365} \right)^{\frac{39 \cdot 38}{2}} = 1 - \left( \frac{364}{365} \right)^{741} \approx 0.87 \]

It just so happens that the 2nd and 3rd presidents, John Adams and Thomas Jefferson died on the same calendar date. Note only that, but it was the very same day, the 4th of July of 1826, the 50 year anniversary of US independence, or at least when we said we were independent. By the way, has anyone else noticed how awesome it is that we don’t celebrate Independence Day on the day we won the war, but on the day we told the British to get lost? Also, isn’t it weird that pretty much nobody but history buffs knows the date(s) we won the war?

1.3) Five people get on an elevator that stops at five floors. Assuming that each has an equal probability of going to any one floor, find the probability that they all get off at different floors.

\[ \frac{5!}{5} = 4! \] by a 5 in both the numerator and denominator cancelling.

1.4) A symphony orchestra has in its repertoire 30 Haydn symphonies, 15 modern works, and 9 Beethoven symphonies. Its program always consists of a Haydn symphony followed by a modern work, and then a Beethoven symphony.

(a) How many different programs can it play?

\[ 30 \cdot 15 \cdot 9 \]

(b) How many different programs are there if the three pieces can be played in any order?

\[ 30 \cdot 15 \cdot 9 \cdot 3! \]

1.5) In how many ways can we choose five people from a group of ten to form a committee?

\[ \binom{10}{5} \]
1.6) Charles claims that he can distinguish between beer and ale 75 percent of the time. Ruth bets that
he cannot and, in fact, just guesses. To settle this, a bet is made: Charles is to be given ten small
glasses, each having been filled with beer or ale, chosen by tossing a fair coin. He wins the bet if he gets
seven or more correct. Find the probability that Charles wins if he has the ability that he claims. Find
the probability that Ruth wins if Charles is guessing.

Because there are two possibilities per glass, we can use Bernoulli trials. If he actually has this ability,
\( \binom{10}{7} \left( \frac{1}{2} \right)^7 \left( \frac{1}{2} \right)^3 + \binom{10}{8} \left( \frac{1}{2} \right)^8 \left( \frac{1}{2} \right)^2 + \binom{10}{9} \left( \frac{1}{2} \right)^9 \left( \frac{1}{2} \right)^1 + \binom{10}{10} \left( \frac{1}{2} \right)^{10} (\frac{1}{2})^0 = \binom{10}{7} \left( \frac{1}{2} \right)^7 \left( \frac{1}{2} \right)^3 + \binom{10}{8} \left( \frac{1}{2} \right)^8 \left( \frac{1}{2} \right)^2 + \binom{10}{9} \left( \frac{1}{2} \right)^9 \left( \frac{1}{2} \right)^1 + \binom{10}{10} \left( \frac{1}{2} \right)^{10} (\frac{1}{2})^0 \)

If he doesn’t have this ability,
\[ 1 - \binom{10}{7} \left( \frac{1}{2} \right)^7 \left( \frac{1}{2} \right)^3 + \binom{10}{8} \left( \frac{1}{2} \right)^8 \left( \frac{1}{2} \right)^2 + \binom{10}{9} \left( \frac{1}{2} \right)^9 \left( \frac{1}{2} \right)^1 + \binom{10}{10} \left( \frac{1}{2} \right)^{10} (\frac{1}{2})^0 = 1 - \binom{10}{7} \left( \frac{1}{2} \right)^7 \left( \frac{1}{2} \right)^3 + \binom{10}{8} \left( \frac{1}{2} \right)^8 \left( \frac{1}{2} \right)^2 + \binom{10}{9} \left( \frac{1}{2} \right)^9 \left( \frac{1}{2} \right)^1 + \binom{10}{10} \left( \frac{1}{2} \right)^{10} (\frac{1}{2})^0 \]

1.7) A die is rolled 30 times. What is the probability that a 6 turns up exactly 5 times? What is the
most probable number of times that a 6 will turn up?

Like the 3 heads out of 8 coin flips problem from the Khan Academy video I sent out, this is a counting problem of how
many ways there are to arrange m things into n slots, over how many total ways there are to fill those n slots. In this
case, it’s how to fit 5 #6s into 30 rolls. So, like the Khan academy video had \( \binom{8}{3} \) ways, we have \( \binom{30}{5} \). However, unlike
the Khan academy video, we don’t only have two possible outcomes per roll. With the Khan coin flipping, we knew that
every “slot” that we didn’t assign a heads to had to be a tails, but each roll that isn’t a 6 could be a 1,2,3,4, or 5 in this
case. So for each arrangement of 5 #6s, we will have to multiply by the numerous ways we can arrange the outcomes of
the other rolls.

This part is like the four-of-a-kind problem from class. We know that there were \( \binom{13}{1} \) ways to pick a type of card, but
then we had to multiply by the 48 remaining possibilities for the 5th card, or “slot”. So, we have 25 remaining rolls of the
die. Because we have already assigned the 5 #6s, we cannot have any of these 25 rolls be a 6, or we would have more
than 5 of them. These can be anything other than 6 though. Thus, there are 25 slots and 5 possibilities for each slot to
give us \( 5^{25} \) ways to fill those 25 other rolls of the die, so now we have a total of \( \binom{30}{5} \cdot 5^{25} \) ways to get 5 #6s in 30 rolls.
As always, in order to determine probability, we need to divide by the total possible number of outcomes. There are 6
outcomes per roll, and there are 30 rolls, so \( 6^{30} \) total outcomes. Thus, the probability of getting exactly 5 #6s in 30 rolls of a fair die is:

\[ \frac{\binom{30}{5} \cdot 5^{25}}{6^{30}} \]

1.8) In a ten-question true-false exam, find the probability that a student gets a grade of 70 percent or
better by guessing. Answer the same question if the test has 30 questions, and if the test has 50
questions.

This is basically the Khan Academy coin flipping problem because there are two equally probable outcomes per question,
with a slight variation. Instead of just finding how to fit 7 Trues into 10 question “slots”, we are fitting 7 or 8 or 9 or 10,
so it’s \( \binom{10}{7} + \binom{10}{8} + \binom{10}{9} \). Similarly, for 30 questions it’s \( \binom{30}{21} + \binom{30}{22} + \ldots + \binom{30}{30} \) and for 50 questions it’s \( \binom{50}{21} + \binom{50}{22} + \ldots + \binom{50}{50} \).

1.9) A poker hand is a set of 5 cards randomly chosen from a deck of 52 cards. Find the probability of a:
(a) four of a kind (four cards of the same face value). (b) full house (one pair and one triple, each of the
same face value).

Explanations were given in class. (a) \( \frac{\binom{13}{1} \cdot \binom{4}{4} \cdot \binom{12}{1}}{\binom{52}{5}} \) (b) \( \frac{\binom{13}{1} \cdot \binom{4}{2} \cdot \binom{12}{1}}{\binom{52}{5}} \)
1.10) Each of the four engines on an airplane functions correctly on a given flight with probability .99, and the engines function independently of each other. Assume that the plane can make a safe landing if at least two of its engines are functioning correctly. What is the probability that the engines will allow for a safe landing?

Each individual engine either fails or doesn’t so we can use Bernoulli trials again.

\[
\begin{align*}
&\binom{4}{2} \left( \frac{99}{100} \right)^2 \left( \frac{1}{100} \right)^2 + \binom{4}{3} \left( \frac{99}{100} \right)^3 \left( \frac{1}{100} \right) \left( \frac{99}{100} \right) \left( \frac{1}{100} \right)^3 + \binom{4}{4} \left( \frac{99}{100} \right)^4 \left( \frac{1}{100} \right)^4 \\
&= 6 \left( \frac{99^2}{100^4} \right) + 4 \left( \frac{99^3}{100^4} \right) + \left( \frac{99^4}{100^4} \right) \\
&= \frac{58,806 + 3,881,196 + 96,059,601}{100,000,000} \\
&= \frac{999,999,603}{100,000,000}
\end{align*}
\]

2.1) A coin is tossed three times. What is the probability that exactly two heads occur, given that:

(a) the first outcome was a head?
Remember, \( P(A|B) = \frac{P(A \cap B)}{P(B)} \). Let \( 2H \) be the event that exactly two heads occur. Let \( H \) be the event that the first outcome was a head. Thus, \( P(2H|H) = \frac{\binom{3}{1}}{\binom{3}{1}} = \frac{1}{2} \).

(b) the first outcome was a tail?
Let \( T \) be the event that the first outcome was a tail. Thus, \( P(2H|T) = \frac{\binom{2}{1}}{\binom{3}{1}} = \frac{1}{4} \).

(c) the first two outcomes were heads?
Let \( H2 \) be the event that the first two outcomes were heads. Thus, \( P(2H|H2) = \frac{\binom{2}{1}}{\binom{3}{1}} = \frac{1}{2} \).

(d) the first two outcomes were tails?
Let \( T2 \) be the event that the first two outcomes were tails. Thus, \( P(2H|T2) = \frac{\binom{2}{1}}{\binom{3}{1}} = 0 \).

(e) the first outcome was a head and the third outcome was a head?
Let \( FT \) be the event that the first two outcomes were heads. Thus, \( P(2H|FT) = \frac{\binom{2}{1}}{\binom{3}{1}} = \frac{1}{2} \).

2.2) A die is rolled twice. What is the probability that the sum of the faces is greater than 7, given that:

(a) the first outcome was a 4?
Remember, \( P(A|B) = \frac{P(A \cap B)}{P(B)} \). Let \( G \) be the event that the sum of the faces is greater than 7. Let \( F \) be the event that the first outcome was a 4. Thus, \( P(G|F) = \frac{\binom{3}{1}}{\binom{6}{1}} = \frac{1}{2} \).

(b) the first outcome was greater than 3?
Let \( T \) be the event that the first outcome was greater than 3. Thus, \( P(G|T) = \frac{\binom{4}{1}}{\binom{6}{1}} = \frac{2}{3} \).

(c) the first outcome was a 1?
Let \( O \) be the event that the first outcome was a 1. Thus, \( P(G|O) = \frac{\binom{0}{1}}{\binom{6}{1}} = 0 \).

(d) the first outcome was less than 5?
Let \( L \) be the event that the first outcome was less than 5. Thus, \( P(G|L) = \frac{\binom{4}{1}}{\binom{6}{1}} = \frac{1}{3} \).
2.3) From a deck of five cards numbered 2, 4, 6, 8, and 10, respectively, a card is drawn at random and replaced. This is done three times. What is the probability that the card numbered 2 was drawn exactly two times, given that the sum of the numbers on the three draws is 12?

Let $T$ be the event of drawing exactly two 2s, and let $S$ be the event of the three cards summing to 12. We know $P(T|S) = \frac{P(T \cap S)}{P(S)}$, and we want to know $P(T|S)$, so we need to find $P(T \cap S)$ and $P(S)$. The total number of combinations of three cards from a set of 5, with replacement, is $5^3 = 125$. Out of those, the only that contain two 2s and sum to 12 are the three ways of drawing two 2s and an 8, so $P(T \cap S) = \frac{3}{125}$.

Now we find $P(S)$. Notice that if one of the cards drawn is a 10, then the minimum sum of three draws is 14, so the set of possible three-draw sets that sum to 12 excludes all sets that contain a 10. Besides the three permutations of two 2s and an 8, there are the six permutations of (2, 4, 6) and the possibility of drawing three 4s, so $P(S) = \frac{10}{125}$. Thus,

$$P(T|S) = \frac{P(T \cap S)}{P(S)} = \frac{\frac{3}{125}}{\frac{10}{125}} = \frac{3}{10}$$

2.4) If $P(\overline{B}) = \frac{1}{4}$ and $P(A|B) = \frac{1}{2}$, what is $P(A \cap B)$?

$P(B) = 1 - P(\overline{B}) = 1 - \frac{1}{4} = \frac{3}{4}$. The definition of $P(A|B)$ is $P(A|B) = \frac{P(A \cap B)}{P(B)}$, so

$$\begin{align*}
\frac{1}{2} & = \frac{P(A \cap B)}{\frac{3}{4}} \\
\frac{1}{2} \cdot \frac{3}{4} & = P(A \cap B) \\
\frac{3}{8} & = P(A \cap B)
\end{align*}$$

2.5) One coin in a collection of 65 has two heads. The rest are fair. If a coin, chosen at random from the lot and then tossed, turns up heads 6 times in a row, what is the probability that it is the two-headed coin?

Let $H$ be the event that the two-headed coin is chosen and let $S$ be the event that 6 heads are tossed in a row. Thus, we want to find $P(H|S)$. One in 65 coins has two-heads so $P(H) = \frac{1}{65}$. If the two-headed coin is chosen, the probability of flipping 6 heads in a row is 1. The probability of choosing a fair coin is $P(\overline{H}) = 1 - P(H) = 1 - \frac{1}{65} = \frac{64}{65}$, and if a fair coin is chosen the probability of flipping 6 heads in a row is $\frac{1}{2^6}$. Thus, the overall probability of flipping 6 heads in a row if we don’t know what coin was chosen is

$$P(S) = 1 \left( \frac{1}{65} \right) + \frac{1}{2^6} \left( \frac{64}{65} \right) = \frac{1}{65} + \frac{1}{64} \left( \frac{64}{65} \right) = \frac{1}{65} + \frac{1}{65} = \frac{2}{65}$$

From the reading, we know that $P(H|S) = \frac{P(H \cap S)}{P(S)}$. Note that $H \subset S$, so $H \cap S = H$. Thus,

$$P(H|S) = \frac{P(H)}{P(S)} = \frac{\frac{65}{65}}{\frac{2}{65}} = \frac{1}{2}$$
2.6) In London, half of the days have some rain. The weather forecaster is correct 2/3 of the time, i.e., the probability that it rains, given that she has predicted rain, and the probability that it does not rain, given that she has predicted that it won’t rain, are both equal to 2/3. When rain is forecast, Mr. Pickwick takes his umbrella. When rain is not forecast, he takes it with probability 1/3. Find (a) the probability that Pickwick has no umbrella, given that it rains. (b) the probability that he brings his umbrella, given that it doesn’t rain.

(a) Let $F$ be the event that rain is forecasted, let $R$ be the event that it rains, and let $U$ be the event that Mr. Pickwick takes his umbrella. Thus, we want to find $P(U|R)$. The forecaster is right 2/3 of the time, so $P(F|R) = \frac{2}{3}$ and $P(\overline{F}|R) = \frac{1}{3}$. It is given that $P(U|F) = 0$ and $P(U|\overline{F}) = \frac{2}{3}$, so

$$P(U|R) = 0 \left( \frac{2}{3} \right) + \frac{2}{3} \left( \frac{1}{3} \right) = \frac{2}{9}$$

(b) We want to find $P(U|\overline{R})$. Again, the forecaster is right 2/3 of the time so $P(F|\overline{R}) = \frac{1}{3}$ and $P(\overline{F}|\overline{R}) = \frac{2}{3}$. It is given that $P(U|F) = 1$ and $P(U|\overline{F}) = \frac{1}{3}$, so

$$P(U|\overline{R}) = 1 \left( \frac{1}{3} \right) + \frac{1}{3} \left( \frac{2}{3} \right) = \frac{3}{9} + \frac{2}{9} = \frac{5}{9}$$

[Extra Credit] A cabinet has three drawers. In the first drawer there are two gold balls, in the second drawer there are two silver balls, and in the third drawer there is one silver and one gold ball. A drawer is picked at random and a ball chosen at random from the two balls in the drawer. Given that a gold ball was drawn, what is the probability that the drawer with the two gold balls was chosen?

Let $F$ be the event that the first drawer is chosen and let $G$ be the event that a gold ball is chosen. We want to find $P(F|G)$. We know from the reading that $P(F|G) = \frac{P(F \cap G)}{P(G)}$. Because $F \subset G$, we know that

$$P(F|G) = \frac{P(F)}{P(G)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} \cdot 2 = \frac{2}{3}$$