1. Let Loves(x, y) mean “x loves y,” Student(x) mean “x is a student,” Friend(x, y) mean “x is a friend of y,” Bicycle(x) mean “x is a bicycle,” Owns(x, y) mean “x owns y,” and Sister(x, y) mean “x is a sister of y.”

a. Translate the following propositions into the most natural equivalent statements in English.

i. \( \forall x \exists y (\text{Student}(x) \rightarrow \text{Sister}(y, x)) \)
   1. All students have a sister.

ii. \( \exists x (\forall z ((x \neq z) \rightarrow \text{Friend}(x, z)) \land \exists y (\text{Bicycle}(y) \land \text{Owns}(x, y))) \)
   1. There’s someone who is friends with everyone (not including themselves) that owns a bicycle.

iii. \( \forall x \exists y ((\text{Student}(x) \rightarrow \text{Friend}(x, y)) \land \forall z (\text{Friend}(x, z) \rightarrow (y = z))) \)
   1. Students have only one friend.

iv. \( \forall x \forall y \exists z ((\text{Student}(x) \land \text{Bicycle}(z) \land \text{Owns}(x, z)) \rightarrow \text{Friend}(x, y)) \)
   1. Students who own a bicycle are friends with everyone.

v. \( \forall x \exists y ((\text{Student}(x) \land \text{Bicycle}(y) \land \text{Owns}(x, y)) \rightarrow \exists w \text{Sister}(x, w)) \)
   1. Students who own a bicycle have a sister.

b. Translate the following statements into predicate logic.

i. Peter has only one friend but at least two bicycles.
   1. \( \exists x (\text{Friend}(\text{Peter}, x) \land \forall y (\text{Friend}(\text{Peter}, y) \rightarrow (x = y))) \land \exists x \exists y (\text{Bicycle}(x) \land \text{Bicycle}(y) \land \text{Owns}(\text{Peter}, x) \land \text{Owns}(\text{Peter}, y) \land (x \neq y)) \)

ii. Every student has at least one friend if and only if they own a bicycle.
   1. \( \forall x (\exists y((\text{Student}(x) \rightarrow \text{Friend}(x, y)) \leftrightarrow \exists z ((\text{Student}(x) \land \text{Bicycle}(z) \land \text{Owns}(x, z)))) \)

iii. If something is a student, it cannot be a bicycle, and vice versa.
   1. \( \forall x (\text{Student}(x) \leftrightarrow \neg \text{Bicycle}(x)) \)

iv. Any student that loves his or her bicycle has at least two friends.
   1. \( \forall x (\exists y (\text{Student}(x) \land \text{Bicycle}(y) \land \text{Owns}(x, y) \land \text{Loves}(x, y) \rightarrow \exists w \exists z \text{Friend}(x, w) \land \text{Friend}(x, z) \land (w \neq z)) \)

v. All bicycle-loving sisters are students.
   1. \( \forall x \exists y \exists z ((\text{Sister}(x, y) \land \text{Bicycle}(z) \land \text{Love}(x, z)) \rightarrow \text{Student}(x)) \)
2. For each of the following, prove that the conclusion follows logically from the premises.
   i. Premises: \( \neg f \), \( (g \land h) \rightarrow (f \lor e) \), \( (\neg f \lor c) \rightarrow h \), \( (\neg f \lor b) \rightarrow g \) Prove: e
   1. \( \neg f \)   Premise
   2. \( (g \land h) \rightarrow (f \lor e) \)   Premise
   3. \( (\neg f \lor c) \rightarrow h \)   Premise
   4. \( (\neg f \lor b) \rightarrow g \)   Premise
   5. \( \neg f \lor c \lor \lor Intro, 1 \)
   6. h   Modus Ponens, 3,5
   7. \( \neg f \lor b \lor \lor Intro, 1 \)
   8. g   Modus Ponens, 4,7
   9. \( g \land h \lor \land Intro, 6, 8 \)
   10. \( (f \lor e) \lor \lor Elim, 2,9 \)
   11. e   \lor Elim, 1, 10

   ii. Premises: \( p \land q \), \( (p \land r) \rightarrow (t \land u) \), \( r \land s \) Prove: \( t \lor m \)
   1. \( p \land q \)   Premise
   2. \( (p \land r) \rightarrow (t \land u) \)   Premise
   3. \( r \land s \)   Premise
   4. p   \land Elim, 1
   5. r   \land Elim, 3
   6. p \land r   \land Intro 4,5
   7. t \land u   Modus Ponens, 2, 6
   8. t   \land Elim, 7
   9. t \lor m   \lor Intro, 8

   iii. Premises: \( r \lor t \), \( r \rightarrow s \), \( \neg t \), \( (p \land q) \leftrightarrow (r \land s) \) Prove: p
   1. \( r \lor t \)   Premise
   2. \( r \rightarrow s \)   Premise
   3. \( \neg t \)   Premise
   4. \( (p \land q) \leftrightarrow (r \land s) \)   Premise
   5. r   \lor Elim, 1,3
   6. s   Modus Ponens, 2,5
   7. r \land s   \land Intro, 5,6
   8. \( p \land q \)   Modus Ponens, 4,7
   9. p   \land Elim, 8
iv. Premises: ¬q, p ∨ q, p → (r ∧ s), ¬u, r ↔ (t ∨ u) Prove: t
1. ¬q Premise
2. p ∨ q Premise
3. p → (r ∧ s) Premise
4. ¬u Premise
5. r ↔ (t ∨ u) Premise
6. p ∨ Elim, 1, 2
7. (r ∧ s) Modus Ponens, 3, 6
8. r ∧ Elim, 7
9. (t ∨ u) Modus Ponens, 8, 5
10. t ∨ 4, 9

v. Premises ¬(p ∨ q) → ¬r, ¬p, r Prove: q
1. ¬(p ∨ q) → ¬r Premise
2. ¬p Premise
3. r Premise
4. ¬(¬(p ∨ q)) Modus Tollens, 1, 3
5. p ∨ q Double negation, 4
6. q ∨ Elim, 2, 5

vi. Premise: w → x, w → ¬y, y ∨ ¬x Prove: ¬w
1. w → x Premise
2. w → ¬y Premise
3. y ∨ ¬x Premise
4. w Assume
5. x Modus Ponens, 1, 4
6. ¬y Modus Ponens, 2, 4
7. y ∨ Elim, 3, 5
8. y ∧ ¬y ∧ Intro, 6, 7
9. Contradiction, therefore the assumption must be false.
10. ¬w
vii. Premise: \((p \lor q) \rightarrow r\), \(\neg r \lor s\), \(p \rightarrow \neg s\) Prove: \(\neg p\)

1. \((p \lor q) \rightarrow r\) Premise
2. \(\neg r \lor s\) Premise
3. \(p \rightarrow \neg s\) Premise
4. \(p\) Assume
5. \(\neg s\) Modus Ponens, 3,4
6. \(\neg r\) \lor Elim, 2,5
7. \(\neg(p \lor q)\) Modus Tollens, 1, 6
8. \(\neg p \land \neg q\) Distributive law
9. \(\neg p\) \land Elim, 8
10. \(p \land \neg p\) \land Intro 4,9
11. Contradiction, therefore the assumption must be false.
12. \(\neg p\)

viii. Premise: \(\neg x \rightarrow \neg z\), \(y \rightarrow z\) Prove: \(y \rightarrow (x \lor w)\)

1. \(\neg x \rightarrow \neg z\) Premise
2. \(y \rightarrow z\) Premise
3. \(y\) Assume
4. \(\neg z\) Modus Tollens, 2,3
5. \(\neg \neg x\) Modus Tollens, 1, 4
6. \(x\) Double negation, 5
7. \(x \lor w\) \lor Intro, 6
8. \(y \rightarrow (x \lor w)\) \rightarrow Intro, 3-7

ix. Premises: \(p \leftrightarrow (q \lor r)\), \(\neg q\) Prove: \(p \rightarrow (r \lor s)\)

1. \(p \leftrightarrow (q \lor r)\) Premise
2. \(\neg q\) Premise
3. \(p\) Assume
4. \((q \lor r)\) Modus Ponens, 1,3
5. \(r\) \lor Elim, 2,4
6. \((r \lor s)\) \lor Intro, 5
7. \(p \rightarrow (r \lor s)\) \rightarrow Intro, 3-6
x. Premises: $q \leftrightarrow s$, $p \lor q$, $p \rightarrow r$, $s \rightarrow t$  Prove: $r \lor t$

1. $q \leftrightarrow s$  Premise
2. $p \lor q$  Premise
3. $p \rightarrow r$  Premise
4. $s \rightarrow t$  Premise
5. $\neg(r \lor t)$  Assume
6. $\neg r \land \neg t$  Distributive law
7. $\neg t$  $\land$ Elim, 6
8. $\neg r$  $\land$ Elim, 6
9. $\neg s$  Modus Tollens, 4, 7
10. $\neg p$  Modus Tollens, 3, 8
11. $q$  $\lor$ Elim, 2, 10
12. $s$  Modus Ponens, 11, 1
13. $s \land \neg s$  $\land$ Intro, 9, 12
14. Contradiction, therefore the assumption must be false.
15. $r \lor t$
3. For each of the following, prove that the conclusion follows logically from the premises.

i. Premises: $\forall x \ (p(x) \rightarrow q(x))$  Prove: $\exists x \ p(x) \rightarrow \exists x \ q(x)$

1. $\forall x \ (p(x) \rightarrow q(x))$  Premise
2. $p(c) \rightarrow q(c)$  $\forall$ Elim, 1
3. $\exists x \ p(x) \rightarrow q(c)$  $\exists$ Intro, 2
4. $\exists x \ p(x) \rightarrow \exists x \ q(x)$  $\exists$ Intro, 3

ii. Premises: $\forall x \ (p(x) \rightarrow q(x))$  Prove: $\forall x \ p(x) \rightarrow \exists x \ q(x)$

1. $\forall x \ (p(x) \rightarrow q(x))$  Premise
2. $p(c) \rightarrow q(c)$  $\forall$ Elim, 1
3. $\forall x \ p(x) \rightarrow q(c)$  $\forall$ Intro, 2
4. $\forall x \ p(x) \rightarrow \exists x \ q(x)$  $\exists$ Intro, 3

iii. Premises: $\exists x \ (p(x) \rightarrow q(x))$  Prove: $\exists x \ p(x) \rightarrow \exists x \ q(x)$

1. $\exists x \ (p(x) \rightarrow q(x))$  Premise
2. $p(c) \rightarrow q(c)$  $\exists$ Elim, 1
3. $\exists x \ p(x) \rightarrow q(c)$  $\exists$ Intro, 2
4. $\exists x \ p(x) \rightarrow \exists x \ q(x)$  $\exists$ Intro, 3
4. Transform each informal argument into predicate logic. Then give a formal proof.
   i. Every dog either likes people or hates cats. Rover is a dog. Rover loves cats. Therefore, some dog likes people.

   \( P(x) = x \text{ likes people. } \ C(x) = x \text{ loves cats.} \)

   Domain of discourse of \( x \) is dogs.

   1. \( \forall x \ (P(x) \lor \neg C(x)) \) Premise
   2. \( C(\text{Rover}) \) Premise
   3. \( P(\text{Rover}) \lor \neg C(\text{Rover}) \) \( \forall \) Elim, 1
   4. \( P(\text{Rover}) \lor \neg \text{C} \) Elim, 2,3
   5. \( \exists x \ P(x) \) \( \exists \) Intro 4

   ii. Every committee member is rich and famous. Some committee members are old. Therefore, some committee members are old and famous.

   \( R(x) = x \text{ is rich. } \ F(x) = x \text{ is famous. } \ O(x) = x \text{ is old.} \)

   Domain of discourse of \( x \) is committee member.

   1. \( \forall x \ (R(x) \land F(x)) \) Premise
   2. \( \exists x \ O(x) \) Premise
   3. \( O(c) \) \( \exists \) Elim, 2
   4. \( R(c) \land F(c) \) \( \forall \) Elim, 1
   5. \( F(c) \) \( \land \) Elim, 4
   6. \( O(c) \land F(c) \) \( \land \) Intro, 3,5
   7. \( \exists x \ (O(x) \land F(x)) \) \( \exists \) Intro, 6

   iii. No human beings are quadrupeds. All men are human beings. Therefore, no man is a quadruped.

   \( Q(x) = x \text{ is quadruped. } \ M(x) = x \text{ is man. } \ H(x) = x \text{ is a human being.} \)

   Domain of discourse of \( x \) is everything.

   1. \( \forall x \ (H(x) \rightarrow \neg Q(x)) \) Premise
   2. \( \forall x \ (M(x) \rightarrow H(x)) \) Premise
   3. \( H(c) \rightarrow \neg Q(c) \) \( \forall \) Elim, 1
   4. \( M(c) \rightarrow H(c) \) \( \forall \) Elim, 2
   5. \( M(c) \rightarrow \neg Q(c) \) Law of Syllogism, 3,4
   6. \( \forall x \ (M(x) \rightarrow \neg Q(x)) \) \( \forall \) Intro, 5
iv. Some freshman like sophomores. No freshman likes any junior. Therefore, no sophomore is a junior.

\( F(x) = x \) is a freshman. \( J(x) = x \) is a junior. \( S(x) = x \) is a sophomore. Likes(x,y) = x likes y.

Domain of discourse of x is students.

1. \( \forall y \left( S(y) \rightarrow \exists x \left( F(x) \wedge \text{Likes}(x,y) \right) \right) \)  
   Premise

Notice that the translation of the sophomores premise above is tricky. We are considering that the sentence “some freshman like sophomores” can be translated as: “For anyone, if they are a sophomore, there is a freshman that likes them.” This is saying that “there is at least one freshman that likes any and all people, as long as they are a sophomore.” We are using the Universal quantifier for the sophomores. We are assuming that if it wasn’t the Universal but the Existential, then it would have been stated as “some freshman like some sophomores.”

2. \( \forall x \forall y \left( \left( F(x) \wedge J(y) \right) \rightarrow \neg \text{Likes}(x,y) \right) \)  
   Premise

After stating the two premises, we proceed to eliminate all the quantifiers from the premises. We do this one quantifier at a time. We start with the existential quantifiers. Only once we’ve eliminated all of the existential quantifiers, do we begin to eliminate the universal quantifiers. This allows us to state the universal truths for the entities for which we know more specific truths.

3. \( \exists y \left( S(y) \rightarrow \left( F(a) \wedge \text{Likes}(a,y) \right) \right) \)  
   \( \exists \) Elim, 1

4. \( S(b) \rightarrow \left( F(a) \wedge \text{Likes}(a,b) \right) \)  
   \( \forall \) Elim, 1

Once we’ve eliminated the quantifiers, the premise that “some freshman like sophomores” has turned into: “if \( b \) is an arbitrary sophomore, then we can find a freshman called \( a \) that likes them.”

5. \( \forall y \left( F(a) \wedge J(y) \right) \rightarrow \neg \text{Likes}(a,y) \)  
   \( \forall \) Elim, 2

6. \( F(a) \wedge J(b) \rightarrow \neg \text{Likes}(a,b) \)  
   \( \forall \) Elim, 2

Notice that with purposefully chose to instantiate J(b) with the same entity as S(b). We do this because of what we need to prove (that if \( b \) is a sophomore, then \( b \) is not a junior - for any arbitrary \( b \)). Next, we are going to assume \( b \) is a sophomore (S(b)), such that if we can arrive at the fact that \( b \) cannot be a junior (\( \neg J(b) \)), then we will be able to say that S(b) \( \rightarrow \neg J(b) \).

7. \( S(b) \)  
   Assume

8. \( F(a) \wedge \text{Likes}(a,b) \)  
   Modus Ponens, 4,7

9. \( F(a) \)  
   \( \wedge \) Elim, 8

10. \( \text{Likes}(a,b) \)  
    \( \wedge \) Elim, 8

11. \( \neg(F(a) \wedge J(b)) \)  
    Modus Tollens, 6, 10

12. \( \neg F(a) \vee \neg J(b) \)  
    Distributive law, 11

13. \( \neg J(b) \)  
    \( \vee \) Elim, 9, 12

14. \( S(b) \rightarrow \neg J(b) \)  
    \( \rightarrow \) Intro, 7-13

Because \( b \) was introduced in step 4 an arbitrary person when we eliminated the universal from the first premise, then we can say that this conditional applies for any student - if they are a sophomore, then they are not a junior.

15. \( \forall y \left( S(y) \rightarrow \neg J(y) \right) \)  
    \( \forall \) Intro, 14
A general point about translating first, and then giving a formal proof: Remember that there are several different ways of translating the problem into logic. And there are several different ways of proving that the argument is valid (or invalid). Your task is first justify your translation. Then, to use the rules of inference and deduction to show that you can depart from the premises and arrive at your conclusion. What we will be evaluating are those two steps: (1) that your translation is well justified. This includes stating the definitions of the predicates and the domain of discourse. It also includes justifications for decisions you made wherever you think the sentence in English was ambiguous. (2) Stating the premises. Enumerating them. Stating other propositions that are either assumed, if necessary, or that follow from a simple rule of inference, as in the lecture notes or the book. Justifying each step, one by one - mentioning which rule was used, and on which previous truths it was applied. And finally arriving at the conclusion.