Sets and Functions
Required reading: FC Sections 2.1, 2.2, and 2.3

1. Overview of the next unit:
   a. Sets
   b. Boolean algebra of sets
   c. Functions
   d. Regular expressions
   e. Finite state automata
   f. Turing machines
   g. Cognition as computation

2. Set theory (FC: 2.1)
   a. In order to deal with the complexity of the world, we categorize things. In mathematics, categories are modeled by sets.
   b. A set is a collection of elements.
   c. We denote sets by elements separated in braces: \{17, New York, Big Ben\}
   d. Anything can be an element:
      i. When we use mathematics to model the world, then we use entities such as ‘New York,’ and ‘Indiana.’
      ii. When we are doing mathematics per se, we generally stick to mathematical entities such as the integers and real numbers. We’ll also use letters such as \(a\) and \(b\) to refer to abstract entities.
   e. There are two important rules about sets:
      i. A set is defined by the elements that it contains. The order in which those elements are listed is irrelevant (e.g., \{a, b, c, d\} and \{b, c, a, d\} define the same set).
      ii. A set can only contain one copy of the same element.
   f. The symbol \(\in\) is used to express the relation “is an element of.”
      i. If \(a\) is an entity and \(A\) is a set, then \(a \in A\) is true if and only if \(a\) is one of the elements of \(A\). We say \(a\) is a member of the set \(A\).
      ii. The symbol \(\notin\) is used to express the relation “is not an element of.”
   g. Both \(a \in A\) and \(a \notin A\) are statements in the sense of propositional logic - they can be true or false. So \(a \notin A\) is equivalent to \(\neg (a \in A)\).
   h. The empty set is the set that contains no elements, and is denoted by \(\emptyset\) or by \(\{\}\).
   i. If \(A\) and \(B\) are sets, then, \(A\) is equal to \(B\) if and only if they contain exactly the same elements. \(A = B\) if and only if \(\forall x (x \in A \rightarrow x \in B)\).
   j. \(A\) is a subset of \(B\) if every element of \(A\) is an element of \(B\). That is, \(A \subseteq B\) if \(\forall x (x \in A \rightarrow x \in B)\).
   k. If \(A = B\), then it’s true that \(A \subseteq B\) and \(B \subseteq A\).
      i. First check that every element of \(A\) is also an element of \(B\). And, every element of \(B\) is also an element of \(A\).
l. The empty set is a subset of all sets.
m. If $A \subseteq B$ but $A \neq B$, we can say $A$ is a proper subset of $B$, or $A \subset B$.
n. A set can contain an infinite number of elements. In this case it’s not possible to list all elements in the set. We use the ellipsis “...” to indicate this. For example, the set of all natural numbers: $N = \{0, 1, 2, ...\}$.
o. That’s an informal notation that is only useful in very well-defined trivial cases. It’s not as useful to say the set $\{17, 42, 105, ...\}$.
p. Clearly we need another way to specify sets besides listing their elements.

3. Expressing sets

a. If $P(x)$ is a predicate, then we can form the set that contains $a$ such that $a$ is in the domain of discourse for $P$ and $P(a)$ is true. We denote it like this: $\{x \mid P(x)\}$. It reads: “The set of $x$ such that $P(x)$.” The name of the variable is irrelevant.
   i. Example. If $E(x)$ is the predicate “$x$ is an even number” and if the domain of discourse is the set of all natural numbers, $N$. Then $\{x \mid E(x)\}$ denotes the set of even natural numbers = $\{0, 2, 4, 6, .. \}$.

b. Often it’s useful to specify the domain of discourse of the predicate: $\{x \in N \mid E(x)\}$. “The set of all $x$ in $N$ such that $E(x)$.” (Or similarly, you can constrain the set and not the domain: $\{x \mid x \in N \land E(x)\}$).

c. We can use this to define, for example, the prime numbers in a rigorous way:
   i. $\{n \in N \mid (n > 1) \land \forall x \forall y((x \in N \land y \in N \land n = xy) \rightarrow (x = n \lor y = n))\}$.

d. Although this is a hard example, it shows that it is possible to define a complex set.

4. Operations on sets

a. Union. The set that contains all the elements of $A$ together with all the elements of $B$. If $A$ and $B$ are sets, then $A \cup B = \{x \mid x \in A \lor x \in B\}$.

b. Intersection. The set that contains every entity that is both a member of $A$ and of $B$. If $A$ and $B$ are sets, then $A \cap B = \{x \mid x \in A \land x \in B\}$.

c. Set difference. $A - B = \{x \mid x \in A \land x \notin B\}$

d. Union and intersection are obviously commutative. Not the set difference.

e. Examples: If $A = \{a,b,c\}$, $B=\{b,d\}$, $C=\{d,e,f\}$, what are
   i. $A \cup B$
   ii. $A \cup C$
   iii. $A \cap B$
   iv. $A \cap C$ (disjoint)
   v. $A - B$
   vi. $A - C$

f. Examples, Let $X = \{x \mid L(x)\}$ and $Y = \{x \mid W(x)\}$, where $L(x)$ is “$x$ is lucky” and $W(x)$ is “$x$ is wise,” then: $X \cup Y$, $X \cap Y$, $X - Y$, and $Y - X$. 
5. Nested sets
   a. Sets can contain other sets as elements.
   b. \{a, \{b\}\}
      i. We can say: \(a \in \{a, \{b\}\}\)
      ii. We can say: \(\{b\} \in \{a, \{b\}\}\)
      iii. But we can’t say: \(b \in \{a, \{b\}\}\).
   c. Power set.
      i. The set that contains all subsets of \(A\), \(P(A)\).
      ii. If \(A = \{a, b\}\), then \(P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}\).
      iii. Notice the empty set is an element of the power set of any set.

6. Boolean Algebra of Sets (FC: 2.2)
   a. Set theory and logic are closely related.
   b. The intersection and union of sets can be defined in terms of logical “and” and “or” operators.
   c. The notation \(\{x \mid P(x)\}\) makes it possible to use predicates to specify sets.
   d. Commutative operations:
      i. We can show that \(A \cup B = B \cup A\), by showing that \(\forall x((x \in A \cup B) \iff (x \in B \cup A))\).
         1. \(x \in A \cup B \iff x \in A \lor x \in B\) (definition of \(\cup\))
         2. \(\iff x \in B \lor x \in A\) (commutativity of \(\lor\))
         3. \(\iff x \in B \cup A\) (definition of \(\cup\))
      ii. We can show similar properties of distributive laws.
         1. \(A \cup (B \cap C) = (A \cup B) \cap (A \cup C)\)
         2. \(A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\)
   e. Universal set. Let \(U\) be a given universal set, and let \(A\) be any subset of \(U\). We can define the complement of \(A\) in \(U\) to be the set \(\overline{A}\) that is defined by \(\overline{A} = \{x \in U \mid x \notin A\}\). Whenever complements are used, there must be some universal set in the background.
f. Some Laws of Boolean algebra for sets.
g. Work out an example of verifying one of those laws (pp. 91).

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<tr>
<th>Double complement</th>
<th>( \overline{A} = A )</th>
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<tr>
<td>Miscellaneous laws</td>
<td>( A \cup \overline{A} = U )</td>
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<td></td>
<td>( A \cap \overline{A} = \emptyset )</td>
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<td>( \emptyset \cup A = A )</td>
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<td>( \emptyset \cap A = \emptyset )</td>
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<td>Idempotent laws</td>
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<td>Commutative laws</td>
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<td>( A \cup B = B \cup A )</td>
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<td>Associative laws</td>
<td>( A \cap (B \cap C) = (A \cap B) \cap C )</td>
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<td>( A \cup (B \cup C) = (A \cup B) \cup C )</td>
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<td>Distributive laws</td>
<td>( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) )</td>
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<td>( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) )</td>
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<td>DeMorgan’s laws</td>
<td>( A \cap \overline{B} = \overline{A} \cup B )</td>
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h. There is a direct correspondence between the basic symbols and operations of propositional logic and certain symbols and operations in set theory:
i. Any valid logical formula or computation involving propositional variables and the symbols \( T, F, \land, \lor, \) and \( \neg \) can be transformed into a valid formula in set theory replacing the propositions with subsets of \( U \) and replacing the logical symbols with \( U, \emptyset, \cap, \cup, \) and the complement operator.

7. Functions (FC: Section 2.4).
a. A function is a relationship between two sets, which associates exactly one element from the second set to each element of the first one.
b. Relationship between two sets. \( f: A \rightarrow B \) \( f \) is a function from the set \( A \) to the set \( B \), \( f \) maps \( A \) to \( B \).
c. There are functions in the real world,
   i. For example, each item in a store has a price. The first set is the items in the store. The second is the list of prices. The question, “What is the price of this item?” has a single, definite answer - for each item in the store.
ii. “Who is the (biological) mother of this person?” also establishes a functional relationship. For each person, there is a single, definite answer. This is not the case for “Who is the child of this person?” The answer to that is not single or definite. A person may not have children, or more than one.

iii. A map is a functional relationship between the physical space and the representation of it on paper. The whole point of the map is that each point in the map represents a single, definite point in physical space. Often functional relationships are called mappings.

d. Same is true in Mathematics,
   i. Every rectangle has an associated area.
   ii. Every natural number $n$ has a square $n^2$.

8. Kinds of functions
   a. Total function: A function $f: A \rightarrow B$ is total if and only if every element of $A$ is assigned a $B$-value by $f$.
      i. $\forall a \in A \ (\exists b \in B (b = f(a)))$
   b. Onto function: A function $f: A \rightarrow B$ is onto iff every element of $B$ is $f$ of something.
      i. $\forall b \in B \ (\exists a \in A (b = f(a)))$
   c. One-to-one function: A function $f: A \rightarrow B$ is 1-1 iff every element of $B$ is $f$ of at most one thing.
      i. $\forall x \in A \ \forall y \in A (x \neq y \rightarrow f(x) \neq f(y))$
      ii. Or the contrapositive: $\forall x \in A \ \forall y \in A (x = y \rightarrow f(x) = f(y))$
   d. Bijection: A function is a bijection iff it is all those things: total, onto, and 1-1. “Exactly one arrow out, exactly one arrow in.”