1. Consider the following deterministic monetary model. Output at date $t$, $y_t$, is produced using capital accumulated at $t-1$:

$$y_t = f(k_{t-1}) = k_{t-1}^\sigma, \quad 0 < \sigma < 1. \quad (1)$$

Assume that capital depreciates completely each period, so investment at $t$ is simply $k_t$.

A representative household derives utility from consumption, $c_t$, and leisure, $1 - l_t$, and is endowed each period with one unit of time. Preferences are

$$\sum_{t=0}^{\infty} \beta^t [\log(c_t) + \gamma \log(1 - l_t)], \quad 0 < \beta < 1, \gamma > 0. \quad (2)$$

In addition to consumption and leisure, at time $t$ households choose investment, $k_t$, the level of nominal government bonds, $B_t$, and the level of money, $M_t$, that they wish to hold. Households also pay taxes that are proportional to output at rate $\tau_t$. Household purchases of consumption and investment goods are subject to the financing constraint (or “shopping-time technology”),

$$(1 - l_t)^\alpha (c_t + k_t) \leq M_{t-1}/P_t, \quad \alpha > 0, \quad (3)$$

with $\gamma/\alpha$ such that an equilibrium exists. The household’s budget constraint is

$$c_t + k_t + \frac{M_t - M_{t-1}}{P_t} + \frac{B_t - (1 + i_{t-1})B_{t-1}}{P_t} \leq (1 - \tau_t)f(k_{t-1}). \quad (4)$$

At time $t$, the government issues one-period bonds that pay gross interest of $1 + i_t$ when the bonds mature at $t+1$, and new money, $M_t - M_{t-1}$, and levies taxes $T_t = \tau_t f(k_{t-1})$ to finance purchases in the amount $g_t$.

It is convenient to express government purchases as a fraction of output: $s^g_t = g_t/f(k_{t-1})$.

(a) Derive the economy’s national income accounting identity.
(b) Let $\mu_t$ be the lagrange multiplier associated with (4) and $\lambda_t$ be the lagrange multiplier associated with (3). Derive the first-order necessary conditions.

(c) Interpret the first-order conditions for $k_t$ and $M_t$.

Private agents take as given the sequences of policy variables $\{\rho_t, \tau_t, s^g_t\}$, where $\rho_t = M_t/M_{t-1}$. Assume the economy is in a stationary equilibrium in dates $t + j, j \geq 1$, but starts from some other position at time $t$. Specifically, assume that

$$\rho_t = \rho, \tau_t = \tau, s^g_t = s^g$$

and

$$\rho_{t+j} = \rho_F, \tau_{t+j} = \tau_F, s^g_{t+j} = s^g_F, \quad j \geq 1.$$  

(d) Show that the decision rule for capital is given by

$$k_t = \left[1 - \sigma \beta \left(\frac{1-\tau_F}{1-s^F_F}\right)\right] \left[1 - \sigma \beta^2 \frac{\gamma}{\alpha} \left(\frac{1-\tau_F}{1-s^F_F}\right)\right] (1 - s^g) f(k_{t-1}).$$

(e) Derive how $k_t$ depends on future taxes and government spending shares, $(\tau_F, s^g_F)$, holding current policies fixed. Explain your results.

(f) Show that the decision rule for velocity, defined as $(1 - l_t)\alpha = \frac{M_{t-1}/P_t}{c_t + k_t}$, solves the difference equation

$$(1 - l_t)\alpha \left(\frac{c_t + k_t}{c_t} - \frac{\gamma}{\alpha}\right) = \beta \left[1 - (1 - l_{t+1})\alpha \left(\frac{c_{t+1} + k_{t+1}}{c_{t+1}} - \frac{\gamma}{\alpha}\right) + \frac{\gamma}{\alpha}\right].$$

(g) Solve (8) and use it together with (7) to characterize how $P_t$ depends on the future policy variables, $(\rho_F, \tau_F, s^g_F)$, holding current policies fixed. Explain your result.

(h) The comparative static-type exercises in (e) and (g) involve incomplete specifications of policy behavior. Explain why.