1. Consider the following deterministic monetary growth model with government debt and lump-sum taxes. A representative household chooses \( \{k_t, c_t, M_t/P_t, b_t\} \) to maximize

\[
\sum_{t=0}^{\infty} \beta^t [u(c_t) + v(M_t/P_t)], \quad 0 < \beta < 1,
\]

subject to

\[
c_t + \frac{M_t}{P_t} + b_t + k_t + \tau_t \leq f(k_{t-1}) + k_{t-1} + \frac{M_{t-1}}{P_t} + (1 + r_{t-1})b_{t-1},
\]

with \( b_t \geq 0 \) and \( M_{-1} > 0, b_{-1} \geq 0, k_{-1} > 0 \). Note that \( b_t \) is real debt and \( r_t \) is the real interest rate on that debt. Capital, \( k \), does not depreciate and the functions \( u, v, \) and \( f \) satisfy the usual conditions (strictly increasing, strictly concave, twice continuously differentiable). \( \tau \) is lump-sum taxes.

The government chooses sequences \( \{g_t, \tau_t, M_t, b_t\} \) to satisfy the constraint

\[
\frac{M_t - M_{t-1}}{P_t} + b_t - (1 + r_{t-1})b_{t-1} = g_t - \tau_t.
\]

(a) Derive necessary and sufficient conditions for a solution to this optimum problem.

(b) Taking the sequence \( \{g_t\} \) as given, show that the policy sequences \( \{\tau_t, b_t, M_t\} \) are irrelevant for solutions for \( \{k_t, c_t, r_t\} \).

(c) Derive the money demand function for this economy and show that under conventional assumptions on the functions \( u \) and \( v \), the interest elasticity is non-positive.

(d) How does higher expected money growth affect the nominal interest rate? Explain your answer.

(e) Assume that \( M_t = M_{t-1}, t \geq 0 \). Prove that it is not feasible to run a fiscal policy with a constant net-of-interest deficit, defined as

\[
d = g_t - \tau_t > 0.
\]

Explain your answer.
(f) Assume that $M_t = M_{t-1}$, $t \geq 0$. Prove that it is feasible to run a fiscal policy with a constant gross-of-interest deficit, defined as
\[ \tilde{d} = r_{t-1} b_{t-1} + g_t - \tau_t > 0. \]
Explain your answer, including an explicit description of how taxes, interest payments on the debt, and the debt-output ratios evolve over time. [Hint: For part (f), conjecture that there exists an equilibrium with constant $\tilde{d}, g, M$ and with $P_t = P, t \geq 0$.]

2. An infinitely lived representative household is endowed with a constant quantity $y$ of goods each period and chooses $\{c_t, M_t, B_t\}$ to maximize
\[
E_0 \sum_{t=0}^{\infty} \beta^t \left[ u(c_t) + v(M_t/P_t) \right], \quad 0 < \beta < 1,
\]
subject to
\[
c_t + \frac{M_t}{P_t} + \frac{B_t}{P_t} + \tau_t \leq y + \frac{M_{t-1}}{P_t} + \frac{(1 + i_{t-1}) B_{t-1}}{P_t},
\]
with $M_{-1} + B_{-1} > 0$.
Government policies finance a stream of spending, $\{g_t\}$, with sequences $\{\tau_t, M_t, B_t\}$. For simplicity, assume $g_t = 0, t \geq 0$. Monetary policy sets the nominal interest rate, $i_t$, as a function of the “gap” between current inflation, $\pi_t$, and a target inflation rate, $\pi^*$, following the rule
\[ i_t = \left( \frac{\pi_t}{\pi^*} \right)^\alpha \theta_t. \]
Tax policy sets lump-sum taxes, $\tau_t$, as a function of the “gap” between past real government debt, $b_{t-1} = B_{t-1}/P_{t-1}$, and a target debt level, $b^*$, following the rule
\[ \tau_t = \left( \frac{b_{t-1}}{b^*} \right)^\gamma \psi_t. \]
$\theta$ and $\psi$ are random components to monetary and tax policy behavior. They are uncorrelated with each other, have mean of 1, and may be serially correlated.
(a) Write down the government’s budget constraint.
In what follows, you are asked to examine a first-order loglinearization of this model around its steady state. For simplicity, assume the following steady state conditions: \( \bar{\pi} = \pi^* = 1 \) and \( \bar{\tau}/\bar{b} = \beta^{-1} - 1 \), where \( \bar{x} \) denotes the deterministic steady state value of \( x \).

(b) Show that the log-linear expression for real money balances, \( \dot{m}_t \), is given by
\[
\dot{m}_t = -\frac{1}{\chi} \left( \frac{\beta}{1 - \beta} \right) \dot{\pi},
\]
where \( 1/\chi \) is the intertemporal elasticity of substitution of real balances, so the interest elasticity of money demand is \( \varphi = -\frac{1}{\chi} \left( \frac{\beta}{1 - \beta} \right) \).

(c) Reduce the model to a dynamical system in \( \{\dot{\pi}_t, \dot{b}_t\} \) and show that the roots are \( (\alpha, \beta^{-1} - \gamma(\beta^{-1} - 1)) \). Give general conditions on \( (\alpha, \gamma) \) such that there exists a unique equilibrium.

(d) Assume values for \( (\alpha, \gamma) \) that deliver a unique equilibrium in which inflation is unaffected by the tax shock, \( \psi \). Derive the solution for inflation in this case. Interpret your findings.

(e) Assume values for \( (\alpha, \gamma) \) that deliver a unique equilibrium in which the tax shock, \( \psi \), affects inflation. Derive the solution for inflation in this case. Interpret your findings. [Hint: Choose \( (\alpha, \gamma) \) wisely and it will save you a lot of time.]

(f) In the solution to (e), how does a higher steady state level of debt affect the elasticity of inflation with respect to taxes? Explain your findings.