Monetary and Fiscal Policy under Imperfect Knowledge
Monash University Mini-course Part III

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Motivation

- Macroeconomic models depend on household and firm expectations

- Policy design
  - Not enough to observe some history of beliefs
  - Need a theory of the determination of beliefs: rational expectations

- Rational expectations policy design
  - Heavy reliance on managing expectations through announced commitments
  - What are the consequences of imprecise control of beliefs?
Motivation II

- Reasonable to suppose agents have limited information about policy regime
  - US Financial Crisis witnessed dramatic change in the nature and conduct of stabilization policy
  - Does imperfect knowledge limit the efficacy of policy?

- For example, Bernanke (2004):

  “[...] most private-sector borrowing and investment decisions depend not on the funds rate but on longer-term yields, such as mortgage rates and corporate bond rates, and on the prices of long-lived assets, such as housing and equities. Moreover, the link between these longer-term yields and asset prices and the current setting of the federal funds rate can be quite loose at times.”
The Agenda

- Part 3 of this mini-course proceeds as follows
  - Lecture 1: Tools and conceptual issues
  - Lecture 2: Consequences of learning for the choice of monetary policy rule
  - Lecture 3: Monetary and fiscal interactions
Introducing Learning Dynamics

- Two foundations of rational expectations equilibrium analysis
  - Optimization
  - Mutual consistency of beliefs

- Strong knowledge assumptions
  - Agents know more than the modeler
Scientific Method

- Principle components
  - Maintained theories
  - Procedures for collecting data
  - Procedures for confronting theory with data
  - Procedures for updating theory given discrepancy between theory and data
Bounded Rationality: Learning Dynamics

- Attempts to place the agent and modeler on equal footing

- Agents might proceed as:
  - Classical econometrician: know model but unsure about parameters
  - Bayesian econometrician: unsure about model and parameters but understand how they are unsure
Learning Dynamics

- Analyze broad class of model that relaxes the second pillar of REE analysis
  - Retain optimization
  - Permit non-rational expectations

- Beware of theorists bearing free parameters!
  - Constrain the analysis by requiring the procedure to nest rational expectations
Why learning?

- Permits analysis of a broader class of problems

- May resolve/elucidate
  - Questions of indeterminacy
  - Questions of robustness
  - Questions of dynamics
  - Questions of regime change/regime uncertainty
Today

- Mathematical foundations
  - Learning renders model dynamics self-referential
  - Requires new set of tools to analyze model properties

- Conceptual issues in modeling multi-period decision problems
A Simple Model

- Consider a variant of the Cobweb model

\[ p_t = \mu + \alpha E_{t-1} p_t + \delta w_{t-1} + \eta_t \]  

(1)

where \( \mu, \alpha, \delta > 0; w_t \) and exogenous process; and \( \eta_t \) a bounded iid disturbance

- Unique bounded solution

\[ p_t = \bar{a} + \bar{b} w_{t-1} + \eta_t \]

where

\[ \bar{a} = (1 - \alpha)^{-1} \mu \text{ and } \bar{b} = (1 - \alpha)^{-1} \delta \]
Suppose agents forecast \( p_t \) as a linear function

\[
\hat{E}_{t-1} p_t = a_{t-1} + b_{t-1} w_{t-1}
\]

(2)

where \( \{a_{t-1}, b_{t-1}\} \) are currently maintained beliefs.

Price dynamics then

\[
p_t = (\mu + \alpha a_{t-1}) + (\delta + \alpha b_{t-1}) w_{t-1} + \eta_t
\]

- Self-referential system
- Data generating process is non-stationary
Least Squares Learning

• Beliefs \( \{a_{t-1}, b_{t-1}\} \) estimates using data \( \{p_i, w_i\}_{i=0}^{i=t-1} \)

• Ordinary least squares implies

\[
\begin{pmatrix}
  a_{t-1} \\
  b_{t-1}
\end{pmatrix}
= \left( \sum_{i=1}^{t-1} z_{i-1} z'_{i-1} \right)^{-1} \left( \sum_{i=1}^{t-1} z_{i-1} p_i \right)
\]

where \( z_i = [1 \ w_i] \)

– Fully specified system: relations (1) - (3)
Question

- Under what conditions do $a_t \to \bar{a}$ and $b_t \to \bar{b}$ as $t \to \infty$?
Theorem

- Consider the model (1) - (3). If $\alpha < 1$ then $(a_t, b_t) \to (\bar{a}, \bar{b})$ with probability one. If $\alpha > 1$ then convergence occurs with probability zero.
  
  - Property $\alpha < 1$ referred to as expectational stability
  
  - E-Stability principle governed by mapping between agent beliefs and true model coefficients
E-Stability principle

- Recall

\[ \hat{E}_{t-1}p_t = a_{t-1} + b_{t-1}w_{t-1} \]  \hspace{1cm} (4)

and

\[ p_t = (\mu + \alpha a_{t-1}) + (\delta + \alpha b_{t-1})w_{t-1} + \eta_t \]

- Latter expression implies optimal rational forecast

\[ E_{t-1}p_t = (\mu + \alpha a_{t-1}) + (\delta + \alpha b_{t-1})w_{t-1} \]
Mapping Defined

- Agents’ beliefs and optimal forecast define the mapping

\[ T \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \mu + \alpha a \\ \delta + \alpha b \end{pmatrix} \]

- An REE is a fixed point of this mapping

- Application of method of undetermined coefficients
E-Stability principle II

- Define the associated ordinary differential equation

\[ \frac{d}{d\tau} \begin{pmatrix} a \\ b \end{pmatrix} = T \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} \]

where \( \tau \) is notional time

- Local stability properties govern E-Stability principle
E-Stability principle III

- In particular, the REE is E-Stable if and only if the ODE is local stable at \((\bar{a}, \bar{b})\)

- Hence

\[
\frac{da}{d\tau} = \mu + (\alpha - 1) a \\
\frac{db}{d\tau} = \delta + (\alpha - 1) b
\]

requires \(\alpha < 1\)
Recursive Least Squares

- Ordinary least squares regression can be written as

\[
\phi_t = \phi_{t-1} + t^{-1}R_t z_{t-1} (p_t - \phi'_{t-1} z_{t-1})
\]

\[
R_t = R_{t-1} + t^{-1} (z_{t-1} z'_{t-1} - R_{t-1})
\]

where \( \phi_t = [a_t \ b_t]' \)
Recursive Least Squares II

Since

\[ p_t = (\mu + \alpha a_{t-1}) + (\delta + \alpha b_{t-1}) w_{t-1} + \eta_t \]

\[ = T(\phi_{t-1})'z_{t-1} + \eta_t \]

we have

\[ \phi_t = \phi_{t-1} + t^{-1}R_{t-1}^{-1}z_{t-1}(z_{t-1}'(T(\phi_{t-1}) - \phi_{t-1}) + \eta_t) \]

\[ R_t = R_{t-1} + t^{-1}(z_{t-1}z_{t-1}' - R_{t-1}) \]

- Question: does this system converge??
Stochastic Approximation Methods

- Provide results characterizing convergence of such systems
  - Ljung (1977), Marcet and Sargent (1989)

- Consider stochastic recursive algorithm

  \[
  \theta_t = \theta_{t-1} + \gamma_t Q(t, \theta_{t-1}, X_t)
  \]

  where

  \(\theta_t\) - parameter estimates \((a_t, b_t, R_t)\)
  \(X_t\) - state vector (effects of \(p_t, z_t\) and \(\eta_t\))
  \(\gamma_t\) - deterministic sequence of gains \((t^{-1})\)
Stochastic approximation approach associates the ODE

\[
\frac{d\theta}{d\tau} = h(\theta(\tau))
\]

where

\[
h(\theta) \equiv \lim_{t \to \infty} EQ(t, \theta, X_t)
\]

- \(E\) denotes the expectation of \(Q(t, \theta, X_t)\) with respect to the invariant distribution of \(X_t\) for fixed \(\theta\)
Stochastic Approximation Results

• Under suitable assumptions:
  
  - If $\bar{\theta}$ is a locally stable equilibrium point of the ODE, then $\bar{\theta}$ is a possible point of convergence of the SRA.
  
  - If $\bar{\theta}$ is not a locally stable equilibrium point of the ODE, then $\bar{\theta}$ is not a possible point of convergence of the SRA. That is $\theta_t \to \bar{\theta}$ with probability zero.
Technical Assumptions

- Stochastic approximation results rely on assumptions regarding
  - Regularity conditions on $Q$
  - Conditions on the rate at which $\gamma_t \to 0$
  - Assumptions on the properties of $X_t$
    - Evans and Honkapohja (2001, Chapter 6 & 7)
Example Revisited

- To write our RLS algorithm in SRA form define $S_{t-1} \equiv R_t$ and write the system as

\[
\phi_t = \phi_{t-1} + t^{-1} S_{t-1}^{-1} z_{t-1} (z_{t-1}' (T (\phi_{t-1}) - \phi_{t-1}) + \eta_t)
\]

\[
S_t = S_{t-1} + t^{-1} \left( \frac{t}{t+1} \right) (z_t z_t' - S_{t-1})
\]

so that

\[
\theta_t = \text{vec}(\phi_t, S_t)
\]

\[
X_t = [1 \ w_t \ w_{t-1} \ \eta_t]'
\]

\[
\gamma_t = t^{-1}
\]
Example Revisited II

- Hence

\[
Q_\phi(t, \theta_{t-1}, X_t) = S_{t-1}^{-1} z_{t-1} \left( z'_{t-1} \left( T(\phi_{t-1}) - \phi_{t-1} \right) + \eta_t \right)
\]

\[
Q_S(t, \theta_{t-1}, X_t) = \text{vec} \left( \left( \frac{t}{t+1} \right) (z_t z'_t - S_{t-1}) \right)
\]

- To do: compute the associated ODE!

- Fix \( \theta_t \) and compute expectation over \( X_t \)
Example Revisited III

- Fixing \((\phi, S)\) implies

\[
    h_\phi (\phi, S) = \lim_{t \to \infty} ES^{-1}z_{t-1} \left( z'_{t-1} (T(\phi) - \phi) + \eta_t \right)
\]

\[
    h_S (\phi, S) = \lim_{t \to \infty} E \left( \frac{t}{t+1} \right) (z_t z'_t - S)
\]

- Since

\[
    Ez_t z'_t = Ez_{t-1} z'_{t-1} = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} \equiv M
\]

where \(\Omega = E \left[ w_t w'_t \right];\) \(Ez_{t-1} \eta_t = 0\) and \(\lim_{t \to \infty} \frac{t}{t+1} = 1\)

\[
    h_\phi (\phi, S) = S^{-1}M (T(\phi) - \phi)
\]

\[
    h_S (\phi, S) = M - S
\]
The Associated ODE is then

\[ \frac{d\phi}{d\tau} = S^{-1}M(T(\phi) - \phi) \]
\[ \frac{dS}{d\tau} = M - S \]

- The latter is globally stable for any initial $S$: $S \rightarrow M$
- Therefore $S^{-1}M \rightarrow I$

Hence

\[ \frac{d\phi}{d\tau} = T(\phi) - \phi \]

- Intuition?
Summary

- Have motivated the concept of E-Stability
  - Governed by eigenvalues of the associated ODE

- Remainder
  - Microfoundations and modeling issues
  - Subsequent lectures: Applications in policy design
Modeling Learning Dynamics: Some Issues

- Intertemporal optimization implies multi-period forecasts
  - Fundamental to macroeconomic analysis

- Transversality conditions central to this theory
  - Partial Equilibrium: Marcet and Sargent (1989)
  - General Equilibrium: Preston (2005)
  - Yet most of the macro learning literature ignores this requirement of optimality
Beliefs

Under rational expectations:

1. Agents optimize given beliefs

2. The probabilities they assign to future state variables coincide with the predictions of the model

This paper retains (1) and replaces (2) with

2’. Future state variables outside agent’s control are forecasted using an econometric model.
Knowledge

- Know own preferences and constraints

- Do not know true economic model of determination of variables outside their control.
  - Uncertain about the equilibrium mechanisms determining prices and policy variables
  - E.g. even if $Y_t^i = Y_t^j = Y_t$ in EQ $\not\Rightarrow$ agents know $Y_t^i = Y_t^j$

- Observe aggregate variables and disturbances

- Forecast variables outside their control using an atheoretical VAR
  - Takes the minimum state variable form
Townsend Investment Problem

- Consider a representative firm solving

$$\max_{\{k_T\}} \mathbb{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[ p_T \gamma k_T - \frac{\delta}{2} (k_T - k_{T-1})^2 \right]$$

where $0 < \beta < 1$ and $\delta, \gamma > 0$. The price of output, $\gamma k_t$, is determined on competitive markets according to

$$p_t = -\gamma K_t + u_t$$
$$u_t = \rho u_{t-1} + \varepsilon_t$$

where $0 < \rho < 1$ and $K_t$ the aggregate capital stock. Equilibrium requires:

$$k_t = K_t = \int_{0}^{1} k_t(i) \, di$$

- Assumption: firms do not know market mechanism determining prices
Townsend Investment Problem II

• To solve this problem a firm requires forecasts of future capital prices and state variables
  
  – Prices are exogenous to their decision problem
  
  – As are the aggregate capital stock and stochastic disturbance

• Denoting the set of variable beyond the firm’s control as

\[ z_t \equiv \{p_t, K_t, u_t\} \]

assume forecasting model of the form

\[ z_t = a_t + b_t z_{t-1} + \eta_t \]

  – Use data available until period \( t - 1 \)
• Forecasts are then constructed according to

\[ \hat{E}_t z_T = (I_3 - \beta)^{-1} \left( I_3 - b_{t-1}^{T-t} \right) a_{t-1} + b_{t-1}^{T-t} z_t \]

– Note that it is assumed that when making decisions over the forecast horizon beliefs are held fixed

– Formally we solve an anticipated utility problem in the sense of Kreps and Sargent

– That agents do not account for subsequent revisions in their beliefs when making decisions in the current period represents the non-rationality of agents decisions
Townsend Investment Problem III

- The first-order conditions of the firm’s problem are

\[ k_t - k_{t-1} = \beta \hat{E}_t (k_{t+1} - k_t) + \frac{\gamma}{\delta} p_t \]

and

\[ \lim_{T \to \infty} E_t \beta^T k_T = 0 \]

- Solving the first condition forward and applying the second gives the optimal investment rule

\[ k_t = k_{t-1} + \frac{\gamma}{\delta} \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} p_T \]

- This is the optimal rule conditional on arbitrary beliefs: holds under rational expectations
Rational Expectations Solution and Learning

- Straightforward to show capital is determined by

\[ k_t = \mu K_{t-1} + \frac{\gamma}{\delta (\beta \mu - \rho)} u_t \]

where \( 0 < \mu < 1 \)

- Under learning dynamics, optimal decisions converge to this allocation with probability 1
Alternative Approaches

- The predominant approach to modeling learning in macroeconomics is to commence with Euler equations implied by a rational expectations analysis.

- In the context of the Townsend problem, commence with

  \[
  k_t - k_{t-1} = \beta \hat{E}_t (k_{t+1} - k_t) + \frac{\gamma}{\delta} p_t
  \]

  as the primitive behavioral assumption.

- Requires a forecasting model

  \[
  k_t = \tilde{a}_t + \tilde{b}_t K_{t-1} + \tilde{c}_t u_{t-1} + \eta_t
  \]

  so that

  \[
  \hat{E}_t k_{t+1} = \tilde{a}_{t-1} + \tilde{b}_{t-1} K_t + \tilde{c}_{t-1} u_t
  \]
Alternative Approaches II

- This second approach is fundamentally distinct from the first. By ignoring the transversality condition agents make suboptimal decisions

- Represents a basic confusion of the economics: firms forecast their own endogenous decisions variable using an arbitrary model
  - Represents a contradiction of what agents know to be true about their decision problem
  - Forecasts of endogenous variable induces by optimal decision rules and primitive assumptions about forecasting variables beyond the control of the decision maker
• It is not that the Euler equation is irrelevant. Note that quasi differencing the optimal decision rule yields

\[
\begin{align*}
k_t &= k_{t-1} + \gamma \frac{\delta}{\delta} \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} p_T \\
&= k_{t-1} + \gamma \frac{\delta}{\delta} p_t + \gamma \frac{\delta}{\delta} \beta \hat{E}_t \hat{E}_{t+1} \sum_{T=t+1}^{\infty} \beta^{T-t} p_T \\
&= k_{t-1} + \gamma \frac{\delta}{\delta} p_t + \gamma \frac{\delta}{\delta} \beta \hat{E}_t (k_{t+1} - k_t)
\end{align*}
\]

which is the Euler equation.

• Optimality requires capital to be forecast in a particular way. Specifically

\[
\hat{E}_t k_{t+1} = k_t + \gamma \frac{\delta}{\delta} \hat{E}_t \sum_{T=t+1}^{\infty} \beta^{T-t} p_T
\]

– In general this will not be equal to

\[
\hat{E}_t k_{t+1} = \tilde{a}_t + \tilde{b}_t K_t + \tilde{c}_t u_t
\]
Model Agents

- Households
- Firms
- Monetary authority
- Fiscal authority
Model Features

- Money or cashless limit
- Monopolistic competition/nominal rigidities
- Incomplete asset markets
- No capital
- Non-rational expectations
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Knowledge

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- Do not know true economic model of determination of variables outside their control.
  - E.g. even if $Y_t^i = Y_t^j = Y_t$ in EQ $\Rightarrow$ agents know $Y_t^i = Y_t^j$

- Observe aggregate variables and disturbances

- Forecast variables outside their control using an atheoretical VAR
  - Takes the minimum state variable form
Household Problem

- Households seek to maximize

\[ \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[ U \left( C_T^i; \xi_T \right) - v \left( h_T^i; \xi_T \right) \right] \]

subject to

\[ B_t^i = (1 + i_{t-1}) B_{t-1}^i + w_t h_t^i + \int_0^1 \Pi_t \left( j \right) dj - T_t - P_t C_t^i \]
\[ = (1 + i_{t-1}) B_{t-1}^i + P Y_t^i - T_t - P_t C_t^i \]

- Beliefs: a-theoretical VAR of exogenous variables — nests MSV REE

- Assume agents forecast period income
First-order conditions

- Log-linear approximation yields

\[ \hat{C}_t^i = \hat{E}_t^i \hat{C}_{t+1}^i - \sigma \left( i_t - \hat{E}_t^i \pi_{t+1} - g_t + \hat{E}_t^i g_{t+1} \right) \]

and

\[ \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \hat{C}_T^i = A_t^i + \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \hat{Y}_T^i \]

where

\[ A_t^i = B_t^i / P_t \tilde{Y}; \quad g_t = \xi_t u_c \xi / u_c; \quad \sigma^{-1} = -u_{cc} \tilde{Y} / u_c \]

- Assume government debt in zero net supply
Deriving the Consumption Decision Rule

- Solving the Euler equation backwards recursively from date $T$ to date $t$ and taking expectations at that time gives

$$\hat{E}_t^i \hat{C}_T^i = \hat{C}_t^i - g_t + \hat{E}_t^i \left[ g_T + \sigma \sum_{T=t}^{T-1} (\hat{\pi}_t - \hat{\pi}_{t+1}) \right]$$

- Use this to eliminate future expected consumption in the intertemporal budget constraint
Optimal decision rule

- First-order conditions imply

\[
\hat{C}_t^i = (1 - \beta) A_t^i + \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left( (1 - \beta) \hat{Y}_T^i - \beta \sigma (\hat{\nu}_T - \pi_{T+1}) + \beta (g_T - g_{T+1}) \right)
\]

- This is an example of permanent income theory

- Forecasts about prices into the indefinite future matter!

- A particularly important source of uncertainty are anticipated future interest rates
Comparison to Existing Approaches

- Are decisions implied by the optimal decision equivalent to those implied by

\[ \hat{C}_t^i = \hat{E}_t^i \hat{C}_{t+1}^i - \sigma (i_t - \hat{E}_t^i \pi_{t+1} - g_t + \hat{E}_t g_{t+1}) \]

- Where forecasts determined by the model

\[ z_t^* = a_{t-1} + b_{t-1} z_{t-1} + \varepsilon_t \]

with \( z_t^* = [\hat{C}_t^i \pi_t i_t g_t]' \) and \( z_t = [\hat{Y}_t \pi_t i_t g_t]' \)
Comparison to Existing Approaches II

- Are decisions implied by the optimal decision equivalent to those implied by

\[ \hat{C}_t^i = \hat{E}_t^i \hat{C}_{t+1}^i - \sigma \left( i_t - \hat{E}_t^i \pi_{t+1} - g_t + \hat{E}_t g_{t+1} \right) \]

- Where forecasts determined by the model

\[ z_t^* = a_{t-1} + b_{t-1} z_{t-1} + \varepsilon_t \]

with \( z_t^* = [\hat{C}_t^i \; \pi_t \; i_t \; g_t]' \) and \( z_t = [\hat{Y}_t \; \pi_t \; i_t \; g_t]' \)

- In general - No!

- Optimal decision rule if expectations treated correctly

  - i.e. the conditional expectation of consumption induced by model primitives and optimality
Forecasting Consumption

- The optimal forecast

\[ \hat{E}_t^i \hat{C}^i_{t+1} = (1 - \beta) \hat{E}_t^i A_{t+1}^i + \]

\[ \hat{E}_t^i \sum_{T=t+1}^{\infty} \beta^{T-t} \left[ (1 - \beta) \hat{Y}_T^i - \beta \sigma (\hat{\nu}_T - \pi_{T+1}) + \beta (g_T - g_{T+1}) \right] \]

- Not the same as

\[ \hat{E}_t^i z^*_{t+1} = a_{t-1} + b_{t-1} z_t \]

Conditional expectations of endogenous variables induced by policy function and forecasts about variable the are exogenous to the agent's decision problem
• Market clearing conditions are part of the REE

• One-period-expectations approach does not deliver optimal decision rule
  – Confuses endogenous decision variables and exogenous state variables
  – The correct distribution with respect to which expectations are taken is induced from the decision problem and properties of exogenous variables
  – In general intertemporal budget constraint will be violated in this formulation
    * Expected future consumption plans are inconsistent with what agents know to be true about their own future behavior given their own decision problem
Aggregate Consumption Dynamics

- Aggregating over the continuum provides

\[ \hat{C}_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[ (1 - \beta) \hat{Y}_T - \beta \sigma (\hat{v}_T - \pi_{T+1}) + \beta (g_T - g_{T+1}) \right] \]

where

\[ \hat{C}_t \equiv \int \hat{C}_t^i di; \quad \hat{E}_t \equiv \int \hat{E}_t^i di; \quad 0 \equiv \int A_t^i di \]

- For any variable \( X_t \), \( \hat{E}_t \hat{E}_{t+1} X_{t+1} \neq \hat{E}_t X_{t+1} \)
Summary

- Under rational expectations equilibrium probability laws by construction satisfy all relevant constraints
  - In particular: Transversality conditions

- Under learning dynamics expectations about future endogenous decision variables are taken with respect to a distribution
  - Induced by beliefs about exogenous states
  - Induced by optimal decisions
Firms

- Continuum of firms maximize

\[ \hat{E}_t^j \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left[ \Pi_T^j (p_t(j)) \right] \]

where

\[ \Pi_T^j (p) = Y_T P_T^{\theta} p^{1-\theta} - w_T f^{-1} \left( Y_T P_T^{\theta} p^{-\theta} / A_T \right) \]

\[ y_t^i = A_t f \left( h_t^i \right) \]
Optimal Price Setting

- Log-linear approximation implies

\[
p^j_t = \hat{E}^j_t \sum_{T=t}^{\infty} (\alpha \beta)^{T-t} \left[ \frac{(1 - \alpha \beta) \left( \omega + \sigma^{-1} \right)}{1 + \omega \theta} x_T + \pi_T \right]
\]

- Infinite horizon forecasts matter

- \( x_t = Y_t - \hat{Y}^n_t \) is the output gap
Model Summary

- Aggregate demand and supply:

\[ x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[ (1 - \beta) x_{T+1} - \sigma (i_T - \pi_{T+1} - r_T) \right] \]

\[ \pi_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha \beta)^{T-t} \left[ \kappa \alpha \beta x_{T+1} + (1 - \alpha) \beta \pi_{T+1} + u_t \right] \]

where

\[ r_t = \rho r_{t-1} + \varepsilon_t \text{ and } 0 < \rho < 1 \]

\[ u_t = \gamma u_{t-1} + \eta_t \text{ and } 0 < \gamma < 1 \]
Under this assumption $\hat{E}_t \hat{E}_{t+1} X_{t+1} = \hat{E}_t X_{t+1}$. Hence

$$x_t = E_t [(1 - \beta) x_{t+1} - \sigma (i_t - \pi_{t+1} - r_t)] + \hat{E}_t \sum_{T=t+1}^{\infty} \beta^{T-t} [(1 - \beta) x_{T+1} - \sigma (i_T - \pi_{T+1} - r_T)]$$

$$= E_t [(1 - \beta) x_{T+1} - \sigma (i_t - \pi_{t+1} - r_t)] + \beta E_t x_{t+1}$$

$$= E_t x_{t+1} - \sigma (i_t - E_t \pi_{t+1} - r_t)$$
• Rational expectations analysis gives

\[ x_t = E_t x_{t+1} - \sigma (i_t - E_t \pi_{t+1} - r_t) \]

\[ \pi_t = k x_t + \beta E_t \pi_{t+1} \]

• Bullard and Mitra (2002, JME) and Evans and Honkapohja (2003, ReStud)

\[ x_t = \hat{E}_t x_{t+1} - \sigma (i_t - \hat{E}_t \pi_{t+1} - r_t) \]

\[ \pi_t = k x_t + \beta \hat{E}_t \pi_{t+1} \]
Does any of this matter?
Objectives

- Implications of Learning for the choice of monetary policy rule

- The role of communication in policy design
Benchmark Model: Optimal Decisions

- Aggregate demand and supply:

\[ x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[ (1 - \beta)x_{T+1} - \sigma (i_T - \pi_{T+1} - r_T) \right] \]

\[ \pi_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha \beta)^{T-t} \left[ \kappa \alpha \beta x_{T+1} + (1 - \alpha) \beta \pi_{T+1} + u_t \right] \]

where

\[ r_t = \rho r_{t-1} + \varepsilon_t \]

\[ u_t = \gamma u_{t-1} + \eta_t \]

are known exogenous processes with \( 0 < \rho, \gamma < 1 \) and \( \varepsilon_t, \eta_t \) iid disturbances

- Implicitly assumes Ricardian fiscal policy

  - Agents need not forecast debt or taxes
Alternative Model: “One-Period-Ahead Forecasts”

- Interested in comparing policy emerging under the anticipated utility approach to the more familiar approach given by

\[ x_t = \hat{E}_t x_{t+1} - \sigma \left( i_t - \hat{E}_t \pi_{t+1} - r_t \right) \]

\[ \pi_t = k x_t + \beta \hat{E}_t \pi_{t+1} \]

- Bullard and Mitra (JME, 2002), Evans and Honkapohja (ReStud, 2003)
Issues in Policy Design

• Early literature concerned with policy robustness and equilibrium selection
  – Given that there are a multiplicity of rules consistent with implementing a given equilibrium, can impose additional desiderata
  – Robustness to small expectational errors

• Models with learning also permit addressing critiques of policy
  – Friedman (1968) — Monetary policy procedures based on interest-rate rules prone to Wicksellian cumulative process
An Interest-Rate Peg: Optimal Decisions

- Consider the policy $i_t = \phi_r r_t$ where $r_t$ is iid and no $u_t$
  
  - Indeterminancy of rational expectations equilibrium: Sargent and Wallace (1975)

- Agent forecasts given by

  $$\hat{E}_t z_T = a_{t-1}$$

  for $z_t = [\pi_t \, x_t \, i_t]'$. Combined with structural model gives mapping

  $$\frac{da}{d\tau} = T(a) - a$$
  $$\frac{db}{d\tau} = T(b) - b$$

  - E-Stability determined by the eigenvalues of the Jacobian of these ordinary differential equation
Conditions on Eigenvalues

- Consider the matrix $A$ with dimension $(3 \times 3)$. From $|A - \lambda I| = 0$ the characteristic equation is

$$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$$

where $c_1 = \text{Trace}(A)$, $c_2$ is the sum of all second-order principal minors of $A$ and $c_3 = |A|$. The following restrictions on the coefficients $c_i$ must be satisfied for all eigenvalues to have negative real parts:

$$c_1 < 0$$

$$c_3 - c_1 c_2 > 0$$

$$c_3 < 0.$$
For a matrix $A$ with dimension $(2 \times 2)$, $|A - \lambda I| = 0$ implies the characteristic equation is

$$\lambda^2 - c_1 \lambda + c_2 = 0$$

where $c_1 = \text{Trace}(A)$ and $c_2 = |A|$. For both eigenvalues to have negative real parts $c_1 < 0$ and $c_2 > 0$ must be satisfied.
Stability Analysis

• Straightforward to show

\[
T'(a) = \begin{bmatrix}
\frac{(1-\alpha)\beta}{1-\alpha\beta} + \frac{\kappa\sigma}{1-\beta} & \frac{\kappa\alpha\beta}{1-\alpha\beta} + \kappa & -\frac{\kappa\alpha\beta}{1-\alpha\beta} \\
\frac{\sigma}{1-\beta} & 1 & \frac{\beta}{1-\beta} \\
0 & 0 & 0
\end{bmatrix}
\]

and

\[
\det(T'(a) - I) = \frac{\kappa\sigma}{1-\beta} + \frac{\alpha\beta\kappa\sigma}{(1-\beta)(1-\alpha\beta)} > 0
\]

– Expectations driven fluctuations

An Interest Rate Peg: One-Period-Ahead Expectations

- Consider the policy $i_t = \phi_r r_t$ with $r_t$ i.i.d.

- Agent forecasts given by
  \[ \hat{E}_t z_{t+1} = a_{t-1} \]
  for $z_t = [\pi_t \ x_t]'$. Combined with structural model gives mapping
  \[
  \frac{da}{d\tau} = T(a) - a \\
  \frac{db}{d\tau} = T(b) - b
  \]
An Interest Rate Peg II

- The true data generating process is

\[
\begin{bmatrix}
    x_t \\
    \pi_t
\end{bmatrix} = \begin{bmatrix}
    1 & \sigma \\
    \kappa & \kappa (\sigma + \beta)
\end{bmatrix} \begin{bmatrix}
    a_{x,t-1} \\
    a_{\pi,t-1}
\end{bmatrix} - \begin{bmatrix}
    \sigma (\phi_r - 1) \\
    \kappa \sigma (\phi_r - 1)
\end{bmatrix} r_t
\]

which imply forecasts

\[
E_t \begin{bmatrix}
    x_{t+1} \\
    \pi_{t+1}
\end{bmatrix} = \begin{bmatrix}
    1 & \sigma \\
    \kappa & \kappa (\sigma + \beta)
\end{bmatrix} \begin{bmatrix}
    a_{x,t-1} \\
    a_{\pi,t-1}
\end{bmatrix}
\]
• Defining the T-map

\[
T \left( \begin{array}{c} a_x \\ a_\pi \end{array} \right) = \begin{bmatrix} 1 & \sigma \\ \kappa & \kappa (\sigma + \beta) \end{bmatrix} \begin{bmatrix} a_x \\ a_\pi \end{bmatrix}
\]

• Expectations stability requires the real parts of the eigenvalues of \( T' (a) - I \) to be negative

  – That is: \( \det (T' (a) - I) > 0 \) and \( \text{trace}(T' (a) - I) < 0 \)

• The determinant is

\[
\det \left( T' (a) - I \right) = -\kappa \beta < 0
\]

  – The minimum-state-variable solution is not learnable
Taylor Rule

- McCallum (1983): conditioning on endogenous variables resolves indeterminacy

- Consider the rule

\[ i_t = \phi_\pi \pi_t + \phi_x x_t \]

- Requires

\[ \kappa (\phi_\pi - 1) + (1 - \beta) \phi_x > 1 \]

for determinacy under REE and stability under learning

Emerging differences

• Policy actions that depend on misspecified beliefs can lead to self-fulfilling expectations

• Investigate a range of policies that are consistent with implementing the optimal commitment equilibrium under rational expectations
  
  – Implementation of such policy involve a dependence of interest-rate policy on expectations
  
  – This will reveal some sharp differences across the anticipated utility approach and the one-period-ahead expectations approach
Central bank is assumed to minimize the loss function

\[ E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left( \pi_t^2 + \lambda x_t^2 \right) \]

where \( 0 < \beta < 1 \) for some weight \( \lambda > 0 \) subject to

\[ \pi_t = \kappa x_t + \beta E_t \pi_{t+1} + u_t \]

\[ \pi_{t_0} = \bar{\pi}_{t_0} = (1 - \mu) \frac{\lambda}{\kappa} x_{t-1} + \frac{\mu}{1 - \beta \mu \gamma} u_t \]

- Optimality from the “timeless perspective”: Woodford (2003, chap. 6) and Giannoni and Woodford (2002, 2012)
Optimal Commitment Solution

- Standard methods show that the optimal state-contingent paths of \( \{\pi_t, x_t, i_t\} \) are given by the following relations for all \( t \geq t_0 \):

\[
\pi_t = (1 - \mu) \frac{\lambda}{\kappa} x_{t-1} + \frac{\mu}{1 - \beta \mu \gamma} u_t
\]

\[
x_t = \mu x_{t-1} - \frac{\kappa}{\lambda} \frac{\mu}{1 - \beta \mu \gamma} u_t
\]

and

\[
i_t = \frac{\sigma \lambda - \kappa}{\sigma \kappa} (1 - \mu) \mu \cdot x_{t-1} + \frac{\sigma \lambda - \kappa}{\sigma \lambda} \cdot \frac{\mu (\mu + \gamma - 1)}{1 - \beta \mu \gamma} \cdot u_t + \frac{1}{\sigma} r^n_t
\]

where \( 0 < \mu < 1 \) is the model’s only eigenvalue within the unit circle.
Optimal Commitment Solution II

- Useful to represent this solution in terms of a particular exogenous state variable

- Solving the output gap relation backwards recursively gives

\[ x_t = -\frac{\kappa}{\lambda} \cdot \frac{\mu}{1 - \beta \mu \gamma} \sum_{j=0}^{\infty} \mu^j u_{t-j}. \]

- Defining the exogenous state variable

\[ \Lambda_t \equiv \sum_{j=0}^{\infty} \mu^j u_{t-j}, \]

which satisfies the process \( \Lambda_t = \mu \Lambda_{t-1} + u_t \)
• Allows the optimal solution to be written as

\[
\pi_t^* = \frac{\mu}{1 - \beta \mu \gamma} [u_t - (1 - \mu) \Lambda_{t-1}]
\]

\[
x_t^* = -\frac{\kappa}{\lambda} \cdot \frac{\mu}{1 - \beta \mu \gamma} [\mu \Lambda_{t-1} + u_t]
\]

\[
i_t^* = \frac{\sigma \lambda - \kappa}{\sigma \lambda} \cdot \frac{\mu}{1 - \beta \mu \gamma} [\gamma u_t - (1 - \mu) (\mu \Lambda_{t-1} + u_t)] + \frac{1}{\sigma} r_t^n
\]

where * denotes the optimal solution when expressed as a linear function of

\[\{\Lambda_{t-1}, u_t, r_t\}\].

– Note the question of what policy implements the optimal commitment equilibrium is yet to be answered

– We have only described what that equilibrium looks like
Candidate Policy Rules

- Consider the following instrument rule as a means to implement the optimal equilibrium:

\[ i_t = i_t^* + \psi_x (E_t^{cb} x_{t+1} - E_t x_{t+1}^*) + \psi_\pi (E_t^{cb} \pi_{t+1} - E_t \pi_{t+1}^*) \]

\[ = \tilde{i}_t^f + \psi_x E_t^{cb} x_{t+1} + \psi_\pi E_t^{cb} \pi_{t+1} \]

where \( E_t^{cb} \) denotes the forecasts that the central bank intends to respond to and

\[ \tilde{i}_t^l = i_t^* - \psi_x E_t x_{t+1}^* - \psi_\pi E_t \pi_{t+1}^* \]

collects exogenous terms and \( E_t x_{t+1}^* \) and \( E_t \pi_{t+1}^* \)

- Recall \( \{\pi_t^*, x_t^*, i_t^*\} \) denote the optimal paths for each endogenous variable in the optimal commitment equilibrium when written as a linear function of the exogenous state variables \( \{A_{t-1}, u_t, r_t\} \).
Candidate Policy Rules II

- Expectations based rules argued to be desirable and given empirical support

- Assume $E_{t}^{cb} = \hat{E}_{t}$ where the forecasts are described below

- The policy parameters $\psi_{\pi}, \psi_{x}$ determine the response to deviations of the actual path of the inflation rate and output gap expectations from that path consistent with the optimal equilibrium

  - In the desired equilibrium it is clear that $E_{t}^{cb}\pi_{t+1} = E_{t}\pi^{*}_{t+1}, E_{t}^{cb}x_{t+1} = E_{t}x^{*}_{t+1}$ and $i_{t} = i^{*}_{t}$
Forecasting

- Agents therefore estimate the linear model

\[ z_t = a_t + b_t \cdot z_{t-1} + c_t \cdot u_t + d_t \cdot r_t + \varepsilon_t \]  \hspace{1cm} (6)

where \( z_t = (\pi_t, x_t, i_t, \Lambda_t)' \), \( \varepsilon_t \) is the usual error-vector term, \( \{a_t, b_t, c_t, d_t\} \) are parameters to be estimated of the form

\[
\begin{align*}
    a_t &= \begin{bmatrix} a_{\pi,t} \\ a_{x,t} \\ a_{i,t} \\ a_{\Lambda,t} \end{bmatrix} ;
    c_t &= \begin{bmatrix} c_{\pi,t} \\ c_{x,t} \\ c_{i,t} \\ c_{\Lambda,t} \end{bmatrix} ;
    d_t &= \begin{bmatrix} d_{\pi,t} \\ d_{x,t} \\ d_{i,t} \\ d_{\Lambda,t} \end{bmatrix}
\end{align*}
\]

and

\[
    b_t \equiv \begin{bmatrix} 0 & 0 & 0 & b_{\pi,t} \\ 0 & 0 & 0 & b_{x,t} \\ 0 & 0 & 0 & b_{i,t} \\ 0 & 0 & 0 & b_{\Lambda,t} \end{bmatrix}
\]
• Believes revised according to recursive least squares formulation

\[
\phi_t = \phi_{t-1} + t^{-1}R^{-1}_t w_{t-1}(z_{t-1} - w'_{t-1}\phi_{t-1}) \\
R_t = R_{t-1} + t^{-1}(w_{t-1}w'_{t-1} - R_{t-1})
\]

where \( \phi_t = (a_t, b_{4t}, c_t, d_t) \) and \( w_t \equiv \{1, \Lambda_{t-1}, u_t, r_t\}' \).

• Given homogeneity of beliefs, average forecasts can then be constructed by solving (9) backward and taking expectations to give

\[
\hat{E}_t z_T = (I_4 - b_{t-1})^{-1} (I_4 - b_{t-1}^{T-t})a_{t-1} + b_{t-1}^{T-t} z_t \\
+ \gamma u_t (\gamma I_4 - b_{t-1})^{-1} (\gamma^{T-t} I_4 - b_{t-1}^{T-t}) c_t \\
+ \rho r_t (\rho I_4 - b_{t-1})^{-1} (\rho^{T-t} I_4 - b_{t-1}^{T-t}) d_t
\]

for \( T \geq t \), where \( I_4 \) is a \((4 \times 4)\) identity matrix.
Properties

• A unique bounded rational expectations equilibrium will obtain iff

\[ 0 < \kappa (\psi_\pi - 1) + (1 - \beta) \psi_x < 2\sigma (1 + \beta) \]

• Under learning a necessary condition for E-Stability is

\[ \psi_\pi + \frac{\psi_x}{\kappa} > \frac{1}{1 - \beta} - \frac{2 - \beta - \alpha \beta}{\kappa \sigma (1 - \alpha \beta)} . \]

  – Clearly violates the Taylor principle; consider the limit case of \( \beta \to 1 \)

  – Difficulty: by having central bank behavior depend on “misspecified” private beliefs makes self-fulfilling expectations more likely

    • Note \( \beta \) central regulator of how shifting expectations impact current outcomes
Properties II

- What about the one-period-ahead expectations models?

- Can show that under the same rule expectation stability obtains if and only if

\[ \kappa (\psi_\pi - 1) + (1 - \beta) \psi_x > 1 \]

- That is — the Taylor principle holds

- Fundamentally different predictions: arises from the one-period-ahead expectations approach gutting the model of economics. For example, expectations of future interest are irrelevant to dynamics in that approach.
Raises important questions

- What is the appropriate use of forecasts in policy design?

- Can instability be mitigated?
  - Optimal policy/targeting rules imply a particular kind of dependency of interest rate decisions on forecasts
  - The role of communication
  - The role of debt policy

- Different statistical properties
  - Forecasts, dynamics and policy evaluation
Optimal Internal Forecasts

- Consider a sophisticated central bank that understands private-sector expectations formation.

- Substituting the private forecasts into the structural relations implies the output gap and inflation to be given by

\[
\pi_t = \bar{a}_\pi + \bar{b}_\pi \Lambda_{t-1} + \bar{c}_\pi u_t + \bar{d}_\pi r_t + \bar{e}_\pi i_t
\]  
\[x_t = \bar{a}_x + \bar{b}_x \Lambda_{t-1} + \bar{c}_x u_t + \bar{d}_x r_t + \bar{e}_x i_t\]  

where the coefficients \((\bar{a}_i, \bar{b}_i, \bar{c}_i, \bar{d}_i, \bar{e}_i)\) for \(i \in \{\pi, x\}\) are functions of the coefficients \((a_t, b_t, c_t, d_t)\).

- These equations give the period \(t\) inflation rate and output gap realizations, conditional on the current choice of the interest rate.
Can then compute “optimal” forecasts as

\[ E_t^{cb} \pi_{t+1} = a_\pi + b_\pi \Lambda_t + c_\pi \gamma u_t + d_\pi \rho r_t + \bar{e}_\pi E_t^{cb} i_{t+1} \]

\[ E_t^{cb} x_{t+1} = a_x + b_x \Lambda_t + c_x \gamma u_t + d_x \rho r_t + \bar{e}_x E_t^{cb} i_{t+1}. \]

Substitution of these relations into

\[ i_t = \bar{i}_t^f + \psi_x E_t^{cb} x_{t+1} + \psi_\pi E_t^{cb} \pi_{t+1} \]

yields an equation of the form

\[ i_t = (\psi_\pi \bar{a}_\pi + \psi_x \bar{a}_x) + (\psi_\pi \bar{b}_\pi + \psi_x \bar{b}_x) \Lambda_t + (\psi_\pi \bar{c}_\pi + \psi_x \bar{c}_x) \gamma u_t + (\psi_\pi \bar{d}_\pi + \psi_x \bar{d}_x) \rho r_t + (\psi_\pi \bar{e}_\pi + \psi_x \bar{e}_x) E_t^{cb} i_{t+1} \]

- Has unique bounded solution if \(|\psi_\pi \bar{e}_\pi + \psi_x \bar{e}_x| < 1\)
- Determines an interest-rate decision that is a function of agents’ learning behavior
Optimal Internal Forecasts II

- If the monetary authority constructs optimal internal forecasts then the Taylor principle

\[ \kappa (\psi_\pi - 1) + (1 - \beta) \psi_x > 0 \]

is necessary and sufficient for stability under learning dynamics.

- Underscores an important point: understanding the details/mechanics of expectations formation critical

- Policy needs to depend on expectations in a very specific way to ensure stability
Implementing Optimal Policy using Target Criteria

- Recent monetary economics literature has proposed implementing policy using target criteria
  - See, for example, Svensson and Woodford (2005), Svensson (2003), Giannoni and Woodford (2002, 2012) and Woodford (2003, chp. 7)

- The approach specifies a criterion that must be checked each time an interest-rate decision is made
  - For example: Bank of England’s inflation forecast targeting
  - Requires assumption about what model the central bank has when implementing policy
  - The approach gives very specific guidance on how forecasts should be incorporated in policy decisions
Recall the optimal commitment equilibrium implies the dynamics

\[ \pi_t = (1 - \mu) \frac{\lambda}{\kappa} x_{t-1} + \frac{\mu}{1 - \beta \mu \gamma} u_t \]

\[ x_t = \mu x_{t-1} - \frac{\kappa}{\lambda} \cdot \frac{\mu}{1 - \beta \mu \gamma} u_t \]

\[ i_t = \frac{\sigma \lambda - \kappa}{\sigma \kappa} (1 - \mu) \mu \cdot x_{t-1} + \frac{\sigma \lambda - \kappa}{\sigma \lambda} \cdot \frac{\mu (\mu + \gamma - 1)}{1 - \beta \mu \gamma} \cdot u_t + \frac{1}{\sigma} r^\alpha_t \]
Deriving the Target Criterion II

- Giannoni and Woodford (2002) and Woodford (2003, chap 7) show

\[ \pi_t = -\frac{\lambda}{\kappa} (x_t - x_{t-1}) \]

fully characterizes the optimal equilibrium from the timeless perspective in the sense that the previous dynamics hold for all \( t \geq t_0 \) if and only if the above target criterion holds.

- Take this as the desired target criterion: under rational expectations implies determinacy for all parameter values.

- It is a restriction on endogenous variables — central bank requires a model to implement.

- History dependence of optimal commitment captured by the lagged output gap.
Forecasts

- Agents therefore estimate the linear model

\[ z_t = a_t + b_t \cdot z_{t-1} + c_t \cdot u_t + d_t \cdot r_t + \epsilon_t \]  \hspace{1cm} (9)

where \( z_t = (\pi_t, x_t, i_t)' \), \( \epsilon_t \) is the usual error-vector term, \( \{a_t, b_t, c_t, d_t\} \) are parameters to be estimated of the form

\[
\begin{bmatrix}
a_{\pi,t} \\
a_{x,t} \\
a_{i,t}
\end{bmatrix};
\begin{bmatrix}
c_{\pi,t} \\
c_{x,t} \\
c_{i,t}
\end{bmatrix};
\begin{bmatrix}
d_{\pi,t} \\
d_{x,t} \\
d_{i,t}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
0 & b_{\pi,t} & 0 \\
0 & b_{x,t} & 0 \\
0 & b_{i,t} & 0
\end{bmatrix}
\]
Believes revised according to recursive least squares formulation

\[ \phi_t = \phi_{t-1} + t^{-1} R_{t-1} w_{t-1} (z_{t-1} - w'_{t-1} \phi_{t-1}) \]

\[ R_t = R_{t-1} + t^{-1} (w_{t-1} w'_{t-1} - R_{t-1}) \]

where \( \phi_t = (a_t, b_{2t}, c_t, d_t) \) and \( w_t \equiv \{1, x_{t-1}, u_t, r_t\}' \).

Given homogeneity of beliefs, average forecasts can then be constructed by solving (9) backward and taking expectations to give

\[ \hat{E}_t z_T = (I_3 - b_{t-1})^{-1} (I_3 - b_{t-1}^{T-t}) a_{t-1} + b_{t-1}^{T-t} z_t \]

\[ + \gamma u_t (\gamma I_3 - b_{t-1})^{-1} (\gamma^{T-t} I_3 - b_{t-1}^{T-t}) c_t \]

\[ + \rho r_t (\rho I_3 - b_{t-1})^{-1} (\rho^{T-t} I_3 - b_{t-1}^{T-t}) d_t \]

for \( T \geq t \), where \( I_3 \) is a \((3 \times 3)\) identity matrix.
The Central Bank’s Model

- A number of assumptions could be made. The central bank could:
  - Make projections based on minimum-state-variable rational expectations equilibrium
  - Make projections based on the structural equations implied by the learning model
  - Make projections based on the assumption that the Euler equations implied by rational expectations are valid.
Projecting under Rational Expectations

- Suppose the central bank mistakenly assumes agents have rational expectations

- If it projects the evolution of the economy assuming the minimum-state-variable equilibrium the target criterion will be satisfied under the interest-rate policy

\[ i_t = \frac{\sigma \lambda - \kappa}{\sigma \kappa} (1 - \mu) \mu \cdot x_{t-1} + \frac{\sigma \lambda - \kappa}{\sigma \lambda} \cdot \frac{\mu (\mu + \gamma - 1)}{1 - \beta \mu \gamma} \cdot u_t + \frac{1}{\sigma r^*_t} \]

- This is not an exogenous instrument rule — depends on lagged output gap

- Concerns of Sargent and Wallace (1975) need not be valid

- Result: \( \theta > \sigma \) sufficient for instability under learning
Projecting under the True Model

- Suppose the central bank understands the true structural model — that is, the aggregate demand and supply equations, and the precise details of agents belief formation.

- The central bank can then guarantee satisfaction of the target criterion by setting interest rates according to

\[
i_t = -\frac{1}{\sigma} \cdot \hat{x}_t + \frac{1}{\sigma} \cdot \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} [(1 - \beta)x_{T+1} - \sigma(\beta \cdot i_{T+1} - \pi_{T+1})] + r_T
\]

where

\[
\hat{x}_t = \frac{\lambda}{\lambda + \kappa^2} \cdot x_{t-1} - \frac{\kappa}{\lambda + \kappa^2} \cdot \hat{E}_t \sum_{T=t}^{\infty} (\alpha \beta)^{T-t} [\kappa \alpha \beta \cdot x_{T+1} + (1 - \alpha) \beta \cdot \pi_{T+1} + u_T]
\]

- Second relation is the value of the output gap that jointly satisfied the aggregate supply curve and the target criterion.
• This approach to implementing policy guarantees satisfaction of the target criterion regardless of the nature of expectations formation
  
  – Requires an interest-rate setting that depends upon expectations into the indefinite future

  – Is consistent with implementing the optimal commitment equilibrium if agents have rational expectations

• Under learning dynamics implies the underlying rational expectations equilibrium is E-Stable for all parameter values
Projecting using Euler Equations

- Suppose the central bank thinks the true model is

\[
x_t = \hat{E}_t x_{t+1} - \sigma \left( i_t - \hat{E}_t \pi_{t+1} - r_t \right)
\]

\[
\pi_t = k x_t + \beta \hat{E}_t \pi_{t+1}
\]

- Then the target criterion implies the instrument rule

\[
i_t = \frac{1}{\sigma} \left[ \hat{E}_t x_{t+1} - \frac{\lambda}{\lambda + \kappa^2} x_{t-1} + \left( \frac{\beta \kappa}{\lambda + \kappa^2} + \sigma \right) \hat{E}_t \pi_{t+1} + \frac{\kappa}{\lambda + \kappa^2} u_t + r_t^n \right]
\]

- What are the stability properties?
Final Remark on Target Criteria

- Under rational expectations the target criterion

\[ p_t = -\frac{\lambda}{\kappa} x_t \]

implies the same state-contingent responses to disturbances as

\[ \pi_t = -\frac{\lambda}{\kappa} (x_t - x_{t-1}) \]

- Under learning they imply different outcomes — that is, out of rational expectations equilibrium they imply different policy responses when the central bank finds its target criterion is not met because expectations evolved in a way that was not anticipated

  - It turns out the price-level targeting is more conducive to stability
Communication as Stabilization Policy

- The discussion so far emphasizes that policy must be conditioned on expectations in certain ways

- An important source of instability is that agents do not know how interest rates are determined
  - This is one of the many equilibrium restrictions that they are attempting to learn
  - Agents beliefs need not be consistent with monetary policy strategy; coherent statement of “unanchored” expectations
  - But what if agents understood some details of policy? Does this improve the stability properties of forecast based instrument rules?
Communication as Stabilization Policy II

• Suppose the central bank declares that its interest rate decisions are determined by

\[ i_t = \psi_\pi E_t \pi_{t+1} \]

• Agents can use this information to restrict their forecasting model. Specifically, policy-consistent forecasts of the nominal interest rate can be determined as

\[ \hat{E}_t i_T = \psi_\pi \hat{E}_t \pi_{T+1} \]

  – Such inflation and interest-rate forecasts are consistent with monetary policy strategy

  – They ensure projections satisfy the Taylor principle
Communication as Stabilization Policy III

- Policy consistent forecasts imply aggregate demand is

\[ x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[ (1 - \beta) x_{T+1} - \sigma \left( \psi \pi \hat{E}_t \pi_{T+1} - \pi_{T+1} - r_T \right) \right] \]

- Agents need only forecast inflation and the output gap

- Can show that E-Stability obtains so long as the Taylor principle holds

- Communication about future policy intentions can resolve important uncertainty

  * Taken up in detail by Eusepi and Preston (2010, AEJM)

  * Now consider simpler analysis by Orphanides and Williams (2005)
Part III: Monetary and Fiscal Interactions

- The analysis so far has pushed fiscal policy to the back ground

- The remainder of the lecture will consider the consequences of debt management policy for the implementation of monetary policy
  - Interested in the role of scale and composition of the public debt
  - What kind of constraints do fiscal policy impose on monetary policy?
Motivation

- US Financial Crisis witnessed dramatic change in the nature and conduct of stabilization policy
  - Large-scale asset purchases intended to influence long interest rates
  - Under what conditions can this be effective?

- Standard models satisfy an irrelevance proposition
  - Ricardian equivalence; more generally state-contingent consumption independent of government liabilities
What we do

- Simple model of output gap and inflation determination

- One informational friction:
  - Agents have an incomplete knowledge about the economy
  - Implication: departures from Ricardian Equivalence

- Explore constraints imposed on stabilization policy by choice of fiscal policy
  - Specifically: Scale and composition of Debt
  - Are there additional consequences of shortening the maturity structure of debt, over and above the presumed stimulus of lower interest rates
Literature

- Monetary and fiscal interactions:
  - Unpleasant monetary arithmetic: Sargent and Wallace (1980)
  - FTPL: Leeper, Sims and Woodford (early 90s)


- Extensive literature on stability of expectations in NK models
  - Howitt (1992), Bullard and Mitra (2002)
  - Evans and Honkapohja (2011): survey of recent literature
Model Agents

- Households
- Firms
- Monetary authority
- Fiscal authority
Beliefs

Under rational expectations:

1. Agents optimize given beliefs

2. The probabilities they assign to future state variables coincide with the predictions of the model

This paper retains (1) and replaces (2) with

2’. Future state variables outside agent’s control are forecasted using an econometric model.
Knowledge

- Know own preferences and constraints

- Do not know true economic model determining variables outside their control.
  - E.g. even if $Y_t^i = Y_t^j = Y_t$ in EQ $\not\Rightarrow$ agents know $Y_t^i = Y_t^j$

- Observe aggregate variables and disturbances

- Forecast variables outside their control using an atheoretical VAR
  - Takes the minimum state variable form
Asset Structure and the Fiscal Authority

- Exogenous purchases of $G_t$ per period

- Issue two kinds of debt
  - $B_t^s$: One period debt in zero net supply with price $P_t^s = (1 + i_t)^{-1}$
  - $B_t^m$: An asset in positive supply that has the payoff structure
    $\rho^{T-(t+1)}$ for $T \geq t + 1$

- Let $P_t^m$ denote the price of this second asset. Asset has the properties:
  - Price in period $t + 1$ of debt issued in period $t$ is $\rho P_{t+1}^m$
  - Average maturity of the debt is $(1 - \beta \rho)^{-1}$
Monetary and Fiscal Authorities

- Flow budget constraint

\[ P_t^m B_t^m = B_{t-1}^m (1 + \rho P_t^m) - P_t S_t \]

- Fiscal policy maintains intertemporal solvency (‘Passive’)

\[ \tau^i_t = i^i \left( \frac{B_{t-1}^m}{B^m} \right)^{\tau^i_l}; \tau^i_l > \tau^*_l, i \]

- Monetary policy controls inflation (‘Active’)

\[ \frac{1 + i_t}{1 + \bar{i}} = \left( \frac{\pi_t}{\pi^*} \right)^{\phi_\pi}; \phi_\pi > 1 \]

- Under rational expectations: standard view of monetary policy
Household Problem

- Saving and work decision. No capital in the model, only gov. bonds.

- Households’ preferences

\[ U(C_t, H_t) = (1 - \sigma)^{-1} \left( C_t - \frac{\psi}{1 + \gamma} C_t^u H_t^{1+\gamma} \right)^{1-\sigma}, \quad \gamma > 0 \]

where

- \( \iota = 0 \rightarrow GHH; \quad \iota = 1 \rightarrow KPR \) preferences

- \( \sigma \geq 1 \rightarrow IES \leq 1, \ C_t, H_t \) complements
Household Problem

- Household $i$ maximizes

$$(1 - \sigma)^{-1} \hat{E}_t^{(i)} \sum_{T=t}^{\infty} \beta^{T-t} \left( C_t(i) - \frac{\psi}{1 + \gamma} C_t^t(i) H_t^{1+\gamma(i)} \right)^{1-\sigma}$$

subject to

$$P_t^s B_t^s(i) + P_t^m B_t^m(i) \leq (1 + \rho P_t^m) B_{t-1}^m(i) + B_{t-1}^s(i) + \left(1 - \tau_t^W\right) W_t H_t(i)$$

$$+ P_t \Gamma_t - \tau_t^{LS} - P_t C_t(i)$$

and No-Ponzi condition

$$\lim_{T \to \infty} \hat{E}_t^{(i)} Q_{t,T}^i \left[ (1 + \rho P_T^m) B_{T-1}^m(i) + B_{T-1}^s(i) \right] / P_T \geq 0$$
Key Equation 1: Consumption Decision

- Combining Euler equations, labor supply, budget constraint to log-linear approximation provides

\[
\hat{C}_t^{(i)} - (1 - \sigma^{-\nu})\Theta \hat{H}_t^{(i)} = \bar{s}_C^{-1}(\sigma, \nu) \left( \frac{\bar{P}_m \bar{B}_m}{\bar{P}_{\bar{Y}}} \right) \times (\beta^{-1} - 1)
\]

\[
\times \left( b_{t-1}^{m,(i)} - \hat{\pi}_t + \rho \beta \bar{P}_t^m - \hat{E}_t^{(i)} \sum_{T=t}^{\infty} \beta^{T-t} \left[ \bar{T}_{LS} \hat{\tau}_{T}^{LS} + \bar{T}_W \hat{\tau}_{T}^{W} - \beta (\hat{\tau}_T - \hat{\tau}_{T+1}) \right] \right)
\]

\[
\bar{s}_C^{-1}(\sigma, \nu) (1 - \beta) \hat{E}_t^{(i)} \sum_{T=t}^{\infty} \beta^{T-t} \left( I_W \hat{w}_T + I_{\Gamma} \hat{\gamma}_T \right) +
\]

\[
- \beta \sigma^{-1} \hat{E}_t^{(i)} \sum_{T=t}^{\infty} \beta^{T-t} (\hat{\tau}_T - \hat{\tau}_{T+1})
\]

Gov. Debt and Taxes
Key Equation 1: Consumption Decision

- Combining Euler equations, labor supply, budget constraint to log-linear approx. provides

\[
\hat{C}_t^{(i)} - (1 - \sigma^{-t})\Theta \hat{H}_t^{(i)} = \bar{s}_C^{-1}(\sigma, \iota) \left( \frac{\check{P}^m \check{B}^m}{\check{P}Y} \right) \times (\beta^{-1} - 1)
\]

\[
\times \left( \hat{b}_{t-1}^{m,(i)} - \hat{\pi}_t + \rho \beta \hat{P}_t^m - \hat{E}_t^{(i)} \sum_{T=t}^{\infty} \beta^{T-t} \left[ \bar{T}_{LS} \hat{\tau}_{LS} + \bar{T}_W \hat{\tau}_W - \beta (\hat{i}_T - \hat{\pi}_{T+1}) \right] \right)
\]

Gov. Debt and Taxes

\[
+ \bar{s}_C^{-1}(\sigma, \iota) (1 - \beta) \hat{E}_t^{(i)} \sum_{T=t}^{\infty} \beta^{T-t} \left( I_W \hat{\omega}_T + I_{\Gamma} \hat{\tau}_T \right) +
\]

\[
- \beta \sigma^{-1} \hat{E}_t^{(i)} \sum_{T=t}^{\infty} \beta^{T-t} (\hat{i}_T - \hat{\pi}_{T+1})
\]
Key Equation 1: Consumption Decision

- Combining Euler equations, labor supply, budget constraint to log-linear approx. provides

\[
\hat{C}^{(i)}_t - (1 - \sigma^{-\nu}) \Theta \hat{H}^{(i)}_t = \bar{s}^{-1}_C (\sigma, \nu) \left( \frac{\bar{P}_m \bar{B}_m}{\bar{P}_Y} \right) \times (\beta^{-1} - 1)
\]

\[
\times \left( \hat{b}^{m,(i)}_{t-1} - \hat{\pi}_t + \rho \beta \hat{P}_t^m - \hat{E}^{(i)}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[ \bar{T}_{LS} \hat{\gamma}_T^{LS} + \bar{T}_W \hat{\gamma}_T^W - \beta (\hat{i}_T - \hat{\pi}_T+1) \right] \right)
\]

Gov. Debt and Taxes

\[
+ \bar{s}^{-1}_C (\sigma, \nu) (1 - \beta) \hat{E}^{(i)}_t \sum_{T=t}^{\infty} \beta^{T-t} \left( I_W \hat{w}_T + I_F \hat{\gamma}_T \right) +
\]

\[
- \beta \sigma^{-1} \hat{E}^{(i)}_t \sum_{T=t}^{\infty} \beta^{T-t} (\hat{i}_T - \hat{\pi}_T+1)
\]
Key Equation 1: Consumption Decision

• Combining Euler equations., labor supply, budget constraint to log-linear approx. provides

$$\hat{C}_t(i) - (1 - \sigma^{-1}) \Theta \hat{H}_t(i) = \bar{s}_C^{-1}(\sigma, \nu) \left( \frac{\bar{P}^m \bar{B}^m}{\bar{PY}} \right) \times (\beta^{-1} - 1)$$

$$\times \left( \hat{b}_{t-1}^{m(i)} - \hat{\pi}_t + \rho \beta \hat{P}_t^m - \hat{E}_t(i) \sum_{T=t}^{\infty} \beta^{T-t} \left[ \hat{T}_{LS} \hat{\hat{L}}_T + \hat{T}_W \hat{\hat{W}}_T - \beta \left( \hat{i}_T - \hat{\pi}_{T+1} \right) \right] \right)$$

Gov. Debt and Taxes

$$+ \bar{s}_C^{-1}(\sigma, \nu) (1 - \beta) \hat{E}_t(i) \sum_{T=t}^{\infty} \beta^{T-t} \left( I_W \hat{w}_T + I_{\Gamma} \hat{\Gamma}_T \right) +$$

$$- \beta \sigma^{-1} \hat{E}_t(i) \sum_{T=t}^{\infty} \beta^{T-t} \left( \hat{i}_T - \hat{\pi}_{T+1} \right)$$
Key Equation 2: Public Debt

- Price of government debt

\[
P_t^m = -\hat{E}_t \sum_{T=t}^{\infty} (\rho \beta)^{T-t} \hat{i}_T
\]

*Expectation Hypothesis*

- Evolution of public Debt

\[
\hat{b}_t^m = \beta^{-1} \left( \hat{b}_{t-1}^m - \hat{\pi}_t \right) + (1 - \rho) \hat{i}_t - (\beta^{-1} - 1) \hat{s}_t
\]

\[
+ (1 - \rho) \rho \beta \hat{E}_t \sum_{T=t}^{\infty} (\rho \beta)^{T-t} \hat{i}_{T+1}
\]
Firms

- Monopolistic competition + nominal rigidities (Rotemberg)

- Firm $j$ has produces according to

\[
Y_t(j) = H_t(j) = (p_t(j)/P_t)^{-\theta} Y_t
\]

- $p_t(j)$ chosen to maximize expected profits, where

\[
\Pi_t(j) = [p_t(j) - W_t] (p_t(j)/P_t)^{-\theta} Y_t - \chi [p_t(j)/p_{t-1}(j) - 1]^2
\]
Firms

- Monopolistic competition + nominal rigidities (Rotemberg)

- In aggregate, the Phillips curve

\[ \hat{\pi}_t = \psi \left( \gamma + \frac{\bar{Y}}{C} \right) \hat{Y}_t + \]

\[ + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[ \psi \alpha \beta \hat{w}_{T+1} + (1 - \alpha) \beta \hat{\pi}_{T+1} \right] \]
Learning and Expectations (use inflation as an example)

- Rational Expectations (RE)

\[ \hat{E}_t^{RE} \pi_{T+1} = 0 + \Omega_b^{RE} \hat{b}_{t-1}^m, \text{ where } T > t \]

- Learning

\[ \hat{E}_t \pi_{T+1} = \Omega_c^{L,t-1} + \Omega_b^{L,t-1} \hat{b}_{t-1}^m + \text{lagged variables,} \]

- Convergence to RE? **E-Stability**

  - Use methods of Marcet and Sargent (1989), Evans and Honkapohja (2001)
E-Stability and Learning

- Actual Law of Motion:

\[ \pi_t = T(\Omega_{c,t-1}^L, \Omega_{b,t-1}^L) \begin{bmatrix} 1 \\ \hat{b}_m \end{bmatrix} + \ldots \]

- Convergence under least-squares learning IFF the associated ODE:

\[
\frac{d}{d\tau} (\Omega_c, \Omega_b) = \underbrace{T(\Omega_c^L, \Omega_b^L)}_{\text{Actual Law of Motion}} - \underbrace{(\Omega_c^L, \Omega_b^L)}_{\text{Perceived Law of Motion}}
\]

is locally stable at the REE of interest: \( T(\Omega_c^{RE}, \Omega_b^{RE}) = (\Omega_c^{RE}, \Omega_b^{RE}) \).

- Source of instability: economy as a self-referential system
E-Stability

- Numerical study: model with LUMP SUM taxation

- Key parameters: $\sigma^{-1} = 1/4$, and $\frac{\bar{p}m \bar{B}m}{4PY} = 1.5$

- Others:
  - Preferences and technology: $\beta = 0.99$, $\alpha = 0.8$ (Price stickiness)
  - Fiscal Policy: $\tau^L_{S} = 1.5$
  - Steady State: $\bar{C}/\bar{Y} = 0.78$
Figure 1: E-stability properties in monetary policy – debt maturity space. Instability regions lie above the contours.
Aggregate Demand: Increase in Inflation Expectations

- Ricardian Households

\[
\hat{C}_t - (1 - \sigma^{-\nu}) \Theta \hat{H}_t =
\]

\[
-\beta \sigma^{-1} \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} (\phi_\pi \hat{\pi}_T - \hat{\pi}_{T+1}) +
\]

\[
+ s_{C}^{-1}(\sigma, \nu) \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} I_T
\]

- we have substituted for the bond price equation and the monetary policy rule,

\[
\hat{\nu}_t = \phi_\pi \hat{\pi}_t
\]
Aggregate. Demand: Increase in Inflation Expectations

- Baseline

\[
\hat{C}_t - (1 - \sigma^{-\nu}) \Theta \hat{H}_t = \\
- \beta \left( \sigma^{-1} - \bar{s}_C^{-1}(\sigma, \nu) \frac{\bar{S}}{\bar{Y}} \right) \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} (\phi_{\pi} \hat{\pi}_T - \hat{\pi}_{T+1}) + \\
- \bar{s}_C^{-1}(\sigma, \nu) \frac{\bar{S}}{\bar{Y}} \cdot \beta \rho \hat{E}_t \sum_{T=t}^{\infty} (\beta \rho)^{T-t} (\phi_{\pi} \hat{\pi}_T) + \\
+ \bar{s}_C^{-1}(\sigma, \nu) \frac{\bar{S}}{\bar{Y}} \hat{b}_{m-1}^{t} + \bar{s}_C^{-1}(\sigma, \nu) (1 - \beta) \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} (I_T - T_T). 
\]
Aggregate Demand: Increase in Inflation Expectations

- Baseline

\[
\hat{C}_t - (1 - \sigma^{-\iota}) \Theta \hat{H}_t = \\
-\beta \left( \sigma^{-1} - \bar{s}_C^{-1}(\sigma, \iota) \frac{\bar{S}}{\bar{Y}} \right) \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} (\phi_{\pi} \hat{\pi}_T - \hat{\pi}_{T+1}) + \\
-\bar{s}_C^{-1}(\sigma, \iota) \frac{\bar{S}}{\bar{Y}} \cdot \beta \rho \hat{E}_t \sum_{T=t}^{\infty} (\beta \rho)^{T-t} (\phi_{\pi} \hat{\pi}_T) + \\
+\bar{s}_C^{-1}(\sigma, \iota) \frac{\bar{S}}{\bar{Y}} \hat{b}_{t-1}^m + \bar{s}_C^{-1}(\sigma, \iota) (1 - \beta) \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} (I_T - T_T).
\]
Aggregate. Demand: Increase in Inflation Expectations

• Baseline

\[ \hat{C}_t - (1 - \sigma^{-\nu}) \Theta \hat{H}_t = \]

\[ -\beta \left( \sigma^{-1} - \bar{s}_C^{-1}(\sigma, \nu) \frac{\bar{S}}{\bar{Y}} \right) \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} (\phi \hat{\pi}_T - \hat{\pi}_{T+1}) + \]

\[ -\bar{s}_C^{-1}(\sigma, \nu) \frac{\bar{S}}{\bar{Y}} \cdot \beta \rho \hat{E}_t \sum_{T=t}^{\infty} (\beta \rho)^{T-t} (\phi \hat{\pi}_T) + \]

\[ + \bar{s}_C^{-1}(\sigma, \nu) \frac{\bar{S}}{\bar{Y}} \hat{b}^{m}_{t-1} + \bar{s}_C^{-1}(\sigma, \nu) (1 - \beta) \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} (I_T - T_T). \]
Government Debt: Increase in Inflation Expectations

- Using the bond price equation and the monetary policy rule, \( \hat{i}_t = \phi_\pi \hat{\pi}_t \), permits the evolution of real debt to be written as

\[
\hat{b}_{tm} = \beta^{-1} (\hat{b}_{tm-1} - \hat{\pi}_t) + (1 - \rho) \phi_\pi \hat{\pi}_t - \left( \beta^{-1} - 1 \right) \hat{s}_t \\
+ (1 - \rho) \rho \beta \hat{E}_t \sum_{T=t}^{\infty} (\rho \beta)^{T-t} \phi_\pi \hat{\pi}_{T+1}
\]

which depends on the maturity structure.
Using the bond price equation and the monetary policy rule, \( \hat{i}_t = \phi_\pi \hat{\pi}_t \), permits the evolution of real debt to be written as

\[
\hat{b}_m^t = \beta^{-1} (\hat{b}_m^{t-1} - \hat{\pi}_t) + (1 - \rho) \phi_\pi \hat{\pi}_t - (\beta^{-1} - 1) \hat{s}_t
\]

\[
+ (1 - \rho) \rho \beta \hat{E}_t \sum_{T=t}^{\infty} (\rho \beta)^{T-t} \phi_\pi \hat{\pi}_{T+1}
\]

which depends on the maturity structure through (with exp. constant inflation)

\[
(1 - \rho) \rho \beta \hat{E}_t \sum_{T=t}^{\infty} (\rho \beta)^{T-t} \phi_\pi \hat{\pi}_{T+1} = \frac{(1 - \rho) \rho \beta}{1 - \rho \beta} \phi_\pi \hat{E}_t \pi
\]

- For \( \beta = 0.99 \) the right hand side peaks \( \rho \simeq 0.9 \): average maturity of 2 years
- For \( \rho = 1 \): dynamics independent of its own price.
Figure 2: E-stability properties in monetary policy – debt maturity space. Instability regions lie above the contours which are indexed by average level of indebtedness.
Figure 3: E-stability properties in monetary policy – elasticity of substitution space. Instability regions lie above the contours which are indexed by average level of indebtedness.
Figure 4: Decomposing wealth and substitution effects. The black lines plot $\frac{1}{s_C}$, which indexes the scale of wealth effects from holdings of the public debt, for KPR and GHH preferences. The blue lines give the elasticity of substitution of consumption demand with respect to the real interest rate. Both are plotted as function of $\sigma$. 
Forward-looking policy rules

- Alternative policy rules that responds to expectations

\[ \hat{\pi}_t = \phi_\pi \hat{E}_t \hat{\pi}_{t+1} \]

- Here we assume that agents know the policy rule
Figure 5: E-stability properties in monetary policy – debt maturity space for a standard Taylor rule and an expectations-based Taylor rule. Instability regions lie above the contours.
Intuition: Debt Dynamics

- Using the bond price equation and the monetary policy rule, \( \hat{\pi}_t = \phi_\pi \hat{E}_t \hat{\pi}_{t+1} \), permits the evolution of real debt to be written as

\[
\hat{b}_t^m = \beta^{-1} (\hat{b}_{t-1}^m - \hat{\pi}_t) + (1 - \rho) \phi_\pi \hat{E}_t \hat{\pi}_{t+1} - (\beta^{-1} - 1) \hat{s}_t
\]

\[
+ (1 - \rho) \rho \beta \hat{E}_t \sum_{T=t}^{\infty} (\rho \beta)^{T-t} \phi_\pi \hat{\pi}_{T+2}
\]

- Debt depends on expectations regardless of \( \rho \)
- Destabilizing effects on consumption and debt decline monotonically with \( \rho \)
Distortionary taxation

• Consider the model labor taxes \((\tau^w_t)\)...

• ...and assume now: \(\sigma^{-1} = 1/2\)
Figure 6: E-stability properties in monetary policy debt maturity space. Instability regions lie above the contours indexed by average indebtedness.
Intuition: Distortionary taxation

- Same mechanism as before but with a ‘twist’

- Phillips curve has an **extra term**

\[
\hat{\pi}_t = \psi \left[ \left( \gamma + \frac{\bar{Y}}{\bar{C}} \right) \hat{Y}_t + \frac{\bar{\pi}^w}{(1 - \bar{\pi}^w)} \hat{\pi}_t^w \right] + \\
\hat{E}_t \sum_{T=t}^{\infty} (\alpha \beta)^{T-t} \left[ \psi \alpha \beta \hat{\omega}_{T+1} + (1 - \alpha) \beta \hat{\pi}_{T+1} \right]
\]

- Changes in tax rates as cost-push “shocks”, reinforcing the drift in expectations
Conclusion

- Uncertainty about policy regime can induce drift in expectations

- High debt levels and short to medium maturity debt induce instability
  - Instability generated through wealth effects

- Instability of expectations as ‘fiscal limit’

- Fundamentally changes the nature of household and firm responses to shocks — even if expectations stable in the long-run
Motivation Revisited

- Macroeconomic models depend on household and firm expectations

- Policy design
  - Not enough to observe some history of beliefs
  - Need a theory of the determination of beliefs: rational expectations

- Rational expectations policy design
  - Heavy reliance on managing expectations through announced commitments
  - What are the consequences of imprecise control of beliefs?
Motivation II

- For example, Bernanke (2004):

  “[…] most private-sector borrowing and investment decisions depend not on the funds rate but on longer-term yields, such as mortgage rates and corporate bond rates, and on the prices of long-lived assets, such as housing and equities. Moreover, the link between these longer-term yields and asset prices and the current setting of the federal funds rate can be quite loose at times.”
The Agenda

- Simple model of output gap and inflation determination

- One informational friction:
  - Agents have an incomplete knowledge about the economy

- Explore constraints imposed on stabilization policy by financial market expectations
  - What if the expectations hypothesis of the yield curve does not hold?
  - Is the zero lower bound on nominal interest rates an important constraint?
Asset Structure and the Fiscal Authority

- Exogenous purchases of $G_t$ per period

- Issue two kinds of debt
  - $B^s_t$: One period debt in zero net supply with price $P^s_t = (1 + i_t)^{-1}$
  - $B^m_t$: An asset in positive supply that has the payoff structure
    \[ \rho^{T-(t+1)} \text{ for } T \geq t + 1 \]

- Let $P^m_t$ denote the price of this second asset. Asset has the properties:
  - Price in period $t + 1$ of debt issued in period $t$ is $\rho P^m_{t+1}$
  - Average maturity of the debt is $(1 - \beta \rho)^{-1}$
Log-linear approximation of Euler equations for each kind of bond holdings implies

\[ \hat{i}_t = -\hat{P}_t^s = -\hat{E}_t^i \left( \hat{P}_t^m - \rho \beta \hat{P}_{t+1}^m \right) \]

- Restriction on asset price movements
- Combined with transversality

\[ \hat{P}_t^m = -\hat{E}_t \sum_{T=t}^{\infty} (\rho \beta)^{T-t} \hat{i}_T \]

- Asset price is a function of fundamentals: the expectations hypothesis of the term structure holds; does not imply interest-rate expectations consistent with monetary policy strategy
- Call: Anchored Financial Market Expectations
Optimal Spending Plans I

- Assume agents understand fiscal policy is Ricardian

- When the expectations hypothesis of the term structure holds — aggregate demand relation of the form

\[
\hat{C}_t^i = -\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} [\beta (i_T - \pi_{T+1})] + \bar{s}_C^{-1} (1 - \beta) \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[ \left( \frac{\bar{\theta} - 1}{\bar{\theta}} \right) (1 + \gamma^{-1}) \hat{w}_T + \bar{\theta}^{-1} \hat{r}_T \right]
\]

- Example of permanent income theory

- Independent of the average maturity structure of debt; Isomorphic to an economy with one-period debt — i.e. \( \rho = 0 \)

- Independent on long-debt-price expectations
Asset Pricing: Theory II

- Under non-rational expectations and incomplete markets: exist alternative ways to impose no-arbitrage

- Consider pricing the asset directly

\[ \hat{i}_t = -\hat{E}_t^i \left( \hat{P}_m^t - \rho \beta \hat{P}_m^{t+1} \right) \]

in all periods \( t \)

- Given monetary policy and expectations, asset price determined. No-arbitrage consistent forecasts of the short-rate can then be determined by

\[ \hat{E}_t^i \hat{\nu}_T = -\hat{E}_t^i \left( \hat{P}_T^m - \rho \beta \hat{P}_T^{m+1} \right) \]

- Because interest rate projections are not directly related to current interest rates --- expectations hypothesis need not hold
Demand then determined by

\[
\hat{C}_t^i = -\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[ \beta \left( \rho \beta \hat{P}_{T+1}^m - \hat{P}_T^m \right) - \pi_{T+1} \right] \\
+ \bar{\sigma}_C^{-1} (1 - \beta) \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[ \left( \frac{\bar{\theta} - 1}{\bar{\theta}} \right) \left( 1 + \gamma^{-1} \right) \hat{\omega}_T + \bar{\theta}^{-1} \hat{\pi}_T \right]
\] (11)

- Nest anchored expectations model when \( \rho = 0 \); since \( \hat{E}_t^i \hat{\pi}_T = -\hat{E}_t^i \hat{P}_T \)

- In this case forecasting bond prices and forecasting interest rates equivalent

  * As maturity increases this divergence increases
Beliefs Formation

- Agents construct forecasts according to

$$\hat{E}_t^i X_{t+T} = a_{t-1}^X$$

where $X = \{\pi, \hat{w}, \hat{r}, \hat{i}, \hat{P}^m\}$ for any $T > 0$.

- In period $t$ forecasts are predetermined.

- Beliefs are updated according to the constant gain algorithm

$$a_t^X = (1 - g) a_{t-1}^X + g X_t$$

where $g > 0$

- With i.i.d. shocks and zero debt nets the REE

  * Learning only about the constant
Beliefs Formation II

• Under anchored expectations, based on interest-rate data:

\[ \hat{E}_t^i \hat{i}_T = a_t^i \]

• Under unanchored expectations, based on bond-price data:

\[ \hat{E}_t \hat{i}_T = -\hat{E}_t (\hat{P}_T^m - \rho \beta \hat{P}_{T+1}^m) \]

\[ = -(1 - \rho \beta) a_{t-1}^{pm} \]

– In general they are not equal

– Equivalent when \( \rho = 0 \)
Calibration

• Discount factor is $\beta = 0.99$

• Labor supply elasticity $\gamma^{-1} = 2$

• Nominal rigidities $\alpha = 0.75$

• Elasticity of demand across differentiated goods $\bar{\theta} = 8$
Unanchored Expectations and Simple Rules

- Consider the case of a simple Taylor rule

\[ \hat{\pi}_t = \phi_\pi \hat{\pi}_t + \phi_y x_t \]

- Use notion of “robust stability”
  - Dynamics converge for a given gain coefficient
  - Distinct from E-Stability
Stability with a Taylor Rule

Figure 7: Robust stability regions for different maturity structures. The three contours correspond to different Taylor distinguished by their respond to the current output gap.
Intuition

- Aggregate demand can be written

\[
x_t = -\hat{\gamma}_t + \hat{E}_t \rho \beta \hat{P}_t^{m} + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[ \beta (1 - \rho) \hat{P}_T^{m} + \pi_{T+1} \right] - \hat{A}_t
\]

\[
+ \bar{s}_C^{-1} (1 - \beta) \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[ \left( \frac{\bar{\theta} - 1}{\bar{\theta}} \right) (1 + \gamma^{-1}) \hat{w}_{T+1} + \bar{\theta}^{-1} \hat{\pi}_{T+1} \right].
\]

- For infinite-period debt \( \rho = 1 \): depends only on \( \hat{E}_t \hat{P}_t^{m} \)

- Requires aggressive adjustment of current nominal interest rates

- Key mechanism: increasing debt maturity implies arbitrage restrictions in the expectations hypothesis are weaker
Optimal Policy under Rational Expectations

- Consider target criterion

\[ \pi_t = -\bar{\theta}^{-1} x_t \]
Optimal Policy under Rational Expectations II

- Target criterion approach implies
  - Interest-rate policy sufficiently aggressive to guarantee satisfaction of the target criterion
  - Adjustment of policy in response to asset prices

- Both might be thought to confer stabilization advantages
Figure 8: Stability regions in gain-maturity space for the optimal rational expectations target criterion under discretion.
Intuition

- Second distinct mechanism at play
  - For a given average maturity of debt — higher gains imply greater volatility in expectations and therefore require more aggressive adjustment in interest rates which has destabilizing feedback effects
  - As the average maturity rises, arbitrage relationship defining term structure weakens — implies more aggressive policy feasible

- Shifting interest-rate expectations represent an additional constraint on policy
  - Limits scope to react to current macroeconomic developments
Optimal Policy

- Central Bank seeks to minimize

\[
\min_{\{x_t, \pi_t, i_t, P^m_t, \alpha_t^\pi, \alpha_t^Y, \alpha_t^w, \alpha_t^\Gamma\}} \mathbb{E}^RE_0 \sum_{t=0}^{\infty} \beta^t \left[ \pi_t^2 + \lambda_x (x_t - x^*)^2 + \lambda_i (i_t - i^*) \right]
\]

where \( \lambda_x, \lambda_i \geq 0 \) and \( x_t \) the output gap and \( Y = \{P^m, i\} \). Minimization subject to the constraints:

- Aggregate demand and supply
- No-arbitrage equation
- Beliefs
- Disturbances: technology and cost-push shocks
Properties of Optimal Policy

Proposition 1  The first-order conditions representing a solution to the minimization of the loss subject to i) the aggregate demand, supply and arbitrage equations; the no-arbitrage condition; and ii) the law of motion for the beliefs $a_t^\pi, a_t^Y, a_t^w, a_t^\Gamma$ have a unique bounded rational expectations solution for all parameter values. In particular, model dynamics are unique for all possible gains.

- First-order conditions constitute a linear rational expectations model
  - Can be solved using standard methods
  - Does not imply that learning is irrelevant for policy outcomes
Special Case: No uncertainty

- Two limiting results of interest
  - When $g \to 0$ and $\beta < 1$ then
    $$\lim_{T \to \infty} E_t \pi_T = \frac{\kappa \lambda_x x^*}{\kappa^2 + \lambda_x (1 - \beta)}$$
  - When $g > 0$ and $\beta \to 1$ then
    $$\lim_{T \to \infty} E_t \pi_T = 0$$

- Patient Central Banker replicates the optimal commitment policy under rational expectations
  - Price stability is optimal in the long run
Impulse Response Functions: Reintroducing Uncertainty

- Assume $\lambda_i = 0$ and $x^* = i^* = 0$
  - Compare with dynamics under rational expectations
  - Cost-push shock
Figure 9: Impulse response functions in response to a cost-push shock. Gain = 0.15. Rational expectations: red dotted line; learning with anchored expectations: blue solid line; and learning with unanchored expectations: green dashed line.
Figure 10: Impulse response functions in response to a cost-push shock. Gain = 0.005. Rational expectations: red dotted line; learning with anchored expectations: blue solid line; and learning with unanchored expectations: green dashed line.
Implications II

- Anchored expectations:
  - Expectations hypothesis holds which imposes a constraint on current interest-rate movements
  - Inflation and output more volatile

- Unanchored expectations:
  - Current interest-rate policy divorced from interest-rate expectations
  - Inflation and output less volatile; interest rates more volatile
Efficient Policy Frontiers

- Compute

\[ \bar{L} = V[\pi] + \lambda_x V[x] + \lambda_i V[i] \]

where \( V[\cdot] \) denotes unconditional variance

- Study variation in

\[ V[\pi] + \lambda_x V[x] \]

as tolerated variance in interest-rates varies: that is \( V[i] \)

  - Equivalent to studying the original loss function as \( \lambda_i \) varies
Figure 11: Policy frontiers as weight on interest rate stability is increased. Exogenous disturbance is a cost-push shock.
The Zero Lower Bound on Nominal Interest Rates

• Compute the unconditional probability of being at the zero lower bound in each model

• Requires two additional parameter assumptions.
  
  – The average level of the short-term nominal interest rate: 5.4% (annualized),
    * corresponds to the average rate of the US 3-month T-bill for the period 1954Q3-2011Q3
  
  – Under RE and $\lambda_i = 0.08$ calibrate the volatility of the technology shocks to deliver a standard deviation of output of 1.5% (in log-deviations from its steady state) and probability of ZLB being 3.5%
Figure 12: The figure shows the unconditional probability of being at the ZLB as a function of $\lambda_i$ for the three different models.
Implications

- If financial market expectations unanchored
  - Zero lower bound likely to a relevant constraint on monetary policy
  - True even if substantial weight placed on losses from such variation

- Contrasts markedly with claims that “Zero lower bound on nominal interest rate is of little quantitative relevance in standard New Keynesian models”
  - Chung et. al. 2010, Schmitt-Grohè and Uribe 2007
Conclusion

• Failure of beliefs to satisfy the expectations hypothesis of the term structure limits the efficacy of monetary policy
  
  – The pricing of public debt places constraints also on optimal monetary policy
  
  – Can make the zero lower bound constraint on nominal interest rates more severe