Norges Bank Mini-Course:
Boundedness of Government Debt

Eric M. Leeper
Indiana University
April 2013
Why Does This Matter?

- Should we restrict attention to equilibria in which debt is bounded?
- Seems like an obscure & technical topic
- It turns out to matter quite a lot
  - permitting debt to grow without bound means imposing one less boundary condition
  - no boundary condition admits many more legitimate solutions
  - these solutions have been interpreted to imply that the fiscal theory is “special” while Ricardian outcomes are “general”
- This lecture explains what is at stake
Sketching the Issue

Consider a log-linearized government budget constraint and tax rule

\[ b_t = \beta^{-1} b_{t-1} - (\beta^{-1} - 1) \tau_t + \text{stuff} \]  \hspace{1cm} (1)

\[ \tau_t = \gamma b_{t-1} \]  \hspace{1cm} (2)

Combining these, get difference equation in real debt

\[ b_t = \Gamma b_{t-1} + \text{stuff} \]  \hspace{1cm} (3)

\[ \Gamma \equiv \beta^{-1} - \gamma (\beta^{-1} - 1) \]

Also a transversality condition

\[ \lim_{j \to \infty} \beta^j E_t b_{t+j} = 0 \]  \hspace{1cm} (4)
Sketching the Issue

- Often we impose \( \{b_t\} \) must be bounded
  - restrict to solutions with \( 0 \leq \Gamma < 1 \) (or \( \gamma > 1 \))
  - sufficient *but not necessary* for transversality
  - TVC requires that debt not grow “too fast” (not that debt is bounded)

- If \( \{\tau_t\} \) is **lump sum**
  - a solution where \( b_t \) and \( \tau_t \) grow without bound, may be an equilibrium
  - it is feasible with growing debt to raise the required revenues

- In such a model, the stability condition is weaker

\[
0 \leq \Gamma < \beta^{-1} \quad \text{or} \quad \gamma > 0
\]

- Permits many more solutions
This Lecture

- Review Canzoneri, Cumby, Diba’s argument
  - show that it relies on unbounded debt

- Review Chung, Davig, Leeper’s setup
  - show that under CCD’s assumptions, the only bounded solution is one that is consistent with the fiscal theory
CCD’s Argument

▶ Write the government’s budget constraint as

\[
\frac{M_t + B_t}{P_{t}y_{t}} = \frac{T_t - G_t}{P_{t}y_{t}} + \frac{M_{t+1}}{P_{t}y_{t}} \left( \frac{i_t}{1 + i_t} \right)
\]

\[
\left( \frac{Y_{t+1}/Y_t}{(1 + i_t)P_t/P_{t+1}} \right) \left( \frac{M_{t+1} + B_{t+1}}{P_{t+1}y_{t+1}} \right)
\]

(6)

▶ Simplify (6) as

\[
w_t = s_t + \delta_t w_{t+1}
\]

(7)

\(w_t\): govt liabilities/nominal GDP; \(s_t\): primary surplus; \(\delta_t\): real discount factor
CCD’s Argument

- Solve (7) forward & take expectations

\[
 w_t = s_t + E_t \sum_{j=t+1}^{\infty} \left( \prod_{k=t}^{j-1} \delta_k \right) s_j
\]

\[
 \iff \lim_{j \to \infty} E_t \left( \prod_{k=t}^{T+t-1} \delta_k \right) w_t = 0
\]

- How does (8) get satisfied?

1 **Ricardian Regime** If \( \{s_t\} \) adjusts to satisfy (8) for all \( (\{\delta_t\}, w_t) \) sequences

2 **Non-Ricardian Regime** If \( \{s_t\} \) unrelated to debt, then \( P_t y_t \) must adjust to satisfy (8)
Suppose fiscal policy obeys

\[ s_t = c_t w_t + \varepsilon_t \]  \hspace{1cm} (9)

Assume \( \{c_t\}, \{\delta_t\}, \{\varepsilon_t\} \) deterministic sequences & \( \{\varepsilon_t\} \) bounded

If (C1) & (C2) are satisfied, then (9) implies Ricardian Regime

\[ 0 \leq c_j < 1, \quad \limsup c_j > c^* > 0 \] \hspace{1cm} (C1)

\[ D_t \equiv 1 + \sum_{j=t+1}^{\infty} \left( \prod_{k=t}^{j-1} \delta_k \right) < \infty \] \hspace{1cm} (C2)
Condition (C2) technical: required for present-value condition to be well defined

Condition (C1) is substantive

- \( c_j \) bounded away from zero infinitely often (FP does not try to rollover debt indefinitely)
- when \( c_j > 0 \): \( s_j \) moves to stabilize \( w_j \) but it need not do so in every period
- for fiscal rule (9) to deliver the Ricardian Regime, just need agents to expect that sooner or later FP will adjust to stabilize debt
- note that the longer policy delays responding to debt, the larger must be the adjustments to \( s_j \)
Proposition: Assume that \( \{c_j\}, \{\delta_j\}, \{\varepsilon_j\} \) are deterministic sequences, \( \delta_j > 0 \), \( \{\varepsilon_j\} \) is bounded, and (C1) & (C2) hold. Then the flow budget constraint, (7), and the fiscal rule, (9), imply that the present-value equilibrium condition, (8), holds for any arbitrary \( w_t \), and fiscal rule (9) delivers a Ricardian Regime.
CCD’s Argument

- Intuition for CCD’s proposition
- Suppose that $c_j = c \geq 0$ & $\delta_j = \delta < 1$ for all $j$
  - if $c = 0$, then (7) unstable with root $1/\delta > 1 \Rightarrow$
  - Non-Ricardian Regime so $w_t$ must jump to suppress the unstable root so (8) can hold in equilibrium
  - use (9) in (7), so root is $(1 - c)/\delta$
  - if $c$ large enough to make $(1 - c)/\delta < 1$, then (7) is stable & (8) holds for any $w_t$
  - $c$ need not make debt stable: (8) requires only that discounted value of $w_{t+T} \rightarrow 0$ as $T \rightarrow \infty$
  - any positive $c$ ensures $PV(w) = 0$

- If the fiscal response is time-varying, (C1) says that the response may be arbitrarily small and infrequent
  - if $c_t$ followed a recurrent Markov chain with $c = 0$ in state 1 & $c > 0$ in state 2, (C1) is satisfied
Comments on CCD’s Argument

1. Ricardian FP imposes no restriction on $P_{tyt}$ because surplus adjusts to ensure (8) holds for all $\{P_{tyt}\}$ sequences
   ▶ raises the specter of indeterminacy

2. CCD’s definition of Ricardian says nothing about whether taxes matter
   ▶ in CCD’s proposition, Ricardian FP consistent with taxes being irrelevant or taxes having wealth effects

3. CCD silent about monetary policy
   ▶ to give the proposition more content, need to specify MP behavior

4. If indeterminate, can construct sunspot equilibria with shock $\varphi_t$
   ▶ “Ricardian FP” & active MP $\Rightarrow$ taxes don’t matter
   ▶ “Ricardian FP” & passive MP $\Rightarrow$
     ▶ $\text{Corr}(\varphi_t, \text{taxes}_t) = 0$: taxes don’t matter
     ▶ $\text{Corr}(\varphi_t, \text{taxes}_t) \neq 0$: taxes matter
5. If debt grows without bound, then interest on the debt grows without bound
   - eventually, debt service will exceed nominal GDP with prob. 1
   - this can happen in many ways, all of which involve the following reasoning
   - fix output at $y$, real interest rate at $r$, and $T_t = cw_{t-1}$, $0 < c < r$
   - as $w_t \to \infty$, $T_t \to \infty$
   - but household income, $y + (1 + r)w_t \to \infty$
   - a small $c > 0$ ensures $w_t$ grows at rate $< \beta^{-1}$
   - government can tax an unbounded amount because it is returning to households unbounded interest payments
6. CCD’s proposition falls apart if one imposes an upper bound on debt/GDP arising from... 
   - distorting taxes that imply a Laffer curve
   - costs of raising revenues that increase with the revenue needs (models of tax evasion)
   - CCD’s proposition *relies* on lump-sum taxes—any proposition that requires lump-sum taxes is immediately suspicious
Canzoneri, Cumby, Diba: Ricardian equilibria more general than non-Ricardian

- if responses of taxes to liabilities is positive infinitely often—however small and infrequent—then eqm exhibits Ricardian equiv
- because fiscal response does not stabilize debt, these are potentially equilibria with unbounded debt-output ratios

Our example satisfies CCD’s assumptions, but delivers a unique eqm in set with bounded debt-output ratios

- this eqm is non-Ricardian
- important conclusions hinge on unboundedness assumption of CCD
An Analytical (Counter) Example

- Bring monetary policy into picture
  - necessary to be precise about determinacy issues
- Permit recurring regime change in MP & FP rules
- Solve for equilibrium inflation & debt processes
- What follows is *not* a general model of regime change
  - it is an example that contradicts CCD’s claims
The Model

- MIUF, constant endowment, log prefs, constant $g$
- Fisher equation
  \[
  \frac{1}{R_t} = \beta E_t \frac{1}{\pi_{t+1}}
  \]
- Money demand
  \[
  m_t = \left[ \frac{R_t - 1}{R_t} \right]^{-1} c
  \]
- Government flow budget constraint
  \[
  \frac{M_t + B_t}{P_t} + \tau_t = g + \frac{M_{t-1} + B_{t-1}}{P_t}
  \]
- Monetary policy
  \[
  R_t = \exp (\alpha_0 + \alpha (S_t) \hat{\pi}_t + \theta_t)
  \]
- Tax policy
  \[
  \tau_t = \gamma_0 + \gamma (S_t) (b_{t-1} + m_{t-1}) + \psi_t
  \]
  $(\theta_t, \psi_t)$ exogenous policy shocks; $\hat{\pi} = \ln \pi$
The Model

- $S_t$ an $N$-state Markov chain with transition probs
  \[ P[S_t = j | S_{t-1} = i] = p_{ij} \]

- Define expectation error (and use Fisher equation)
  \[ \eta_{t+1} = \frac{1/\pi_{t+1}}{E_t[1/\pi_{t+1}]} = \beta \frac{R_t}{\pi_{t+1}} \]

- Then the inflation process is given by
  \[ \hat{\pi}_{t+1} = \alpha(S_t)\hat{\pi}_t + \alpha_0 + \theta_t - \hat{\eta}_{t+1} + \ln \beta \]

- Let $l_t = b_t + m_t$, real govt liabilities
- Use tax rule & money demand in govt budget constraint
  \[ l_t = \left[ \frac{R_{t-1}}{\pi_t} - \gamma(S_t) \right] l_{t-1} - \frac{R_{t-1}}{\pi_t} c + D - \psi_t \]

\[ D = g - \gamma_0 \]
Solution

- Assume that
  1. \( E_t[\gamma_{t+1}] = \gamma \)
  2. \( \gamma \) satisfies \( |1/\beta - \gamma| > 1 \)
  3. Inflation process is stable in expectation (i.e., there exists a \( 0 < \xi < \infty \) such that \( |E_t \pi_{t+k}| < \xi \) for all \( k \))

- (I)-(II): on average FP active; (III): on average MP passive

- Iterate on \( l \) equation and take \( E_{t-1} \) and law of iterated expectations

\[
E_{t-1} [l_{t+k}] = (1/\beta - \gamma)^{k+1} \left[ l_{t-1} - c \left( \frac{1/\beta - D/c}{1/\beta - \gamma - 1} \right) \right] + c \left( \frac{1/\beta - D/c}{1/\beta - \gamma - 1} \right)
\]

*Boundedness* requires that \( l_{t-1} = c \left( \frac{1/\beta - D/c}{1/\beta - \gamma - 1} \right) \), which is positive if \( D/c < 1/\beta \)
Solution

- The value of $\eta_t$ is obtained from the budget constraint after substituting in the value of $l$:

$$
\eta_t = \beta \left( 1 + \gamma(S_t) \right) \left( \frac{1}{\beta} - \frac{D}{c} \right) \left( \frac{1}{\beta} - \gamma - 1 \right) \frac{1}{1 + \gamma - D/c} \\
+ \frac{\beta}{c} \left( \frac{1}{\beta} - \gamma - 1 \right) \psi_t
$$

- The unique equilibrium mapping from $\psi_t$ and $\gamma(S_t)$ to forecast error in inflation

- $\eta_t$ and $\pi_t$ process yields unique solution for inflation

- Both tax shocks, $\psi_t$, and tax regime, $\gamma(S_t)$, matter for inflation
Concrete Example

- Two regimes, \( N = 2 \), and policy parameters take on the values

\[
\alpha(S_t) = \begin{cases} 
\alpha_1 & \text{for } S_t = 1 \\
\alpha_2 & \text{for } S_t = 2 
\end{cases} 
\quad \gamma(S_t) = \begin{cases} 
\gamma_1 & \text{for } S_t = 1 \\
\gamma_2 & \text{for } S_t = 2 
\end{cases}
\]

- Suppose \( \alpha_1 \) and \( \alpha_2 \) are sufficiently small such that the inflation process is stable in expectation

- What does the assumption \( E_t \gamma_{t+1} = \gamma \) mean?

\[
E[\gamma_{t+j} | S_t = 1, \Omega_t] = \gamma_1 p_{11} + \gamma_2 (1 - p_{11}) = \gamma \\
E[\gamma_{t+j} | S_t = 2, \Omega_t] = \gamma_1 (1 - p_{22}) + \gamma_2 p_{22} \equiv \gamma \\
\Rightarrow p_{11} + p_{22} = 1
\]

- eliminates state-dependence in the probabilities
Concrete Example

- If either $\gamma_1$ or $\gamma_2$ is positive, then the model satisfies CCD's premise that taxes adjust to debt infinitely often.
- But negative tax shocks generate wealth effects that raise inflation.
- The only equilibrium with *bounded* debt is one in which Ricardian equiv breaks down: counterexample to CCD.
- Also makes the general point that short-run policy behavior—in the current regime—may not accurately describe the long-run behavior that matters for nature of the equilibrium.
Messages

1. Lump-sum taxes/transfers is a modeling device—not a description of reality
   ▶ use them to ease the analysis
   ▶ not to be taken literally
2. Always need to check if “assumptions of convenience” are essential to model’s outcomes
   ▶ if they are, then need to rethink the use of those assumptions
3. Critical results hinge on whether debt is bounded or unbounded
   ▶ unbounded debt leaves terminal condition unspecified
   ▶ admits many more solutions
4. If FP is active in the sense that surpluses respond positively to debt, but by less than the real interest rate, then...
   ▶ the unique bounded solution is one in which taxes affect aggregate demand & inflation