Linear Analysis

- Generalize policy behavior with a conventional parametric representation
- Allow us to characterize the MP/FP behavior that is consistent with existence & uniqueness of a bounded equilibrium
- Cost of generality: focus on local dynamics within a neighborhood of steady state, rather than previous global results
- Same cashless endowment economy
Model’s Critical Equations

Fisher: \( \frac{1}{R_t} = \beta E_t \left[ \frac{1}{\pi_{t+1}} \right] \)

Fiscal Policy: \( \tau_t = \tau^* \left( \frac{b_{t-1}}{b^*} \right)^\gamma \exp(z^\tau_t) \ (\text{AR}(1) \text{ coeff } \rho^\tau) \)

Monetary Policy: \( R_t = R^* \left( \frac{\pi_t}{\pi^*} \right)^\alpha \exp(z^R_t) \ (\text{AR}(1) \text{ coeff } \rho^R) \)

GBC: \( b_t + \tau_t = \frac{R_{t-1} b_{t-1}}{\pi_t} \)

where: \( \pi_t = P_t/P_{t-1}, b_t = B_t/P_t \)

innovations: \( \varepsilon^R_t & \varepsilon^\tau_t \)

- Linearize & reduce to system in \((\pi_t, b_t)\) with exogenous variables \((z^\tau_t, z^R_t)\)
  - reflects two tasks: control \(\pi_t\); stabilize \(b_t\)
- Policy parameters \((\alpha, \gamma)\) critical to existence & uniqueness of equilibrium
Solving the Model

- Reduce the model to a dynamical system in
  \( x_t \equiv (\pi_t, b_t, z_t^R, z_t^\tau)' \)
- Define the forecast error \( \eta_{t+1} = \pi_{t+1} - E_t \pi_{t+1} \)
- Define vector of innovations as \( \xi_t \equiv (\varepsilon_t^R, \varepsilon_t^\tau, 0, \eta_t)' \)
- Write the linearized system to be solved for \( t \geq 0 \) as

\[
x_{t+1} = Ax_t + C\xi_{t+1}
\]

\[
A \equiv \begin{bmatrix}
\alpha & 0 & 1 & 0 \\
0 & \Gamma & 0 & -\rho_\tau(\beta^{-1} - 1) \\
0 & 0 & \rho_R & 0 \\
0 & 0 & 0 & \rho_\tau
\end{bmatrix}, \quad \Gamma \equiv \beta^{-1} - \gamma(\beta^{-1} - 1)
\]

\[
B \equiv \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & -(\beta^{-1} - 1) & 0 & -\beta^{-1} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]
Solving the Model

- Eigenvalues are $\alpha \& \Gamma \equiv \beta^{-1} - \gamma(\beta^{-1} - 1)$
- With a single forecast error, $\eta_{t+1}$, need one unstable root for a unique bounded eqm to exist
  - this *anchors expectations*
- Four regions of policy parameter space are of interest

  I: $|\alpha| > 1$ and $|\gamma| > 1 \Rightarrow$ Unique Eqm

  II: $|\alpha| < 1$ and $|\gamma| < 1 \Rightarrow$ Unique Eqm

  III: $|\alpha| < 1$ and $|\gamma| > 1 \Rightarrow$ Multiple Eq/Sunspots

  IV: $|\alpha| > 1$ and $|\gamma| < 1 \Rightarrow$ No Bounded Eqm
Solving the Model

- Nature of eqm very different across regions
  - Region I: monetarist & Ricardian
    - the usual regime in which “monetary policy determines price level”
  - Region II: non-monetarist & FTPL
    - the fiscal theory regime in which “fiscal policy determines price level”
  - Region III: non-monetarist & FTPL in all eq
  - Region IV: no bounded eqm unless $z_t^R$ & $z_t^T$ correlated perfectly
    - with lump-sum taxes, debt-GDP can grow without bound, so long as debt satisfies transversality condition
- Language that one policy “determines price level” is misleading and inaccurate
- Monetary & fiscal policy always jointly determine the price level
The system

\[ x_{t+1} = Ax_t + C\xi_{t+1}, \text{ for } t \geq 0 \] (1)

which implies

\[ x_t = A^t x_0 + \sum_{s=0}^{t-1} A^s C\xi_{t-s} \]

Jordan decomposition of \( A \) implies \( A^s = P\Lambda^s P^{-1} \), where the eigenvalues are along the diagonal of \( \Lambda \)

Let \( P^j \) be the \( j \)th row of \( P^{-1} \) and let \( P_{.j} \) be the \( j \)th column of \( P \)

Then system (1) is

\[ x_t = \sum_{j=1}^{n} P_{.j} \lambda_j^t P^j x_0 + \sum_{j=1}^{n} P_{.j} \sum_{s=0}^{t-1} \lambda_j^s P^j C\xi_{t-s} \]
Characterizing Equilibria

\[ x_t = \sum_{j=1}^{n} P_j \lambda_j^t P^j x_0 + \sum_{j=1}^{n} P_j \sum_{s=0}^{t-1} \lambda_j^s P^j C \xi_{t-s} \]

To eliminate explosive eigenvalues (ones where \( |\lambda_j| > 1 \)) we need to impose for each explosive \( j \):

**Stability Conditions**

\[ P^j x_t = 0, \quad t = 0, 1, 2, \ldots \]  \hspace{0.5cm} (2)

or, equivalently,

\[
\begin{align*}
P^j x_0 &= 0 \\
P^j C \xi_t &= 0, \quad t = 1, 2, \ldots
\end{align*}
\]  \hspace{0.5cm} (3a) (3b)
For the $A$ & $C$ matrices in (1), the eigenvectors are

$$P_1^\cdot = \begin{pmatrix} 1 & 0 & \frac{1}{\alpha - \rho_R} & 0 \end{pmatrix}$$

$$P_2^\cdot = \begin{pmatrix} 0 & 1 & 0 & -\frac{\rho_\tau (\beta^{-1} - 1)}{\Gamma - \rho_\tau} \end{pmatrix}$$

We employ these two eigenvectors to obtain the stability conditions

- Region I: use $P_1^\cdot$, (4)
- Region II: use $P_2^\cdot$, (5)

In each region, these eigenvectors yield the linear combination of variables that produce the respective determinate equilibrium.
Active & Passive Policy Behavior

- An active policy authority is free to pursue its objectives, unconstrained by the state of government debt
  - decision rule may depend on past, current, or expected future variables

- A passive policy authority is constrained by the behavior of the active authority and the private sector and must be consistent with equilibrium
  - decision rule necessarily depends on state of government indebtedness, as summarized by current and past variables

- Active forward-looking & passive backward-looking consistent with Simon’s “rule vs. discretion” perspective as put forth by Friedman and with Sargent-Wallace’s terminology
Active MP & Passive FP: Regime M

- Applying eigenvector (4) to stability condition (2) yields

\[ P^1 \cdot x_t = 0 \Rightarrow \pi_t = -\frac{1}{\alpha - \rho_R} z_t^R \]  

- Monetary contraction—higher nominal interest rate—reduces inflation

- Applying eigenvector (4) to stability condition (3b) yields

\[ P^1 \cdot C \xi_{t+1} = 0 \Rightarrow \eta_{t+1} = -\frac{1}{\alpha - \rho_R} \varepsilon_{t+1}^r \]
Active MP & Passive FP: **Regime M**

- Equilibrium is

\[
\pi_t = -\frac{1}{\alpha - \rho_R} z_t^R \\
E_t \pi_{t+1} = -\frac{\rho_R}{\alpha - \rho_R} z_t^R \\
\eta_{t+1} = -\frac{1}{\alpha - \rho_R} \varepsilon_{t+1}^R
\]

- Inflation entirely a monetary phenomenon

- What is FP doing?
  - adjusts future surpluses to cover interest plus principal on debt
  - any shock that changes debt must create the expectation that eventually future surpluses will adjust to stabilize debt’s value
  - for MP to target inflation, expectations must be anchored on FP adjusting to maintain value of debt
Active MP & Passive FP: **Regime M**

- Equilibrium sequences of $\{\tau_t, b_t\}$ determined by the (stable) difference equation in debt and the tax rule
- Taxes & debt irrelevant for inflation $\Rightarrow$ equilibrium exhibits Ricardian equivalence

\[
b_{t+1} = (1 - \Gamma L)^{-1} \left[ - (\beta^{-1} - 1) z_{t+1}^\tau + \frac{1}{\beta(\alpha - \rho_R)} \varepsilon_{t+1}^R \right]
\]

- In expectation, debt converges back to steady state

\[
E_t b_{t+k} = \Gamma^k b_t - (\beta^{-1} - 1) \left( \Gamma^{k-1} \rho_{\tau} + \Gamma^{k-2} \rho_{\tau}^2 + \ldots + \rho_{\tau}^k \right) z_t^\tau
\]

since $|\gamma > 1| \Rightarrow |\Gamma < 1|$, it is straightforward to show that $\lim_{k \to \infty} E_t b_{t+k} = 0$

- Higher debt induced by fiscal *or monetary* shocks is expected to bring forth higher taxes that retire debt back to steady state
- Tax policy provides essential support to monetary policy
Passive MP & Active FP: **Regime F**

- Applying eigenvector (5) to stability condition (2) yields

\[ P^2 \cdot x_t = 0 \Rightarrow b_t = \frac{(\beta^{-1} - 1) \rho_{\tau}}{\Gamma - \rho_{\tau}} z_t^\tau \tag{8} \]

- fiscal contraction—higher current & future taxes—raises debt

- Applying eigenvector (5) to stability condition (3b) yields

\[ P^2 \cdot C\xi_{t+1} = 0 \Rightarrow \eta_{t+1} = -(1 - \beta) \left( \frac{\Gamma}{\Gamma - \rho_{\tau}} \right) \varepsilon_{t+1}^\tau \tag{9} \]
Passive MP & Active FP: Regime F

Equilibrium is

\[ \pi_t = \alpha \pi_{t-1} + z_{t-1}^R - (1 - \beta) \left( \frac{\Gamma}{\Gamma - \rho \tau} \right) \varepsilon_t^\tau \]

\[ E_t \pi_{t+1} = \alpha \pi_t + z_t^R \]

\[ \eta_{t+1} = -(1 - \beta) \left( \frac{\Gamma}{\Gamma - \rho \tau} \right) \varepsilon_{t+1}^\tau \]

Surprise inflation a fiscal phenomenon

What is MP doing?

- permits inflation to adjust to revalue & stabilize debt
- any shock that would raise debt, instead raises inflation because taxes do not respond to debt
- for FP to determine inflation, expectations must be anchored on MP adjusting to maintain value of debt
Passive MP & Active FP: Regime F

- Take $\gamma = 0$: taxes exogenous, so $\tau_t = z_t^\tau$
  - iterate forward on GBC, take expectations at $t$
    $$b_t = \beta E_t b_{t+1} + (1 - \beta) E_t \tau_{t+1} - E_t (R_t - \pi_{t+1})$$
  - using $E_t \tau_{t+k} = \rho_{\tau}^k z_t^\tau$ & $E_t (R_{t+k} - \pi_{t+k+1}) = 0$ yields
    $$b_t = \left( \frac{(1 - \beta) \rho_{\tau}}{1 - \beta \rho_{\tau}} \right) z_t^\tau \quad (8)$$
  - identical to stability condition (8) when $\gamma = 0$
- Suppose economy in steady state up to period $t$ and $\varepsilon_t^\tau \neq 0$
  - if $\rho_{\tau} = 0$, then $b_t = 0$ and from GBC
    $$\left( \beta^{-1} - 1 \right) \tau_t = -\beta^{-1} \pi_t \Rightarrow$$
    $$\left( \beta^{-1} - 1 \right) \varepsilon_t^\tau = -\beta^{-1} (1 - \beta) \varepsilon_t^\tau$$
  - GBC satisfied by having inflation exactly offset tax change
Passive MP & Active FP: Regime F

- Under active FP, tax cuts generate wealth effects
  - at pre-shock prices, lower taxes with no increase in expected taxes, makes households feel wealthier
  - higher wealth induces households to try to raise their consumption paths
  - higher aggregate demand raises goods prices
  - price level rises until wealth effect disappears
  - in equilibrium, no change in real wealth & tax cut raises current inflation
How does MP behavior influence fiscal effects?
Consider \( i.i.d. \) tax cut with \( 0 < \alpha < 1 \)

\[
R_t = \alpha \pi_t + z_t^R
\]

\[
\pi_t = \alpha \pi_{t-1} + z_{t-1}^R - (1 - \beta) \varepsilon_t^T
\]

\( \varepsilon_t^T < 0: \pi_t \uparrow \Rightarrow R_t \uparrow \Rightarrow E_t \pi_{t+1} \uparrow \) & \( \pi_{t+k} = -\alpha^k (1 - \beta) \varepsilon_t^T \)

if MP tries to combat inflation with higher interest rates, it \emph{amplifies} inflation effects of FP

total effect on inflation is

\[
\sum_{j=0}^{\infty} \pi_{t+j} = - \left( \frac{1 - \beta}{1 - \alpha} \right) \varepsilon_t^T
\]

higher \( R_t \) raises nominal debt service; if \( \pi_{t+1} \) were unchanged, this would raise household’s real income and aggregate demand at \( t + 1 \), raising prices at \( t + 1 \) until real income unchanged
Effects of exogenous MP shocks

- consider a MP contraction: $\varepsilon_t^R > 0$
- raises $R_t$; has no effect on $b_t$ & $\pi_t$
- higher $R_t$ raises household’s nominal interest receipts at $t + 1$
- at initial prices, this raises real income and aggregate demand at $t + 1$
- higher aggregate demand raises inflation at $t + 1$
- because $R_{t+j}$ & $\pi_{t+j+1}$ rise by equivalent amounts, the GBC continues to hold with unchanged taxes & debt
- note that the higher price level raises nominal tax revenues to support the expansion in nominal debt
- same phenomenon as in unpleasant arithmetic, but very different mechanism

This is the sense in which in this regime, MP loses its ability to influence the economy in the usual ways
For Completeness: Other Regimes

- Passive MP & Passive FP
  - Eqm is indeterminate and admits bounded sunspot solutions
  - Both MP & FP are stabilizing debt
  - Neither policy is controlling inflation
  - This is the policy regime in Sargent-Wallace’s result about indeterminacy under an interest rate peg
  - Work this out on your own

- Active MP & Active FP
  - No eqm exists with bounded debt
  - Both MP & FP are controlling inflation
  - Neither policy is stabilizing debt
  - An “unsustainable” policy mix
  - Work this out on your own
Decoupled Roots

Inflation & Debt Dynamics Decoupled
Decoupled Roots

- The simplicity of this model makes it appear as though inflation and debt dynamics are decoupled
  - inflation dynamics depend only on $\alpha$, MP parameter
  - debt dynamics depend only on $\gamma$, FP parameter
  - all monetary-fiscal interactions emerge *in equilibrium*
- It is tempting to say that
  - active MP/passive FP implies MP alone determines inflation
  - active FP/passive MP implies FP alone determines inflation
- This is nonsense, but it is widespread
  - Galí & Woodford texts entirely about MP
  - “consensus assignment”: monetary policy controls demand and inflation and fiscal policy controls government debt
  - “monetary policy dominates fiscal policy as a means of controlling inflation”