Example that a valid variogram in Euclidean space yields negative variance on the sphere

Consider an intrinsic stationary process \( X(P) \) on the sphere with power variogram

\[
\text{var}(X(P_1) - X(P_2)) = \theta(P_1, P_2)^\alpha,
\]

where \( P_1, P_2 \in S^2 \) and \( \theta(P_1, P_2) \) is the spherical distance between \( P_1 \) and \( P_2 \). This variogram is valid in Euclidean space when \( \alpha \in [0, 2] \). (Cressie, 1993).

If one considers four points on the sphere: north pole \( (P_N) \), south pole \( (P_S) \) and two opposites points on the equator: \( P_E \) and \( P_W \) with \( \theta(P_E, P_W) = \pi \). The variance of \( X(P_N) + X(P_S) - X(P_E) - X(P_W) \) can be computed as (assuming that mean is constant)

\[
\text{var}(X(P_N) + X(P_S) - X(P_E) - X(P_W)) = \text{E}[(a + b - c - d)^2],
\]

where \( a, b, c, d \) are \( X(P_N), X(P_S), X(P_E), X(P_W) \) respectively.

Note that

\[
(a + b - c - d)^2 = -(a - b)^2 + (a - c)^2 + (a - d)^2 + (b - c)^2 + (b - d)^2 - (c - d)^2,
\]

one can compute

\[
\begin{align*}
\text{var}(X(P_N) + X(P_S) - X(P_E) - X(P_W)) &= -\text{E}(a - b)^2 + \text{E}(a - c)^2 + \text{E}(a - d)^2 + \text{E}(b - c)^2 + \text{E}(b - d)^2 - \text{E}(c - d)^2 \\
&= -\text{var}(X(P_N) - X(P_S)) + \text{var}(X(P_N) - X(P_E)) + \text{var}(X(P_N) - X(P_W)) \\
&\quad + \text{var}(X(P_S) - X(P_E)) + \text{var}(X(P_S) - X(P_W)) - \text{var}(X(P_E) - X(P_W)) \\
&= -\pi^\alpha + (\pi/2)^\alpha + (\pi/2)^\alpha + (\pi/2)^\alpha + (\pi/2)^\alpha - \pi^\alpha = 2\pi^\alpha(2^{1-\alpha} - 1).
\end{align*}
\]

It is clear that this variance is non-negative when \( \alpha \leq 1 \). If \( \alpha = 1 \), the variance is zero, which indicates that the power variogram \( \gamma(\theta) = \theta \) is not strictly conditionally negative definite. In addition, the variances become \( -(2 - \sqrt{2})\pi^{3/2} \) and \( -\pi^2 \) when \( \alpha = 3/2 \) and 2, respectively. Both are negative.