Why use quantum probability to model cognition & decision?

Quantum probability

Quantum dynamics

Quantum entanglement
Organization

• **First**, we will describe a proposed experiment on bi-stable perception that distinguishes quantum vs. classic dynamic based on the temporal Bell inequality.

• **Second**, we will compare Markov and Quantum models by investigating interference effects found in several areas including
  – Categorization and decisions
  – Violations of the sure thing principle of decision making
  – Dynamic inconsistency
Path Diagram Approach

- a Hilbert space view of Dirac & Von Neumann
- a path integration representation view of Feynman
- Mathematically equivalent
Suppose you are a juror trying to judge whether a defendant is guilty or innocent.
Is the defendant Guilty or Innocent?
Single Trajectory Principle:

- Our beliefs jump from one definite state to another.
- At one point definitely favoring guilt, at another point definitely favoring innocent.

Figure 2: Schematic representation of the bistable switching between two states 1 and 2 as a function of time $t$. 
Quantum Information Processing

Is the defendant Guilty or Innocent?
Superposition State

\[ |S\rangle = \text{superposition state} \]
Superposition principle:
• Our beliefs don’t jump from one definite state to another
• Instead we experience a feeling of ambiguity about all of the states simultaneously
Consider the ways to realized the event:

Start at $|0\rangle$

Return to $|0\rangle$ after 2 transitions
Classic Events

-1 → 0 → +1

|0⟩ → |0⟩ → |0⟩
Classic Events

\[ |0 \rangle \rightarrow |+1 \rangle \rightarrow |0 \rangle \]
Classic Events

\[ |0 \rangle \rightarrow |-1 \rangle \rightarrow |0 \rangle \]
Classic Events

\[
\begin{align*}
\text{-1} & \quad \quad \text{0} & \quad \quad \text{+1} \\
\text{Start at } |0\rangle & \quad \quad \text{Return to } |0\rangle \text{ after 2 transitions} \\
\text{When unobserved, either} & \\
|0\rangle & \rightarrow |0\rangle & \rightarrow |0\rangle \\
|0\rangle & \rightarrow |0\rangle & \rightarrow |0\rangle \\
|0\rangle & \rightarrow |+1\rangle & \rightarrow |0\rangle & \rightarrow |0\rangle \\
\text{but no others}
\end{align*}
\]
Quantum Events

Start at $|0\rangle$
Return to $|0\rangle$ after 2 transitions

Cannot say what happens when system is not observed
Quantum Events

Start at $|0\rangle$ at $t = 0$
Return to $|0\rangle$ after 2 transitions
How can we empirically test these two competing principles?

• Test of Temporal Bell Inequality
• Test of Law of Total Probability
1. Test of Temporal Bell Inequality

Bi-stable Perception: Spontaneous switching of perceptions for ambiguous figures

Figure 1: The Necker cube (left) and the two perspectives under which it can be perceived (right).

Figure 5: A classical trajectory assumes at each moment in time a definite state (here one of two possible states). With respect to three instances $t_1$, $t_2$, and $t_3$ it falls into one of $2^3 = 8$ possible classes (cf. Tab. 1, left). For the shown history the states are $(-1, +1, +1)$. 
An X appears in each column when the state changes sign across two time points.

<table>
<thead>
<tr>
<th>$s(t_1)$</th>
<th>$s(t_2)$</th>
<th>$s(t_3)$</th>
<th>$N^-(t_1, t_3)$</th>
<th>$N^-(t_1, t_2)$</th>
<th>$N^-(t_2, t_3)$</th>
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<tr>
<td>+1</td>
<td>+1</td>
<td>+1</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>+1</td>
<td>+1</td>
<td>−1</td>
<td>×</td>
<td>×</td>
<td>×</td>
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<tr>
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<td>−1</td>
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<td>×</td>
<td>×</td>
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<tr>
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<td>−1</td>
<td>×</td>
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</tr>
<tr>
<td>−1</td>
<td>+1</td>
<td>+1</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
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<tr>
<td>−1</td>
<td>−1</td>
<td>+1</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>−1</td>
<td>−1</td>
<td>−1</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
</tbody>
</table>

Table 1: A classical trajectory for a two-state system falls into one of eight possible classes with respect to the states assumed at three different moments of time.

$$N^-(t_1, t_3) < N^-(t_1, t_2) + N^-(t_2, t_3)$$
Predictions

• Temporal Bell inequality

\[ N^-(t_1,t_3) < N^-(t_1,t_2) + N^-(t_2,t_3) \]

• **Classic** theory must **satisfy** temporal Bell inequality

• **Quantum** model can **violate** temporal Bell inequality
2. Test of Total Probability
The Law of Total Probability

• From the **distributive axiom** of Boolean logic, the **law of total probability** can be derived:

\[
p(A) = p(A \cap (B \cup \sim B)) \\
= p(A \cap B \cup A \cap \sim B) \\
= p(A \cap B) + p(A \cap \sim B) \\
= p(B)p(A | B) + p(\sim B)p(A | \sim B)
\]

• This law is fundamental to all **Bayesian** modeling.
Interference Effects

• An empirical test: the single event A alone vs. measuring the joint events (A, B)

• Violations of this law are called *interference effects*
Cumulated Evidence of Interference Effects

• Interference btw categorization- decision (Townsend et al., 2000)
• The disjunction effect (Tversky & Shafir, 1992)
• Dynamic inconsistency in decision making (Barkan & Busemeyer, 1999, 2003)
Cumulated Evidence of Interference Effects

- Interference btw categorization- decision (Townsend et al., 2000)
A recent study by Busemeyer, Wang, & Lambert-Mogiliansky (2009) studied interference effects of categorization on decisions.

The Experiment Task

Participants were shown pictures of faces
1. **Categorize** as ‘Good guy’ or ‘Bad guy’;
2. **Decide** to act Friendly or Attack.
The Experimental Conditions

- **Condition 1**: Make an action decision after categorizing a face (C-then-D)
- **Condition 2**: Make an action decision without reporting any categorization (D-only)
Classic Markov Transition Probabilities

\[ \Pr[G|S] = \text{probability for transiting from } S \rightarrow G. \]
\[ \Pr[G|S] + \Pr[B|S] = 1 \]

\[ \Pr[A|G] = \text{probability for transiting from } G \rightarrow A. \]
\[ \Pr[A|G] + \Pr[F|G] = 1 \]

\[ \Pr[A|B] = \text{probability for transiting from } B \rightarrow A. \]
\[ \Pr[A|B] + \Pr[F|B] = 1 \]
• Suppose we do observe the category inference and we then observe the action
Observe First Path

- **S**: Face
- **G**: categorize as Good guy
- **B**: categorize as Bad guy
- **F**: decide to Act Friendly
- **A**: decide to Attack

Pr[G|S] and Pr[A|G] represent the probabilities of categorization and decision, respectively.
Observe Second Path

S: Face

G: categorize as Good guy

B: categorize as Bad guy

F: decide to Act Friendly

A: decide to Attack

Pr[B|S]

Pr[A|B]
S: Face

G: categorize as Good guy

B: categorize as Bad guy

F: decide to Act Friendly

A: decide to Attack

Pr[G|S] \cdot Pr[A|G] + Pr[B|S] \cdot Pr[A|B]

Probability of Attack from either path for C-D condition:
Single Path Assumption of Classic Models for the D-only Condition

• Suppose we do **not** observe the category inference and we **only** observe the action
  
  – One path from **two possible paths** is used to get from the face to the attack decision
  
  – We **don’t know** which one and so we assign probabilities to each of these **two possibilities**
  
  – The state **is a probability mixture** of these two possibilities
First Path taken (unobserved)

S: Face

G: categorize as Good guy

B: categorize as Bad guy

F: decide to Act Friendly

A: decide to Attack

Pr[G|S]

Pr[A|G]
Second Path taken (unobserved)

- S: Face
- B: categorize as Bad guy
- G: categorize as Good guy
- F: decide to Act Friendly
- A: decide to Attack

Pr[B|S] → Hidden → Pr[A|B]
Model For D- alone (categorization unobserved)

\[
\Pr[A|S] = \Pr[G|S] \cdot \Pr[A|G] + \Pr[B|S] \cdot \Pr[A|B]
\]

Law of Total Probability
Experimental Results

• The **D-only** Condition
  \[ \Pr[A | S] = .69 \]

• The **C-then-D** Condition
  \[ \Pr[G | S] = .17 \]; \[ \Pr[A | G] = .42 \]
  \[ \Pr[B | S] = .83 \]; \[ \Pr[A | B] = .63 \]

• Law of total probability:
  \[ \Pr[A | S] = (.17)(.42) + (.83)(.63) = .59 \]

• **Something is wrong! ??**
Table 1. Choice Probability and Interference Effects in Four Experimental Studies

<table>
<thead>
<tr>
<th>Study</th>
<th>Face stimuli*</th>
<th>n</th>
<th>CD or XD conditions</th>
<th>D-only</th>
<th>C-only</th>
<th>Interference</th>
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<tr>
<td></td>
<td></td>
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<td>P(G)</td>
<td>P(D</td>
<td>G)</td>
<td>P(B)</td>
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<td>S1</td>
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<td>.3801</td>
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<tr>
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<tr>
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<td>.7818</td>
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<td>Total</td>
<td>C-then-D</td>
<td>395</td>
<td>.2303</td>
<td>.3832</td>
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<td>.6055</td>
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<tr>
<td>Total</td>
<td>C-then-D</td>
<td>403</td>
<td>.7803</td>
<td>.3556</td>
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<td>.5258</td>
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<tr>
<td>S4</td>
<td>X-then-D</td>
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<tr>
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<td>242</td>
<td>.6048</td>
<td>.2646</td>
<td>.3952</td>
<td>.5941</td>
</tr>
</tbody>
</table>

Note: * The face stimuli show facial features which were more likely to be associated with the “bad guy” category. n is the number of participants excluding optimizers.
Quantum Transition **Amplitudes**

\[ \langle G | S \rangle = \text{amplitude} \text{ for transiting from } S \rightarrow G. \]

\[ |\langle G | S \rangle|^2 = \text{probability of } S \rightarrow G \text{ transition} \]

\[ |\langle G | S \rangle|^2 + |\langle B | S \rangle|^2 = 1 \]

\[ \langle A | G \rangle = \text{amplitude} \text{ for transiting from } G \rightarrow A. \]

\[ |\langle A | G \rangle|^2 = \text{probability of } G \rightarrow A \text{ transition} \]

\[ |\langle A | G \rangle|^2 + |\langle F | G \rangle|^2 = 1 \]

\[ \langle A | B \rangle = \text{amplitude} \text{ for transiting from } B \rightarrow A. \]

\[ |\langle A | B \rangle|^2 = \text{probability of } B \rightarrow A \text{ transition} \]

\[ |\langle A | B \rangle|^2 + |\langle F | B \rangle|^2 = 1 \]
Quantum Model for the C-D Condition

• Suppose we do observe the **category** inference and we then observe the **action**
Q model when First Path observed

- **S**: Face
  - **G**: categorize as Good guy
  - **B**: categorize as Bad guy
- **A**: decide to be Aggressive
- **F**: decide to Act Friendly

Top Path Probability = $|\langle G | S \rangle \langle A | G \rangle|^2$
Q Model when Second Path Observed

\[ \text{Bottom Path Probability} = \left| \langle B | S \rangle \langle A | B \rangle \right|^2 \]
Quantum Probabilities when path is observed

Probability of Attack when categorization is observed:

Probability of first path = $|<G|S>|^2 \cdot |<A|G>|^2$

Probability of sec path = $|<B|S>|^2 \cdot |<A|B>|^2$

Probability of either path

$= |<G|S>|^2 \cdot |<A|G>|^2 + |<B|S>|^2 \cdot |<A|B>|^2$
Quantum rejection of single path assumption for the D-only condition

- Suppose we do **not** observe the category inference and we **only** observe the action.
- Then we **cannot assume** that one of only two possible paths are taken.
- In this case, the state is **superposed** between the two possible paths.
Quantum Model for D-only Condition

\[ |S\rangle = \langle G|S\rangle \cdot |G\rangle + \langle B|S\rangle \cdot |B\rangle \]

S: Face

F: decide to act Friendly

A: decide to be Aggressive
Quantum Model for D-only Condition

Law of Total Amplitude

\[ \Pr[ A | S ] = |<G|S><A|G> + <B|S><A|B> |^2 \]
Quantum probabilities **disobey** Law of Total Probability

\[
\Pr[A|S] = |<G|S><A|G> + <B|S><A|B>|^2
\]

= \[
|<G|S><A|G>|^2 + |<B|S><A|B>|^2
\]

+ \[
2 \cdot |<G|S><A|G><B|S><A|B>| \cdot \cos(\theta)
\]

**Classic Part**

**Interference Part**
Interference Term: $\langle G|S\rangle\langle A|G\rangle\langle B|S\rangle\langle A|B\rangle$

$\text{Int} = \text{point in Complex Plane}$
Results

\[ |<G|S>| = \sqrt{.17} \; ; \; |<B|S>| = \sqrt{.83} \; ; \]
\[ |<A|B>| = \sqrt{.60} \; ; \; |<A|G>| = \sqrt{.40} \]

**Classic Part** (same as the Law of Total Probability):
\[ |<G|S><A|G>|^2 + |<B|S><A|B>|^2 \]
\[ = (.17)(.42) + (.83)(.63) = .59 \]

Magnitude of **Interference part**:
\[ 2 \cdot |<G|S><A|G><B|S><A|B>| \]
\[ = 2[(.17)(.42)(.83)(.63)^5 = .39 \]

\[ \Pr[ A|S] = .59 + \cos(\theta) \cdot (.39) = .69. \]
\[ \cos(\theta) = .2564 \text{ (amplitudes are } \text{complex}) \]
What determines the angle $\theta$?

- $\theta = 0 \rightarrow$ no interference, classic probabilities
- $\theta > 0 \rightarrow$ positive interference
- $\theta < 0 \rightarrow$ negative interference
A coffee break?
Comparison of Markov and quantum dynamic models

The disjunction effect and Violations of Sure Thing Principle

(Tversky & Shafir, 1992; Shafir & Tversky, 1992)
Savage (1954) Sure Thing Principle

• If option A is preferred over B under the state of the world X
• And option A is also preferred over B under the complementary state Y
• Then option A should be preferred over B even when it is unknown whether state X or Y obtains.
Disjunction effect using Tversky & Shafir (1992) Gambling Paradigm

Chance to play the following gamble twice: Even chance to win $250 or lose $100

• Condition Win:
  – Subjects told ‘Suppose you won the first play’
  – Result: 69% choose to gamble

• Condition Lost:
  – Subjects told ‘Suppose you lost the first play’
  – Result: 59% choose to gamble

• Condition Unknown:
  – Subjects told: ‘Don’t know if you won or lost’
  – Result: 35% choose to gamble
Failure of a 2-D Markov Model

Law of Total Probability:

\[ \Pr[G \mid U] = P[W \mid U] \cdot P[G \mid W] + P[L \mid U] \cdot P[G \mid L] \]
Failure of a 2-D Markov Model

• Total Probability

\[ \Pr(G | U) = P(W | U) \cdot P(G | W) + P(L | U) \cdot P(G | L) \]

⇒ \( P(G | W) = .69 > \Pr(G | U) > P(G | L) = .59 \)

But Tversky and Shafir (1992) found that

\( P(G | U) = .35 < P(G | L) = .59 < P(G | W) = .69 \)

violating law of total probability
2-D Quantum Model

Law of total amplitude:

\[
\text{Pr}[ G \mid U ] = | \langle W \mid U \rangle \langle G \mid W \rangle + \langle L \mid U \rangle \langle G \mid L \rangle |^2
\]
Quantum model account violation of Sure Thing Principle

\[ \text{Pr}[G | U] = |<W|U><G|W> + <L|U><G|L>|^2 \]
\[ = |<W|U>|^2 \cdot |<G|W>|^2 \]
\[ + |<L|U>|^2 \cdot |<G|L>|^2 \]
\[ + \text{Int} \]

\[ \text{Int} = 2 \cdot \text{Re}[<W|U><G|W><L|U><G|L>] \]

To account for Tversky and Shafir (1992) we require
\[ \text{Int} < 0 \]
Tversky and Shafir’s Intuition?

• If you win on first play, you play again because you have extra “house” money
• If you lose on first play, you play again because you need to make up for your losses
• If you don’t know, these two reasons interfere and living you without any reason coming to mind
Quantum Model must satisfy **Double stochasticity**

In particular

\[ |<G|W>|^2 + |<G|L>|^2 = 1 \]

But Tversky & Shafir found that

\[ \text{Pr}[G|W] = 0.69; \, \text{Pr}[G|L] = 0.59 \]

Violates double stochasticity!
What is double stochasticity?
2-D Transition Matrix


- Columns of $T$ must sum to unity
- Row of $T$ do not have to sum to unity
Markov Process

\[
P(t) = T(t) \cdot P(0)
\]

\[
\begin{bmatrix}
P[N|S] \\
P[G|S]
\end{bmatrix} =
\begin{bmatrix}
P[N|W] & P[N|L] \\
P[G|W] & P[G|L]
\end{bmatrix} \cdot 
\begin{bmatrix}
P[W|S] \\
P[L|S]
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
\end{bmatrix}
\]

→ **Obeyes** law of total probability but allows for general transition matrix
Quantum Process

\[ \psi(t) = U(t) \cdot \psi(0) \]

\[
\begin{bmatrix}
\langle A | S \rangle \\
\langle F | S \rangle
\end{bmatrix} =
\begin{bmatrix}
\langle A | G \rangle & \langle A | B \rangle \\
\langle F | G \rangle & \langle F | B \rangle
\end{bmatrix}
\cdot
\begin{bmatrix}
\langle G | S \rangle \\
\langle B | S \rangle
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\langle A | G \rangle \langle G | S \rangle + \langle A | B \rangle \langle B | S \rangle \\
\langle F | G \rangle \langle G | S \rangle + \langle F | B \rangle \langle B | S \rangle
\end{bmatrix}
\]

\(\rightarrow\text{Obeys law of total amplitude and not law of total probability. But } U \text{ must transform a unit length vector } \psi(0) \text{ into another unit length vector } \psi(t).\)

\(\rightarrow\text{To preserve lengths, } U \text{ must be unitary.}\)
Quantum Unitary Matrix

\[ U = \begin{bmatrix} \langle G|W \rangle & \langle G|L \rangle \\ \langle N|W \rangle & \langle N|L \rangle \end{bmatrix} \]

Unitary Matrix

\[ T = \begin{bmatrix} |\langle G|W \rangle|^2 & |\langle G|L \rangle|^2 \\ |\langle N|W \rangle|^2 & |\langle N|L \rangle|^2 \end{bmatrix} \]

Transition Matrix

\[ U^T U = I \]

→ \( T \) must be Doubly stochastic: Both rows and columns of \( T \) must sum to unity
Disjunction effect using prisoner dilemma game (Shafir & Tversky, 1992)

<table>
<thead>
<tr>
<th></th>
<th>you defect</th>
<th>you cooperate</th>
</tr>
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<tbody>
<tr>
<td><strong>other</strong></td>
<td><strong>You: 10</strong></td>
<td><strong>You: 5</strong></td>
</tr>
<tr>
<td><strong>defects</strong></td>
<td><strong>Other: 10</strong></td>
<td><strong>Other: 25</strong></td>
</tr>
<tr>
<td><strong>other</strong></td>
<td><strong>You: 25</strong></td>
<td><strong>You: 20</strong></td>
</tr>
<tr>
<td><strong>cooperates</strong></td>
<td><strong>Other: 5</strong></td>
<td><strong>Other: 20</strong></td>
</tr>
</tbody>
</table>
Condition 1: You *know* the other *defected*, and now you must decide whether to defect or cooperate

Condition 2: You *know* the other *cooperated*, and now you must decide whether to defect or cooperate

Condition 3: You *do not know*, and now you must decide whether to defect or cooperate
Results from 4 Experiments  
(Entries show % to defect)

<table>
<thead>
<tr>
<th>Study</th>
<th>Defect</th>
<th>Cooperate</th>
<th>Unknown</th>
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<tbody>
<tr>
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</tr>
<tr>
<td>Matthew</td>
<td>91</td>
<td>84</td>
<td>66</td>
</tr>
</tbody>
</table>

Violates law of total probability
Violates law of double stochasticity
Another Failure: Both 2-D Models fail to explain PD Game results

- The Markov model fails because the results once again violate the law of total probability
- The quantum model fails because the results once again violate the law of double stochasticity
Compatible vs. Incompatible Measures

• The failed QP model assumes beliefs and actions are **incompatible**

• Previously we assumed that beliefs and actions were represented by different bases within the same 2-D vector space.

• Now we need to switch to a **compatible** representation which requires a 4-D space.
Inference – Action
State Space

I_1A_1 \quad I_1A_2

I_2A_1 \quad I_2A_2

4 dimensional space
Suppose
Observe start at t=0 in state $I_1A_1$
Do not observe during $t = 1$
Observe end at $t=2$ in state $I_2A_2$

These 4 are the only possibilities in 2 steps;
We just don’t know which is true.
Suppose
Observe start at $t=0$ in state $I_1A_1$
Do not observe during $t = 1$
Observe end at $t=2$ in state $I_2A_2$

We cannot say there are only 4 possible ways to get there;
At $t=1$, the state is a superposition of all four;
There is deeper uncertainty.
Compare 4-D Markov and Quantum Models for PD game
Markov Model Assumption 1

Four basis states: \{ |DD\>, |DC\>, |CD\>, |CC\> \}

e.g. \( |DC\> \rightarrow \) you infer that opponent will defect but you decide to cooperate.

\[
\psi = \begin{bmatrix}
\psi_{DD} \\
\psi_{DC} \\
\psi_{CD} \\
\psi_{CC}
\end{bmatrix}
\]

E.g. \( \psi_{DC} \) = Initial probability that the Markov system starts in state \( |DC\> \).

\[
\sum \psi_i = 1
\]
Initial inferences affected by prior information (Markov)

Condition 1
Known Defect

\[ \psi_D = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} \]

Condition 2
Known Coop

\[ \psi_C = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix} \]

Condition 3
Unknown

\[ \psi_U = (.5) \cdot \psi_D + (.5) \cdot \psi_C \]
Quantum Model Assumption 1

Four basis states: \{ |DD\rangle, |DC\rangle, |CD\rangle, |CC\rangle \}

e.g. \: |DC\rangle \rightarrow \text{you infer that opponent will defect but you decide to cooperate.}

\[ \psi = \begin{bmatrix} \psi_{DD} \\ \psi_{DC} \\ \psi_{CD} \\ \psi_{CC} \end{bmatrix} \]

\[ \text{e.g. } \psi_{DC} = \text{Initial probability amplitude that the Quantum system starts in state } |DC\rangle. \]

Probability = \[ |\psi_{DC}|^2 \]

\[ |\psi|^2 = 1 \]
Initial Inferences affected by prior information (Quantum)

- **Condition 1**
  Known Defect
  \[
  \psi_D = \begin{bmatrix}
  \sqrt{1/2} \\
  \sqrt{1/2} \\
  0 \\
  0
  \end{bmatrix}
  \]

- **Condition 2**
  Known Coop
  \[
  \psi_C = \begin{bmatrix}
  0 \\
  0 \\
  \sqrt{1/2} \\
  \sqrt{1/2}
  \end{bmatrix}
  \]

- **Condition 3**
  Unknown
  \[
  \psi_U = \sqrt{.5} \cdot \psi_D + \sqrt{.5} \cdot \psi_C
  \]
Markov Model Assumption 2

Strategy Selection

Output vector $\phi = T \cdot \psi$ Input vector

Transition matrix $T = [T_{ij}]$

Input state $|j\rangle$  \downarrow

Output state $|i\rangle$ \leftarrow

$0 \leq T_{ij} \leq 1$, and $\sum_i T_{ij} = 1$

guarantees that $\phi$ remains a probability distribution.
Strategies affected by game payoffs and processing time

\[ dT(t) / dt = K \cdot T(t) \quad \text{(Kolmogorov Equation)} \]
\[ T(t) = \exp(t \cdot K), \quad K = \text{intensity matrix} \]

The processing time parameter, \( t \), is a free parameter in the model, but it can be manipulated by deadline pressure.

The intensities must satisfy \( k_{ij} \geq 0 \) for \( i \neq j \) and \( \sum_i k_{ij} = 0 \) to guarantee that \( \exp(t \cdot K) \) is a transition matrix.
\[ K_C = K_A + K_B \]

\[ K_A = \begin{bmatrix} K_{Ad} & 0 \\ 0 & K_{Ac} \end{bmatrix} \quad K_{Ai} = \begin{bmatrix} -1 & \mu_i \\ 1 & -\mu_i \end{bmatrix} \]

\[ K_B = \left( \begin{bmatrix} -1 & 0 & +\gamma & 0 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & +\gamma & 0 & -1 \end{bmatrix} \right) \]
Quantum Model Assumption 2

Strategy Selection

\[ \phi = U \cdot \psi \]

unitary matrix

output vector \[ \phi \] = \text{output vector} \[ \psi \]

input vector

Input state \[ |j\rangle \]

output state \[ |i\rangle \]

\[ U = [U_{ij}] \]

\[ U^TU = I \]

guarantees that \( \phi \) remains a probability distribution.
Strategies affected by Game Payoffs and processing time

\[ \frac{dU(t)}{dt} = -i \cdot H \cdot U(t) \] (Schrödinger Equation)

\[ U(t) = \exp(-i \cdot t \cdot H), \quad H = \text{Hamiltonian Matrix} \]

The processing time parameter, \( t \), is a free parameter in the model, but it can be manipulated by deadline pressure.

The Hamiltonian \( H \) must be Hermitian \( H^\top = H \), to guarantee that \( \exp(-i \cdot t \cdot U) \) is a unitary matrix.
\[ H_C = H_A + H_B \]

\[
H_A = \begin{bmatrix}
H_{Ad} & 0 \\
0 & H_{Ac}
\end{bmatrix}
\quad
H_{Ai} = \frac{1}{\sqrt{1+\mu_i^2}} \begin{bmatrix}
\mu_i & 1 \\
1 & -\mu_i
\end{bmatrix}
\]

\[
H_B = \frac{\gamma}{\sqrt{2}} \cdot \left( \begin{bmatrix}
+1 & 0 & +1 & 0 \\
0 & 0 & 0 & 0 \\
+1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & +1 \\
0 & 0 & 0 & 0 & 0 \\
0 & +1 & 0 & 0 & +1
\end{bmatrix} \right)
\]
Markov Model Assumption 3

output vector

\[ T \cdot \psi = \phi = \begin{bmatrix} \phi_{DD} \\ \phi_{DC} \\ \phi_{CD} \\ \phi_{CC} \end{bmatrix} \]

e.g. \( \phi_{DC} \) = final probability that the Markov system ends in state \( |DC\rangle \).

measurement operator for decision to defect

\[ L = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \]

Probability defect = \( L \cdot \phi \)
Markov Prediction

If the opponent is known to defect:
\[ L \cdot \phi_D = L \cdot T_t \psi_D \]

If the opponent is known cooperate:
\[ L \cdot \phi_C = L \cdot T_t \psi_C. \]

Under the unknown condition:
\[
L \cdot \phi_U = L \cdot T_t \psi_U = L \cdot T_t (p \cdot \psi_D + q \cdot \psi_C) \\
= p \cdot L \cdot T_t \psi_D + q \cdot L \cdot T_t \psi_C \\
= (p \cdot L \cdot \phi_D + q \cdot L \cdot \phi_C)
\]

<table>
<thead>
<tr>
<th></th>
<th>Known Defect</th>
<th>Known Coop</th>
<th>Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matthew</td>
<td>91</td>
<td>84</td>
<td>66</td>
</tr>
<tr>
<td>Markov</td>
<td>91</td>
<td>84</td>
<td>between 91 and 84</td>
</tr>
</tbody>
</table>
Markov Prediction

Under the unknown condition:
= (p \cdot L \cdot \phi_D + q \cdot L \cdot \phi_C)

<table>
<thead>
<tr>
<th></th>
<th>Known Defect</th>
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<td>84</td>
<td>between 91 and 84</td>
</tr>
</tbody>
</table>

Quantum Model Assumption 3

output vector

\[
U \cdot \psi = \phi = \begin{bmatrix}
\phi_{DD} \\
\phi_{DC} \\
\phi_{CD} \\
\phi_{CC}
\end{bmatrix}
\]

e.g. \( \phi_{DC} = \) final probability amplitude that the Quantum system ends in state \( |DC\rangle \).

Probability = \( |\phi_{DC}|^2 \)

measurement operator for decision to defect

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Probability defect = \( |M \cdot \phi|^2 \)
Quantum Predictions

If the opponent is known to defect:
\[ |\phi_D|^2 = \phi_D^\dagger \phi_D = (M\phi_D)^\dagger (M\phi_D) = (MU_t\psi_D)^\dagger (MU_t\psi_D) \]

If the opponent is known to cooperate:
\[ |\phi_C|^2 = \phi_C^\dagger \phi_C = (M\phi_C)^\dagger (M\phi_C) = (MU_t\psi_C)^\dagger (MU_t\psi_C). \]

Under the unknown condition:
\[ |\phi_U|^2 = \phi_U^\dagger \phi_U = (M\phi_U)^\dagger (M\phi_U) = (MU_t\psi_U)^\dagger (MU_t\psi_U) \]
\[ = [MU_t(\sqrt{p}\cdot\psi_D + \sqrt{q}\cdot\psi_C)]^\dagger [MU_t(\sqrt{p}\cdot\psi_D + \sqrt{q}\cdot\psi_C)] \]
\[ = (\sqrt{p}\phi_D + \sqrt{q}\phi_C)^\dagger (\sqrt{p}\phi_D + \sqrt{q}\phi_C) \]
\[ = (p\phi_D^\dagger \phi_D + q\phi_C^\dagger \phi_C) + (\sqrt{p}\sqrt{q}\phi_D^\dagger \phi_C + \sqrt{q}\sqrt{p}\phi_C^\dagger \phi_D) \]

<table>
<thead>
<tr>
<th></th>
<th>Known Defect</th>
<th>Known Coop</th>
<th>Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matthew</td>
<td>91</td>
<td>84</td>
<td>66</td>
</tr>
<tr>
<td>Quantum</td>
<td>91</td>
<td>84</td>
<td>69</td>
</tr>
</tbody>
</table>
Quantum Predictions

During the unknown condition:
\[ = (p\phi_D^\dagger \phi_D + q\phi_C^\dagger \phi_C) + (\sqrt{p}\sqrt{q}\phi_D^\dagger \phi_C + \sqrt{q}\sqrt{p}\phi_C^\dagger \phi_D) \]

<table>
<thead>
<tr>
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</tr>
<tr>
<td>Quantum</td>
<td>91</td>
<td>84</td>
<td>69</td>
</tr>
</tbody>
</table>
Cumulated Evidence of Interference Effects

• Dynamic inconsistency in decision making (Barkan & Busemeyer, 1999, 2003)
Dynamic Consistency

• Planned choice = final choice
• Also known as the principle of optimality in dynamic programming
• Required for backward induction used in dynamic programming
Two Stage Decision Task used to Study Dynamic Consistency

The person makes two choices

• a **planned** choice at G1 (before knowing the outcome)

• a **final** choice after experiencing the outcome after G1

G1

+200

G2

0

-100

200

-100

-100

0

G2

0

200

-100
Dynamic Inconsistency Effect

• Following **actual loss**
  Preference change:
  Plan_Not Play $\rightarrow$ Final_Play

• Following **actual gain**
  Preference change:
  Plan_Play $\rightarrow$ Final_Not Play
<table>
<thead>
<tr>
<th>Order of appearance</th>
<th>Gamble values</th>
<th>EV</th>
<th>STD</th>
<th>Planned acceptance (shown as %)</th>
<th>Final acceptance (shown as %)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Anticipated gain</td>
<td>Anticipated loss</td>
</tr>
<tr>
<td>10</td>
<td>200, -220</td>
<td>-10</td>
<td>210</td>
<td>46%</td>
<td>46%</td>
</tr>
<tr>
<td>5</td>
<td>180, -200</td>
<td>-10</td>
<td>190</td>
<td>42</td>
<td>47</td>
</tr>
<tr>
<td>2</td>
<td>200, -200</td>
<td>0</td>
<td>200</td>
<td>56</td>
<td>61</td>
</tr>
<tr>
<td>4</td>
<td>120, -100</td>
<td>10</td>
<td>120</td>
<td>68</td>
<td>73</td>
</tr>
<tr>
<td>14</td>
<td>140, -100</td>
<td>20</td>
<td>120</td>
<td>59</td>
<td>64</td>
</tr>
<tr>
<td>15</td>
<td>200, -140</td>
<td>30</td>
<td>170</td>
<td>60</td>
<td>65</td>
</tr>
<tr>
<td>6</td>
<td>200, -120</td>
<td>40</td>
<td>160</td>
<td>74</td>
<td>73</td>
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<tr>
<td>13</td>
<td>200, -100</td>
<td>50</td>
<td>150</td>
<td>78</td>
<td>79</td>
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<tr>
<td>16</td>
<td>80, -100</td>
<td>-10</td>
<td>90</td>
<td>32</td>
<td>40</td>
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<tr>
<td>8</td>
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<td>-10</td>
<td>110</td>
<td>41</td>
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<td>0</td>
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<td>56</td>
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<td>160, -140</td>
<td>10</td>
<td>200</td>
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<td>68</td>
</tr>
<tr>
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<td>200, -160</td>
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<td>180</td>
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<tr>
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<td>7</td>
<td>180, -100</td>
<td>40</td>
<td>140</td>
<td>66</td>
<td>70</td>
</tr>
<tr>
<td>1</td>
<td>200, -100</td>
<td>50</td>
<td>150</td>
<td>85</td>
<td>85</td>
</tr>
</tbody>
</table>

*The proportions are based on 200 observations for each gamble (except for gamble 1—the practice problem—for which there were 100 observations).
Two models proposed to explain the dynamic inconsistency effect

• **Reference point model**
  
  — Tversky & Kahneman, 1992, *Psychological Science*

• **Quantum model (exactly same model presented earlier in this tutorial)**
  
Model Comparison Problem

• Quantum probability models relax some of the usual restrictions of “classic” probability to account for paradoxical decisions.

• Deciding which model, traditional versus quantum, is better requires consideration of model complexity

• Model comparisons based on Bayes factors provide a balance of accuracy and parsimony
Reference point model

- **Different utility function** used for plan vs. final
  - **Plan**
    - $U(G) = (.5)u(200) + (.5)u(-100)$
    - $U(nG) = 0$
  - **Final after Loss**
    - $U(G) = (.5)u(200 - 100) + (.5)u(-100 - 100)$
    - $U(nG) = u(-100)$
  - **Final after Win**
    - $U(G) = (.5)u(200 + 200) + (.5)u(-100 + 200)$
    - $U(nG) = u(200)$
Three parameters for Reference Point model

• \( x > 0 \)  \( u(x) = x^a \)
  - \( a \) = risk aversion parameter  \( < 1 \)

• \( x < 0 \)  \( u(x) = b \cdot |x|^a \)
  - \( b \) = loss aversion parameter  \( > 1 \)

• \( D = U(G) - U(nG) \)

• \( \text{Prob}(G) = 1/(1 + \exp(-\gamma \cdot D)) \)
  - \( \gamma \) = choice consistency parameter  \( \sim 2 \)

• \((a, b, \gamma)\)
Quantum Model

- **Same utility function** used for plan vs. final
  - Final after Loss
    - $U(G) = (0.5) \cdot u(200 -100) + (0.5) \cdot u(-100 -100)$
    - $U(nG) = u(-100)$
  - Final after Win
    - $U(G) = (0.5) \cdot u(200+200) + (0.5) \cdot u(-100+200)$
    - $U(nG) = u(200)$

- Plan
  - A **Superposition** of winning or losing conditions
Quantum Probability to Gamble following a win

$\psi_w$ is a unit length vector representing the decision maker’s state after a win. Determined by utility for gambling following a win.

$P_G$ is a projector representing the decision to gamble

$$\text{Prob}[\text{gamble after win}] = |P_G \cdot \psi_w|^2$$
Quantum Probability to Gamble following a Loss

$\psi_L$ is a unit length vector representing the decision maker’s state after a loss. Determined by utility for gambling following a loss.

$P_G$ is a projector representing the decision to gamble

$$\text{Prob[ gamble after loss]} = |P_G \cdot \psi_L|^2$$
Interference from Superposition

Plan state is a superposition: \( \sqrt{.5} \cdot \psi_W + \sqrt{.5} \cdot \psi_L \)

\[
\text{Prob[ plan to play second gamble]} \nonumber \\
= |P_G \cdot (\sqrt{.5} \cdot \psi_W + \sqrt{.5} \cdot \psi_L)|^2 \\
= |\sqrt{.5} \cdot P_G \psi_W + \sqrt{.5} \cdot P_G \psi_L|^2 \\
= (.5) \cdot |P_G \psi_W|^2 + (.5) \cdot |P_G \psi_L|^2 + (P_G \psi_W \cdot P_G \psi_L)
\]

Interference = \( (P_G \psi_W \cdot P_G \psi_L) \)
\[
= |P_G \psi_W| \cdot |P_G \psi_L| \cdot \text{Cosine}(W,L)
\]
Three Quantum model parameters

• $x > 0$ $u(x) = x^a$ $a =$ risk aversion parameter
• $x < 0$ $u(x) = b \cdot |x|^a$ $b =$ loss aversion
• $\gamma$ quantum state rotation parameter which produces interference effects (same $\gamma$ used in $H_B$ of the Hamiltonian presented earlier)
• $(a, b, \gamma)$
Comparison of Parameters

- Parameters $a$ and $b$ have the same interpretation for both models
  - Risk aversion for parameter $a < 1$
  - Loss aversion for parameter $b > 1$
- $\gamma$ affects choice probability for both models but in different ways
Model fits to Mean data for 34 gambling conditions

- **Reference point model**
  - $a = 0.8683$, $b = 0.9223$, $\gamma = 2.7$, $R^2 = 0.77$

- **Quantum model with $\gamma = 0$ (no interference)**
  - $a = 0.86$, $b = 2.30$, $R^2 = 0.79$

- **Quantum model $\gamma \neq 0$ (interference)**
  - $a = 0.71$, $b = 2.54$, $\gamma = -4.40$, $R^2 = 0.82$
Bayesian Analysis of Individual Data
Barkan Data used for Model Comparison

• N = 100 participants
• K = 17 different gambles
• Each person played each gamble twice (except the first problem) for a total 33 trials
• Each person made two choices (plan, final) on each trial for 66 trial per person
• (100 people)(66 choices) = 6600 data points
Likelihood grid analysis

- Likelihood for each person calculated from 33 choice pairs (plan & final) (66 choices)
- 100 people \times 66 choices = 6600 data points
- Grid of 41 values for each parameter (described later)
- Compute Bayes factor for each person
Bayes Factor Model Comparison

• Compute the Bayes factor for each person
  \[ BF = \frac{P(D|Q)}{P(D|R)} \]
  \[ P(D|M) = \text{expected likelihood given model} \]

• Uniform prior
• Normal prior
Uniform Prior Results

- Prior Utility
- Prior Loss aversion
- Prior Memory
- Prior GammaQ
- Prior Prospect Threshold
- Bayes Factor pQ/pR
Normal Prior Results
Conclusions

• Bayes factor favors Quantum over Reference point using both priors

• Cannot conclude that quantum model is more complex as measured by a Bayes factor

• **Same** Quantum model applies to the disjunction effects obtained with the two stage gamble **AND** the prisoner dilemma game.

• The reference point model only applies to the two stage gamble and does **NOT** apply to the prisoner dilemma game.
Why use quantum probability to model cognition & decision?

Quantum probability

Quantum dynamics

Quantum entanglement
A coffee break?

Spooky action at a distance? Just give me a good cup of Quantum Coffee and I am good!
Signal Detection Task

• Image is presented at time $t_0$.
  – satellite or radar or sonar images for military
  – X-ray or mammogram or MRI images for medicine
  – meter readings for a nuclear engineer
  – scanning baggage or checking cars for security
  – etc.

• Decision maker samples information across time until some deadline time $t_1$.

• Decide whether signal is present or not, and rate confidence on a 7 point scale.

• Rewarded for **Hits** or **Correct Rejections**, punished **False Alarms** or **Misses**.
Random Walk/Diffusion Models of Decision Making

- Neural Activation
  - Shadlen (2003)
  - Schall (2003)
- Sensory Processing
  - Smith (1995)
  - Rudd (1996)
- Perceptual Discrimination
  - Usher & McClelland (2001)
  - Link & Heath (1975)
- Memory Recognition
  - Ratcliff (1978)
- Categorization
  - Nosofsky & Palmeri (1997)
  - Ashby (2000)
- Risky Decision Making
  - Busemeyer & Townsend (1993)
- Multiattribute Decisions
  - Roe, Busemeyer, Townsend (2001)
  - Diederich (1997)
Random Walk Model of Decision

Seven states:

-3 \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow +1 \rightarrow +2 \rightarrow +3

p = probability step up toward signal
q = probability step down to no signal

-3 \rightarrow strongly confident NO signal
-2 \rightarrow confident NO signal
-1 \rightarrow weakly confident NO signal is not present
0 \rightarrow uncertain
+1 \rightarrow weakly confident YES signal
+2 \rightarrow confident YES signal
+3 \rightarrow strongly confident YES signal
100 states of confidence
Hypothetical Sample Path

Graph showing evidence state over time with decision time and signal/noise thresholds.
The set of confidence level state vectors
\[ \Omega = \{ |-m \rangle, |-m+1 \rangle, \ldots, |-1 \rangle, |0 \rangle, |+1 \rangle, \ldots, |j \rangle, \ldots, |m-1 \rangle, |m \rangle \} \]
forms an **orthonormal basis** of a \(2m+1\) vector space for both systems.

Each vector represents a level of confidence.
For the **Markov** model, the distribution of **probability** at time $t$ is a linear transformation of the initial distribution, transformed by a **transition** matrix $T$

$$P(t) = T(t) \cdot P(0)$$

For the **Quantum** model, the distribution of **amplitudes** at time $t$ is a linear transformation of the initial distribution, transformed by a **unitary** matrix $U$

$$\psi(t) = U(t) \cdot \psi(0)$$
For the **Markov** model, the transition probabilities satisfy an important property known as the **Chapman–Kolmogorov** Equation:

\[ T(s+t) = T(s) \cdot T(t). \]

For the **Quantum** model, the Unitary operator satisfies the same **group** property

\[ U(s+t) = U(s) \cdot U(t) \]

This is the Quantum analogue of the Chapman-Kolmogorov Equation.
For the **Markov** model, the group property implies that one can derive a differential equation known as the **Kolmogorov** Forward Equation

\[
\frac{d}{dt} P(t) = Q \cdot P(t) \quad \text{with a solution} \quad P(t) = e^{Q \cdot t} \cdot P(0)
\]

For the **Quantum** model, the group property implies that one can derive a differential equation known as the **Schrödinger** Equation:

\[
\frac{d}{dt} \psi(t) = -i \cdot H \cdot \psi(t) \quad \text{with a solution} \quad \psi(t) = e^{-i \cdot H \cdot t} \cdot \psi(0)
\]
For the Markov model, \[ \sum_i q_{ij} = 0 \]

For the Quantum model, \[ h_{ij} = h_{ji}^* \text{ (H is Hermitian)} \]

Distance dependent **Intensities** for the Markov model:
\[ q_{ij} \neq 0 \text{ only for } |i - j| \leq 1. \]

Distance dependent **Hamiltonian** of the Quantum model:
\[ h_{ij} \neq 0 \text{ only for } |i - j| \leq 1 \]
The **Intensity Matrix** $Q$ of the Markov model is defined by

$$q_{j-1,j} = \frac{1}{2} \left( \frac{\sigma^2}{\Delta^2} - \frac{\mu}{\Delta} \right), \quad q_{j+1,j} = \frac{1}{2} \left( \frac{\sigma^2}{\Delta^2} + \frac{\mu}{\Delta} \right), \quad q_{jj} = -\frac{\sigma^2}{\Delta^2}$$

For example with $2m+1 = 5$ states with reflective bounds

$$Q = \begin{bmatrix}
-\frac{\sigma^2}{\Delta^2} & \frac{\sigma^2}{2\Delta^2} - \frac{\mu}{2\Delta} & 0 & 0 & 0 \\
\sigma^2 & \frac{\sigma^2}{2\Delta^2} - \frac{\mu}{2\Delta} & \frac{\sigma^2}{2\Delta^2} - \frac{\mu}{2\Delta} & 0 & 0 \\
\frac{\sigma^2}{\Delta^2} & \frac{\sigma^2}{2\Delta^2} - \frac{\mu}{2\Delta} & \frac{\sigma^2}{2\Delta^2} - \frac{\mu}{2\Delta} & 0 & 0 \\
0 & \frac{\sigma^2}{2\Delta^2} + \frac{\mu}{2\Delta} & -\frac{\sigma^2}{\Delta^2} & \frac{\sigma^2}{2\Delta^2} - \frac{\mu}{2\Delta} & 0 \\
0 & 0 & \frac{\sigma^2}{2\Delta^2} + \frac{\mu}{2\Delta} & -\frac{\sigma^2}{\Delta^2} & \frac{\sigma^2}{\Delta^2} \\
0 & 0 & 0 & \frac{\sigma^2}{2\Delta^2} + \frac{\mu}{2\Delta} & -\frac{\sigma^2}{\Delta^2}
\end{bmatrix}$$
The Hamiltonian $H$ of the Quantum model is defined by

$$h_{ij} \neq 0 \text{ only for } |i - j| \leq 1$$

$$h_{ii} = \mu_i \text{ and } h_{i,i+1} = h_{i,i-1} = \sigma$$

For example with $2m+1 = 5$ states

$$H = \begin{bmatrix}
\mu_{-2} & \sigma^2 & 0 & 0 & 0 \\
\sigma^2 & \mu_{-1} & \sigma^2 & 0 & 0 \\
0 & \sigma^2 & \mu_0 & \sigma^2 & 0 \\
0 & 0 & \sigma^2 & \mu_1 & \sigma^2 \\
0 & 0 & 0 & \sigma^2 & \mu_2
\end{bmatrix}$$
• If we define the level of confidence as $x = j \cdot \Delta$ and we let the step size $\Delta \to 0$ in the limit, then the discrete state process converges to a continuous state process.

• For continuous states, the **Kolmogorov Forward** Equation corresponding to the Markov process is

$$\frac{\partial}{\partial t} P(t, x) = \frac{\sigma^2}{2} \cdot \frac{\partial^2}{\partial x^2} P(t, x) - \mu \cdot \frac{\partial}{\partial x} P(t, x)$$

• The **Schrödinger** equation corresponding to the Quantum process is

$$\frac{\partial}{\partial t} \psi(t, x) = i\{ \lambda^2 \frac{\partial^2}{\partial x^2} \psi(t, x) - v(x) \cdot \psi(t, x) \}$$
Response time distributions of Markov Model with absorbing bounds

Stoping Time Density

Stoping Time Distribution

Correct
Incorrect

Correct
Incorrect
Quantum model
Response time distribution of Quantum model with Measurement at each 10 msec time step
Critical Test

Condition 1: Only take **one measure** at time \( t_f \).
Condition 2: Take **two measures** \( t_i < t_f \).

Condition 1: Estimate the probability distribution over responses at time \( t_f \):
\[
\Pr[R(t_f) = k \mid \text{C1}].
\]

Condition 2: Estimate the probability distribution over responses at time \( t_f \) *(law of total probability)*:
\[
\Pr[R(t_f) = k \mid \text{C2}]
= \sum_{k'} \Pr[R(t_f) = k \mid R(t_i) = k'] \cdot \Pr[R(t_i) = k']
\]
Markov Model:

\[
\begin{align*}
\Pr[ R(t_f) = k \mid C2] &= \sum_{k'} \Pr[ R(t_f) = k \cap R(t_i) = k'] \\
&= \sum_{k'} 1 \cdot M^k \cdot T(t_f - t_i) \cdot M^{k'} \cdot P(t_i) \\
&= 1 \cdot M^k \cdot T(t_f - t_i) \cdot \sum_{k'} M^{k'} \cdot P(t_i) \\
&= 1 \cdot M^k \cdot T(t_f - t_i) \cdot (\sum_{k'} M^{k'}) \cdot P(t_i) \\
&= 1 \cdot M^k \cdot T(t_f - t_i) \cdot I \cdot P(t_i) \\
&= 1 \cdot M^k \cdot T(t_f - t_i) \cdot P(t_i) \\
&= 1 \cdot M^k \cdot T(t_f - t_i) \cdot T(t_i) \cdot P(0) \\
&= 1 \cdot M^k \cdot T(t_f) \cdot P(0) \\
&= \Pr[ R(t_f) = k \mid C1].
\end{align*}
\]
Markov model

- Obeys the law of total probability
- Violates the law of double stochasticity
- Violates the law of reciprocity
Quantum Model

\[
\Pr[ R(t_f) = k \mid C2] = \sum_{k'} \Pr[ R(t_f) = k \cap R(t_i) = k' ] ,
= \sum_{k'} |M^k \cdot U(t_f - t_i) \cdot M^{k'} \cdot U(t_i) \psi(0) |^2 ,
\neq | \sum_{k'} M^k \cdot U(t_f - t_i) \cdot M^{k'} \cdot U(t_i) \psi(0) |^2 ,
= | M^k \cdot U(t_f - t_i) \cdot \sum_{k'} M^{k'} \cdot U(t_i) \psi(0) |^2 ,
= | M^k \cdot U(t_f - t_i) \cdot (\sum_{k'} M^{k'}) \cdot U(t_i) \psi(0) |^2 ,
= | M^k \cdot U(t_f - t_i) \cdot I \cdot U(t_i) \psi(0) |^2 ,
= | M^k \cdot U(t_f - t_i) \cdot U(t_i) \psi(0) |^2 ,
= | M^k \cdot U(t_f) \cdot \psi(0) |^2 ,
= \Pr[ R(t_f) = k \mid C1] .
\]
Interference Effects with Quantum Model

Three different Initial States

Position

Probability

Initial
Final
Quantum Dynamic models

- Violate the law of total probability
- Violate the temporal Bell inequality
- Obey the law of double stochasticity
- Obey Law of reciprocity