KNOTS WHICH ARE NOT CONCORDANT TO THEIR REVERSES

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[Received 30th March 1981]

If $K$ is an oriented knot in $S^3$, the reverse of $K$, $K^*$, is the knot $K$ with its orientation reversed. (This has traditionally been called the inverse of $K$. We call it the reverse to distinguish it from the inverse to $K$ in the knot concordance group, denoted by $-K$ and represented by the mirror image of $K$ with orientation reversed.) Fox [3] asked for an example of a knot which is not isotopic to its reverse. Trotter provided the first examples in [8]. In this paper we will give examples of knots which are not concordant to their reverses.

Finding such an example is difficult on two accounts. A knot and its reverse have equivalent images in the algebraic concordance group defined by Levine [6]. This follows from the fact that if $V$ is a Seifert matrix for $K$, then $V^t$ is a Seifert matrix for $K^*$. In [7] Levine defined a complete set of invariants for the algebraic concordance group, none of which are changed by taking the transpose of an element. A second difficulty is that the branched covers of $S^3$ branched along $K$ and $K^*$ are identical. Hence a direct application of the techniques of Casson and Gordon [1, 2] will not work.

Our approach is to use the refinement of the Casson–Gordon technique which was developed by Gilmer [4]. Throughout this work we will use the results and notation of [4]. Thanks are due to Pat Gilmer for many informative and helpful conversations, and to Cameron Gordon for pointing out this problem.

1. Statement of results

Consider the knot $J$ illustrated in Figure 1. It is constructed as the boundary of a surface $F$ built by adding two untwisted bands to a disk as indicated. One of the bands is tied in a knot $L_1$, the other in a knot $L_2$. The main result of this paper, the proof of which is contained in sections 2 and 3, is the following.

**Theorem 1.** If $J$ is concordant to $J^*$ then either $\sigma_{(1/3)}(L_1) = 0$, $\sigma_{(1/3)}(L_2) = 0$, or $\sigma_{(1/3)}(L_1) = \sigma_{(1/3)}(L_2)$.

**Remark.** Notice that if either $L_1$ or $L_2$ is a slice knot then $J$ is also a slice knot and is certainly concordant to $J^*$. If $L_1 = L_2$ then $J$ is isotopic.
to $J^*$. The isotopy consists of a 180 degree rotation about a vertical axis in the plane of the picture, followed by appropriate twists in each band.

The proof of this theorem consists of showing that if the knot $J\#-J^*$ is slice then one of these three conditions is satisfied. Figure 2 illustrates $J\#-J^*$ along with a basis $\{a, b, c, d\}$ for $H_1(F)$, where $F$ is the evident Seifert surface.

We will now summarize the main result of [4]. Let $K$ be a knot with Seifert surface $F$. If $V$ is a Seifert matrix for $F$ then $V$ determines a pairing $\theta: H_1(F) \times H_1(F) \to \mathbb{Z}$ and a symmetric bilinear form on $H_1(F)$, $\beta$, given by $V + V^t$. Define $\varepsilon: H_1(F) \to H^1(F)$ by $\varepsilon(x)(y) = \beta(x, y)$ and let $A = \ker (\varepsilon \otimes \text{id}(\mathbb{Z}/\mathbb{Z}))$. $H^1(L; \mathbb{Z}/\mathbb{Z})$ is isomorphic to $A$, where $L$ is the 2-fold branched cover of $S^3$ branched over $K$. Let $A'$ be the subset of
elements of $A$ with prime power order, and $\tau(K, \chi)$ the invariant defined by Casson and Gordon [1, 5]. We have the following.

**Proposition** (Gilmer [4]). If $K$ is a slice knot there is a direct summand $H$ of $H_1(F)$ such that 1) $2 \operatorname{rank}(H) = \operatorname{rank}(H_1(F))$, 2) $\theta(H \times H) = 0$, and 3) for all $\chi \in \Lambda' \cap H \otimes Q/Z$, $\tau(K, \chi) = 0$.

In section 2 of this paper we will find all summands $H$ satisfying conditions 1 and 2 of the above proposition for the knot $J \# J^\ast$. In section 3 the value of $\tau(J \# J^\ast, \chi)$ will be computed for the appropriate $\chi$.

2. Null summands for the Seifert form of $J \# J^\ast$

With respect to the ordered basis $(a, b, c, d)$ of $H_1(F)$ indicated in Figure 2, the Seifert matrix for $J \# J^\ast$ is given by the $4 \times 4$ matrix

$$V = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix}$$

According to Levine [7] finding 2 dimensional null summands for $V$ is equivalent to finding 2 dimensional subspaces of $Q^4$ on which the bilinear form $\beta$ vanishes and which are invariant under the transformation $T = V^{-1}V$. Notice that if two direct summands of $Z^n$ have the same span in $Q^n$ they are identical.

In the present case we have

$$T = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \oplus \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

and $\beta$ is given by the matrix

$$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$$

The transformation $T$ has eigenvectors $v_1 = (1, 0, 0, 0)$ and $v_2 = (0, 0, 1, 0)$ with eigenvalue 2, and eigenvectors $w_1 = (0, 1, 0, 0)$ and $w_2 = (0, 0, 0, 1)$ with eigenvalue $\frac{1}{2}$. In order to find the invariant subspaces of $T$ in $Q^4$ we first find the invariant subspaces of $T$ considered as a transformation on $C^4$.

**Lemma 1.** Let $T$ be a transformation of a 4 dimensional complex vector space $W$ such that $W$ splits as the direct sum of 2 dimensional eigenspaces with distinct eigenvalues. Any 2 dimensional invariant subspace $Y$ of $W$ is spanned by eigenvectors.

**Proof.** As $Y$ is invariant under $T$, $T$ restricted to $Y$ has an eigenvector,
which must be an eigenvector of $T$ acting on $W$. Denote it by $x_1$. $W$ is spanned by eigenvectors $x_1$ and $x_2$ of eigenvalue $\lambda_1$ and eigenvectors $y_1$ and $y_2$ of eigenvalue $\lambda_2$. $Y$ is hence spanned by $x_1$ and $a_1x_1 + a_2x_2 + b_1y_1 + b_2y_2$. A change of basis shows that $Y$ is also spanned by $x_1$ and $a_2x_2 + b_1y_1 + b_2y_2$. As $Y$ is invariant under $T$, $T(a_2x_2 + b_1y_1 + b_2y_2) = m_1x_1 + m_2(a_2x_2 + b_1y_1 + b_2y_2)$. This gives \( \lambda_1a_2x_2 + \lambda_2b_1y_1 + \lambda_2b_2y_2 = m_1x_1 + m_2(a_2x_2 + b_1y_1 + b_2y_2) \). Clearly $m_1 = 0$, so $a_2x_2 + b_1y_1 + b_2y_2$ must be an eigenvector also.

Letting \( \langle \ ) \rangle \) denote the span of vectors, the possible invariant subspaces of $T$ acting on $C^4$ are \( \langle v_1, v_2 \rangle \), \( \langle w_1, w_2 \rangle \), \( \langle v_2, \alpha_1w_1 + \alpha_2w_2 \rangle \), \( \langle w_2, \beta_1v_1 + \beta_2v_2 \rangle \), and \( \langle v_1 + t_1v_2, w_1 + t_2w_2 \rangle \). If we use $\beta$ to denote the Hermitian form on $C^4$ determined by $\beta = V + V^*$, we find

\[
\beta(v_i, v_j) = 0 \quad \beta(w_i, w_j) = 0 \quad \beta(v_i, w_j) = (-1)^3 \delta_{ij}
\]

for any $i$ and $j$, with $\delta_{ij}$ being $1$ or $0$ depending on whether or not $i$ equals $j$.

Applying this calculation immediately gives the following.

**Lemma 2.** The 2 dimensional subspaces of $C^4$ which are invariant under $T$ and on which $\beta$ vanishes are: A) \( \langle v_1, v_2 \rangle \), B) \( \langle w_1, w_2 \rangle \), C) \( \langle v_2, w_1 \rangle \), D) \( \langle v_1, w_2 \rangle \), and E) \( \langle v_1 + tv_2, w_1 + (1/t)w_2 \rangle \), where $t$ is any nonzero complex number.

In each of these cases we must now determine all 2 dimensional rational subspaces, and then generators of the associated $Z$ summand of $H_i(F)$. In case A, if $av_1 + bv_2 \in Q^4$ it is clear that both $a$ and $b$ must be rational. The $Z$ direct summand contained in the rational span of $v_1$ and $v_2$ is $H_A = \langle (1, 0, 0, 0), (0, 0, 1, 0) \rangle$. Cases B, C, and D are similar, giving $H_B = \langle (0, 1, 0, 0), (0, 0, 1, 0) \rangle$, $H_C = \langle (0, 1, 0, 0), (0, 0, 1, 0) \rangle$, and $H_D = \langle (1, 0, 0, 0), (0, 0, 1, 0) \rangle$. In case E, if $a(v_1 + tv_2) + b(w_1 + (1/t)w_2) \in Q^4$, consideration of the first coordinate gives that $a$ is rational. From the second coordinate we see that $b$ is rational. Looking at the third coordinate then shows that $t$ is necessarily rational also. Writing $t = p/q$ in lowest terms we have the rational space $\langle (1, 0, 0, q/p), (0, 1, 0, q/p) \rangle$. The associated $Z$ summand is $H_E = \langle (q, 0, p, 0), (0, p, 0, q) \rangle$.

Summarizing, we have

**Lemma 3.** The rank 2 direct summands of $H_i(F)$ on which the Seifert form for $J \# -J^*$ vanishes are: $H_A = \langle (1, 0, 0, 0), (0, 0, 1, 0) \rangle$, $H_B = \langle (0, 1, 0, 0), (0, 0, 1, 1) \rangle$, $H_C = \langle (0, 1, 0, 0), (0, 0, 1, 0) \rangle$, $H_D = \langle (1, 0, 0, 0), (0, 0, 1, 0) \rangle$, and $H_{E,2a} = \langle (q, 0, p, 0), (0, p, 0, q) \rangle$. ($p$ and $q$ relatively prime integers).
3. Calculation of $\tau(J \# - J^*, \chi)$

In order to apply the proposition stated in Section 1 we must identify $A'$. It is a straightforward calculation to show that $A'$ is generated by $\{v_1 \otimes (1/3), v_2 \otimes (1/3), w_1 \otimes (1/3), w_2 \otimes (1/3)\}$. Using Lemma 3 there are the following subgroups of $H_1(F) \otimes Q/Z$ to consider:

\[
\begin{align*}
G_A &= \langle v_1 \otimes (1/3), v_2 \otimes (1/3) \rangle \\
G_B &= \langle w_1 \otimes (1/3), w_2 \otimes (1/3) \rangle \\
G_C &= \langle v_2 \otimes (1/3), w_1 \otimes (1/3) \rangle \\
G_D &= \langle v_1 \otimes (1/3), w_2 \otimes (1/3) \rangle \\
G_{E,p,q} &= \langle v_1 \otimes (q/3) + v_2 \otimes (p/3), w_1 \otimes (p/3) + w_2 \otimes (q/3) \rangle
\end{align*}
\]

Notice that in the last case if either $p$ or $q$ is divisible by 3 (one of the two is not divisible by 3 as $p$ and $q$ are relatively prime) $G_{E,p,q}$ is equal to one of the other four subgroups. From now on we will assume that both $p$ and $q$ are relatively prime to 3.

In order to evaluate $\tau(J \# - J^*, \chi)$ we use the additivity of $\tau$, ([4, Proposition 3.2]) $\tau(K_0 \# K_1, \chi_0 + \chi_1) = \tau(K_0, \chi_0) + \tau(K_1, \chi_1)$, and the fact that $\tau(K, 0) = 0$. The character determined by $v_1 \otimes (1/3)$ is in $G_A$ and $G_D$, the character determined by $w_1 \otimes (1/3)$ is in $G_B$ and $G_C$, and the character determined by $v_1 \otimes (q/3) + v_2 \otimes (p/3)$ is in $G_{E,p,q}$. Hence, one of the following conditions must hold if $J \# - J^*$ is slice.

1) $\tau(J, v_1 \otimes (1/3)) = 0$
2) $\tau(J, w_1 \otimes (1/3)) = 0$
3) $\tau(J, v_1 \otimes (q/3)) + \tau(-J^*, v_2 \otimes (p/3)) = 0$

According to [4, Theorem 3.5], if genus $(F) = 1$ and $\chi = x \otimes (s/m)$ with $x$ primitive, then

$\tau(K, \chi) = \rho(2\sigma_{(w/m)}(J_x) + \frac{4(m-s)s}{m^2} \theta(x, x) - \sigma_{(1/2)}(K))$,

where $J_x$ is any simple closed curve on $F$ representing $x$ and $\rho$ is the injection of $Q$ into the Witt group in which $\tau(K, \chi)$ is defined.

In the case we are studying, $J$ is algebraically slice, so $\sigma_{1/2}(J) = \sigma_{1/2}(-J^*) = 0$. Also, $\theta(v_i, v_i) = \theta(w_i, w_i) = 0$, $i = 1$ or 2. Finally, $J_{v_1} = L_2, J_{w_1} = L_1, J_{v_2} = -L_1$, and $J_{w_2} = -L_2$. Hence, as $\rho$ is an injection, the 3 conditions above become:

1) $\sigma_{(1/3)}(L_2) = 0$
2) $\sigma_{(1/3)}(L_1) = 0$
3) $\sigma_{(w/3)}(L_2) + \sigma_{(p/3)}(-L_1) = 0$

Using the facts that $\sigma_{(1/3)}(K) = \sigma_{(2/3)}(K)$ for any knot $K$, and $\sigma_{(1/3)}(-K) = -\sigma_{(1/3)}(K)$, we finally achieve the result stated in Theorem 1.

The simplest example of a knot which is not concordant to its reverse
which this theorem gives is the knot which is formed when $L_1$ is a right handed trefoil knot and $L_2$ is a left handed trefoil knot.

REFERENCES


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