CONCORDANCE CROSSCAP NUMBERS OF KNOTS
AND THE ALEXANDER POLYNOMIAL

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Abstract. For a knot $K$ the concordance crosscap number, $c(K)$, is the minimum crosscap number among all knots concordant to $K$. Building on work of G. Zhang, which studied the determinants of knots with $c(K) < 2$, we apply the Alexander polynomial to construct new algebraic obstructions to $c(K) < 2$.

With the exception of low crossing number knots previously known to have $c(K) < 2$, the obstruction applies to all but four prime knots of 11 or fewer crossings.

Every knot $K \subset S^3$ bounds an embedded surface $F \subset S^3$ with $F \cong nP^2 - B^2$ for some $n \geq 0$, where $P^2$ denotes the real projective plane. The crosscap number of $K$, $\gamma(K)$, is defined to be the minimum such $n$. The careful study of this invariant began with the work of Clark in [Cl]; other references include [HT, MY1]. The study of the 4–dimensional crosscap number, $\gamma_4(K)$, defined similarly but in terms of $F \subset B^4$, appears in such articles as [MY2, Vi, Ya].

Gengyu Zhang [Zh] recently introduced a new knot invariant, the concordance crosscap number, $\gamma_c(K)$. This is defined to be the minimum crosscap number of any knot concordant to $K$. This invariant is the nonorientable version of the concordance genus, originally studied by Nakanishi [Na] and Casson [Ca], and later investigated in [Li].

In [Zh], Zhang presented an obstruction to $\gamma_c(K) \leq 1$ based on the homology of the 2–fold branched cover of the knot, or equivalently, $\det(K)$. Inspired by her work, in this note we will observe that the obstruction found in [Zh] extends to one based on the Alexander polynomial of $K$, $\Delta_K(t)$, and the signature of $K$, $\sigma(K)$.

**Theorem 1.** Suppose $\gamma_c(K) = 1$ and set $q = |\sigma(K)| + 1$. For all odd prime power divisors $p$ of $q$, the $2p$–cyclotomic polynomial $\phi_{2p}(t)$ has odd exponent in $\Delta_K(t)$. Furthermore, every other symmetric irreducible polynomial $\delta(t)$ with odd exponent in $\Delta_K(t)$ satisfies $\delta(-1) = \pm 1$.

**Proof.** Any knot $K'$ with $\gamma(K') = 1$ bounds a Mobius band and is thus a $(2, r)$–cable of some knot $J$ for some odd $r$. If $K$ is concordant to $K'$, then $\sigma(K) = \sigma(K') = \pm| r | - 1$; the signature $\sigma(K')$ is given by a formula of Shinohara [Sh] for the signature of 2–stranded cable knots. It follows that $| \sigma(K) | = | r | - 1$, so $| r | = | \sigma(K) | + 1 = q$.

According to a result of Seifert [Se], the Alexander polynomial of $K'$ is given by $\Delta_{2,q}(t)\Delta_J(t^2)$, where $\Delta_{2,q}(t)$ is the Alexander polynomial of the $(2, q)$–torus knot.
A standard result states that \( \Delta_{2,q}(t) = \frac{(t^q-1)(t^{q-1})}{(t-1)(t^{q-1})} = \frac{t^q+1}{t+1} \). This can be written as the product of cyclotomic polynomials,

\[
\Delta_{2,q}(t) = \prod_{p|q, ~ p > 1} \Phi_p(t).
\]

Since \( K \) is concordant to \( K' \), \( K \neq \overline{K} \) is slice, and thus has Alexander polynomial of the form \( g(t)g(t^{-1}) \). That is, with \( q = |\sigma(K)| + 1 \),

\[
\Delta_K(t)\Delta_j(t^2)\Delta_{2,q}(t) = g(t)g(t^{-1}).
\]

We now make two observations: (1) Any symmetric irreducible polynomial has even exponent in \( g(t)g(t^{-1}) \), and thus even exponent in \( \Delta_K(t)\Delta_j(t^2)\Delta_{2,q}(t) \); (2) since \( \Delta_j(t) \) is an Alexander polynomial, \( \Delta_j(1) = \pm 1 \), and thus \( \Delta_j(t^2)|_{t=-1} = \pm 1 \).

By Lemma 2 \( \Phi_{2p}(-1) = p \) if \( p \) is an odd prime power, and \( \Phi_{2p}(-1) = \pm 1 \) if \( p \) is an odd composite. Thus, for \( p \) an odd prime power divisor of \( q \), \( \Phi_{2p}(t) \) has odd exponent in \( \Delta_{2,q}(t) \) and does not divide \( \Delta_j(t^2) \), so has odd exponent in \( \Delta_K(t) \).

Any other irreducible factor of \( \Delta_K(t) \) with odd exponent is either a factor \( \delta(t) \) of \( \Delta_{2,q}(t) \), and thus of the form \( \Phi_{2p}(t) \) with \( p \) an odd composite (and so \( \delta(-1) = \pm 1 \)), or else is not a factor of \( \Delta_{2,q}(t) \) and so has odd exponent in \( \Delta_j(t^2) \), and again must satisfy \( \delta(-1) = \pm 1 \). This completes the argument. \( \square \)

**Lemma 2.** The cyclotomic polynomial \( \Phi_{2p}(t) \) satisfies \( \Phi_{2p}(-1) = p \) if \( p \) is an odd prime power and \( \Phi_{2p}(-1) = \pm 1 \) if \( p \) is an odd composite.

**Proof.** For an odd \( r \), \( h_r(t) = \frac{t^r+1}{t+1} \) satisfies \( h_r(-1) = r \) by l'Hôpital’s rule. We have that \( h_r(t) \) is the product

\[
h_r(t) = \prod_{p|r, ~ p > 1} \Phi_p(t).
\]

For \( p \) a prime power, \( s^n \), \( \Phi_{2p}(t) = \frac{t^{r^n}+1}{t^n+1} \), and so, again by l'Hôpital’s rule, \( \Phi_{2s^n}(-1) = s \). Thus, the product

\[
\prod_{p|r, ~ p > 1, ~ p \text{ a prime power}} \Phi_{2p}(-1) = r.
\]

It follows that all the other terms in the product expansion of \( h_r(t) \) must equal \( \pm 1 \) when evaluated at \( t = -1 \), as desired. \( \square \)

**Example.** Theorem 1 is quite effective in ruling out \( \gamma_c(K) = 1 \). For instance, there are 801 prime knots with 11 or fewer crossings. Of these, 51 are known to be topologically slice, and 23 are known to be concordant to a \((2,q)\)-torus knot for some \( q \) and thus have \( \gamma_c = 1 \). Of the remaining 727 knots, all but four can be shown to have \( \gamma_c \geq 2 \). These four are 11n45 and 11n145, both of which are possibly slice, and 940 and 11n66, both of which are possibly concordant to the trefoil. Of the collection of 727 knots, Yasuhara’s result [41] applies to show that 207 of them have 4–ball crosscap number \( \gamma_4(K) \geq 2 \). The 4–ball crosscap numbers of the rest are unknown.

As a second set of examples, consider knots \( K \) with \( \Delta_K(t) \) of degree 2. It follows immediately from Theorem 1 that there are only two possibilities: either \( \sigma(K) = 0 \) and \( \Delta_K \) is reducible (an irreducible symmetric quadratic \( f(t) \) cannot satisfy \( f(1) = \pm 1 \) and \( f(-1) = \pm 1 \)) or \( \sigma(K) = \pm 2 \) and \( \Delta_K(t) = t^2 - t + 1 \).
We conclude with the further special case consisting of \((p, q, r)\)-pretzel knots, \(P(p, q, r)\), with \(p, q\), and \(r\) odd; some of these were studied in [Zh]. If we let 
\[D = D(p, q, r) = pq + qr + rp,\]
then
\[\Delta_{P(p,q,r)}(t) = \frac{D + 1}{4}t^2 - \frac{D - 1}{2}t + \frac{D + 1}{4},\]
which has discriminant \(-D\). Thus, by the previous argument we have:

**Corollary 3.** If \(\gamma_4(P(p, q, r)) = 1\), then either \(\sigma(P(p, q, r)) = 0\) and \(D(p, q, r) = -l^2\) for some integer \(l\) or \(\sigma(P(p, q, r)) = \pm 2\) and \(D(p, q, r) = 3\).

These pretzel knots include some shown by Zhang [Zh] to have 4-dimensional crosscap number \(\gamma_4(K) = 1\).

**REFERENCES**


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