A Strong Künneth Theorem for Topological Periodic Cyclic Homology

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Workshop on K-Theory and Related Fields

Hausdorff Research Institute for Mathematics

June 27, 2017

Overview

Topological periodic cyclic homology (*TP*) is the analogue of periodic cyclic homology (*HP*) using *THH* in place of *HH*. If k is a finite field, then smooth and proper d.g. categories over k satisfy a strong Künneth theorem:

$$TP(X) \wedge^{L}_{TP(k)} TP(Y) \to TP(X \otimes_k Y)$$

is an isomorphism in the derived category of $TP(k)$-modules.

- Joint work with Andrew Blumberg
- Preprint arXiv:1706.06846

Outline

1. Non-commutative derived algebraic geometry
2. Introduction to *TP*
3. The Künneth theorem

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Non-commutative derived algebraic geometry

Basic objects: [small] differential graded (or spectral) categories
Equivalences: Morita equivalences

Example: $A \xrightarrow{\sim} \mathfrak{Mod}_A \xrightarrow{\sim} \mathfrak{Mod}_{\mathfrak{Mod}_A}$

Example: Tilting ${}_AM_B$, ${}_BN_A$ with $M \otimes^L_N N \simeq A$, $N \otimes^L_B M \simeq B$

$$A \xrightarrow{\sim} \mathfrak{Mod}_A \xrightarrow{\sim} \mathfrak{Mod}_B \xleftarrow{\sim} B$$

Goal: Construct/study invariants

Example: K theory of subcategory of compact objects

Example: algebraic variety $X \leftrightarrow$ d.g. cat of perfect complexes $\mathscr{D}^{\mathrm{dg}}_{\mathrm{perf}}(X)$

$$K(X) = K(\mathscr{D}^{\mathrm{dg}}_{\mathrm{perf}}(X))$$

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$\mathscr{C}(X, -)$ commutes with $\amalg$

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How is this algebraic geometry?

Use $\mathscr{D} = \mathscr{D}^{\mathrm{dg}}_{\mathrm{perf}}(X)$ for X

For reasonable X, [some] properties of X equivalent to properties of $\mathscr{D}$

Example: X is proper over $\operatorname{spec} k$ if and only if $\mathscr{D}(a, b)$ is a compact d.g. k-module for all $a, b \in \mathscr{D}$. ($\sum \dim H^n(\mathscr{D}(a, b)) < \infty$)

Example: X is smooth over $\operatorname{spec} k$ if and only if $\mathscr{D}$ is a compact $\mathscr{D}^{\mathrm{op}} \otimes_k \mathscr{D}$-module. ($\mathrm{RHom}_{\mathscr{D}^{\mathrm{op}} \otimes_k \mathscr{D}}(\mathscr{D}, -)$ commutes with $\bigoplus$)

Definition

Let A be a d.g. (or spectral) R-algebra. Then A is:

- *proper* if it is compact as an R-module
- *smooth* if it is compact as an $A^{\mathrm{op}} \otimes^L_R A$-module (or $A^{\mathrm{op}} \wedge^L_R A$-module)

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- Results are about algebraic varieties (and generalizations)
- Proofs often just need the case of d.g. algebras (or ring spectra)

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$$\mathrm{End}\left(\bigoplus_{r=0}^{m} \mathcal{O}(-r)\right) \xrightarrow{\simeq} \mathscr{D}^{\mathrm{dg}}_{\mathrm{perf}}(\mathbb{P}^n)$$

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Hochschild Homology

Cyclic bar construction

$$N^{cy}_q A = \underbrace{A \otimes \cdots \otimes A}_{q \text{ factors}} \otimes A$$

$$A \otimes \cdots \otimes A$$

$$\otimes \qquad\qquad \otimes$$

$$A$$

Chain complex

Cyclic structure $\Longrightarrow$ Connes' B operator $\quad B\colon N^{cy}A \to N^{cy}A[-1]$

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Morita Invariance

Tilting situation ${}_A M_B$, ${}_B N_A$

Dennis-Waldhausen Argument

$$A \otimes \cdots \otimes A$$
$$\otimes \qquad\qquad \otimes$$
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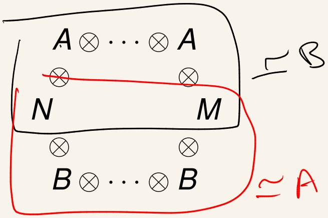

Cyclic structure $\Longrightarrow$ Connes' B operator $\qquad B \colon N^{cy} A \to N^{cy} A[-1]$

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Chain complex

Construct Double Complex:

$$\begin{array}{c} \vdots \\ \downarrow \\ \bullet \\ \downarrow \\ \bullet \\ \downarrow \\ \bullet \\ \downarrow \\ \bullet \end{array}$$

$$HH$$

Cyclic structure $\Longrightarrow$ Connes' B operator $\qquad B \colon N^{cy} A \to N^{cy} A[-1]$

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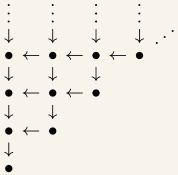

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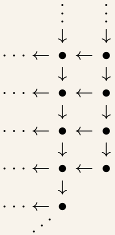


HN

Cyclic structure $\Longrightarrow$ Connes' B operator $\qquad B\colon N^{cy}A \to N^{cy}A[-1]$

Hochschild Homology and Cyclic Homology

Cyclic bar construction

$$N^{cy}_q A = \underbrace{A \otimes \cdots \otimes A}_{q \text{ factors}} \otimes A$$

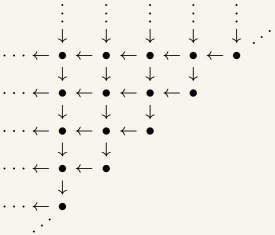
$$A \otimes \cdots \otimes A$$

$$\otimes \qquad\qquad \otimes$$

$$A$$

Chain complex

Construct Double Complex:



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Topological Hochschild Homology

Cyclic bar construction (Bökstedt)

$$N^{cy}_q A = \underbrace{A \wedge \cdots \wedge A}_{q \text{ factors}} \wedge A$$

$$\begin{array}{ccc} & A \wedge \cdots \wedge A & \\ \wedge & & \wedge \\ & A & \end{array}$$

Spectrum

Cyclic structure $\Longrightarrow$ circle group action

Topological Hochschild Homology

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HH corresponds to *THH*

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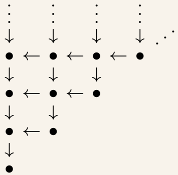

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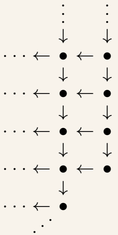

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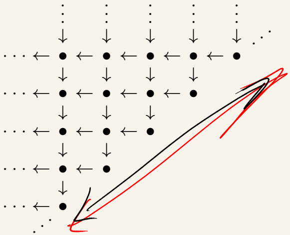

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Topological Periodic Cyclic Homology

Definition

For a ring spectrum A, define the Topological Periodic Cyclic Homology of A by $TP(A) = THH(A)^{t\mathbb{T}}$.

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- Characteristic p replacement for HP (?)

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2017 Tabuada

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Realization functor / Weil cohomology theory

$$HP_*(X) \otimes_{k[t,t^{-1}]} HP_*(Y) \to HP_*(X \otimes_k Y)$$

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Künneth Theorem

Theorem

Lax symmetric monoidal functor

$$TP(X) \wedge^{L}_{TP(R)} TP(Y) \to TP(X \wedge^{L}_{R} Y)$$

Corollary

There are short exact sequences of graded $\mathbb{W}k$*-modules*

$$0 \to (TP_*(X) \otimes_{TP_*(k)} TP_*(Y))_n \to TP_n(X \otimes_k Y) \to \mathrm{Tor}^{TP_*(k)}_{1,n-1}(TP_*(X), TP_*(Y)) \to 0$$

for all n, which split but not naturally.

Corollary

$TP_*(X)[1/p] \otimes_{TP_*(k)[1/p]} TP_*(Y)[1/p] \to TP_*(X \otimes_k Y)[1/p]$ *is an isomorphism.*

Künneth Theorem

Theorem

Let k be finite field. The lax symmetric monoidal ~~functor~~ natural transf.

$$TP(X) \wedge^{L}_{TP(k)} TP(Y) \to TP(X \otimes_k Y)$$

is an isomorphism when X and Y are smooth and proper over k.

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$TP(k) \cong W k[v, v^{-1}]$

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$\hat{\mathbb{Z}}_p$ or ext.

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There are short exact sequences of graded $\mathbb{W}k$-modules $TP_*(k)[1/p] = \mathbb{Q}_p^\wedge[v, v^{-1}]$

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$k = \mathbb{F}_p$ $\quad$ $\mathbb{W}k = \mathbb{Z}_p^\wedge$

$[1/p]$ $\quad$ $\mathbb{Q}_p^\wedge$

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Review of Tate Construction

$E\mathbb{T}$ $\qquad$ $E\mathbb{T}_+ \to S^0 \to \widetilde{E\mathbb{T}}$

Smash with $Z^{E\mathbb{T}}$ and take fixed points

$$(Z^{E\mathbb{T}} \wedge E\mathbb{T}_+)^{\mathbb{T}} \to (Z^{E\mathbb{T}})^{\mathbb{T}} \to (Z^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$

$(X^{E\mathbb{T}} \wedge E\mathbb{T}_+)^{\mathbb{T}} \simeq \Sigma(X^{E\mathbb{T}})_{h\mathbb{T}} \simeq \Sigma X_{h\mathbb{T}}$ (Adams Isomorphism)

Definition

For Z a $\mathbb{T}$-equivariant spectrum $Z^{t\mathbb{T}} = (Z^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$.
(Composite of derived functors.)

$$\Sigma Z_{h\mathbb{T}} \to Z^{h\mathbb{T}} \to Z^{t\mathbb{T}} \to \Sigma^2 Z_{h\mathbb{T}}$$

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- Use diagonal map $E\mathbb{T} \to E\mathbb{T} \times E\mathbb{T}$
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- $\widetilde{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \simeq \widetilde{E\mathbb{T}}$
- Use diagonal map $E\mathbb{T} \to E\mathbb{T} \times E\mathbb{T}$ $\longleftarrow$ This is coherent
- $THH(X) \wedge THH(Y) \cong THH(X \wedge Y)$

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- $\widetilde{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \simeq \widetilde{E\mathbb{T}}$ $\longleftarrow$ This can be made coherent
- Use diagonal map $E\mathbb{T} \to E\mathbb{T} \times E\mathbb{T}$ $\longleftarrow$ This is coherent
- $THH(X) \wedge THH(Y) \cong THH(X \wedge Y)$

$$TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$$
$$TP(X) \wedge TP(R) \wedge TP(Y) \to TP(X \wedge R \wedge Y)$$

The Filtration

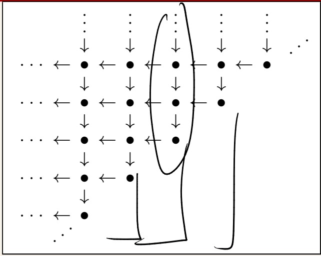
Filtration on $TP(X)$ with associated graded

$$F^i/F^{i-1} \simeq \Sigma^{2i} THH(X)$$

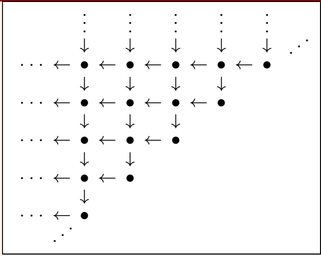
$$TP(X) = (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$

Simplicial filtration on $E\mathbb{T}$

$\mathbb{T}_+, \Sigma^2\mathbb{T}_+, \Sigma^4\mathbb{T}_+, \ldots$



The Filtration



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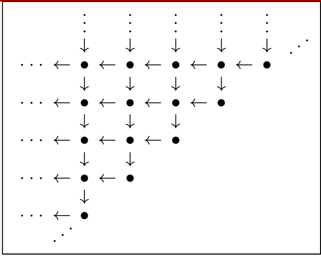

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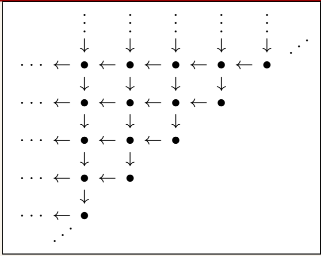

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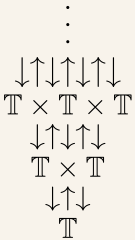

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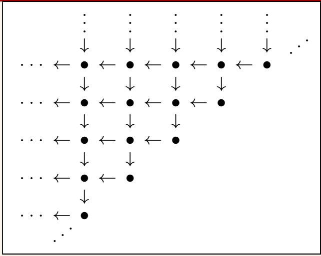

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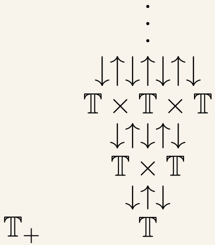

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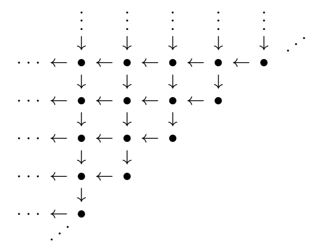


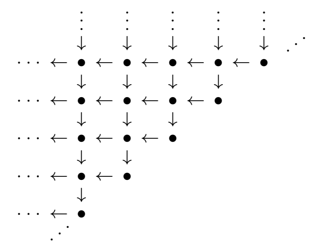
Simplicial filtration on $E\mathbb{T}$

$\mathbb{T}_+$, $\Sigma^2\mathbb{T}_+$, $\Sigma^4\mathbb{T}_+, \ldots$

$$\mathbb{T}_+ \wedge (\mathbb{T}/\{1\}) \wedge \Delta[1]/\partial\Delta[1]$$

$$\begin{array}{c} \vdots \\ \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \\ \mathbb{T}\times\mathbb{T}\times\mathbb{T} \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ \mathbb{T}\times\mathbb{T} \\ \downarrow\uparrow\downarrow \\ \mathbb{T} \end{array}$$

The Filtration



Filtration on $TP(X)$ with associated graded

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Simplicial filtration on $E\mathbb{T}$

$\mathbb{T}_+, \Sigma^2\mathbb{T}_+, \Sigma^4\mathbb{T}_+, \dots$

$$\mathbb{T}_+ \wedge (\mathbb{T} \times \mathbb{T}/(\mathbb{T} \vee \mathbb{T})) \wedge \Delta[2]/\partial\Delta[2] \qquad \begin{array}{c} \vdots \\ \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \\ \mathbb{T}\times\mathbb{T}\times\mathbb{T} \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ \mathbb{T}\times\mathbb{T} \\ \downarrow\uparrow\downarrow \\ \mathbb{T} \end{array}$$

The Filtration

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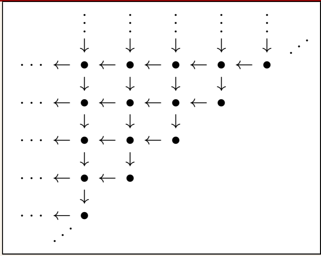

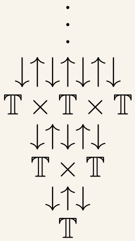

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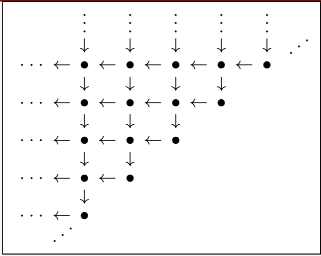

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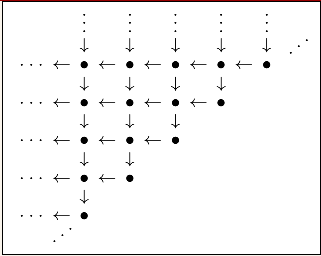
The Filtration



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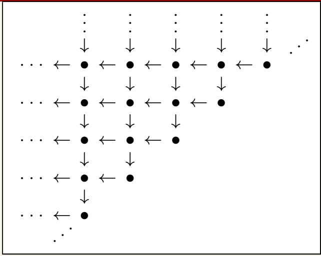
Filtration on $TP(X)$: $F^i TP(X) = \begin{cases} (THH(X)^{(E\mathbb{T}, E\mathbb{T}_{-i-1})} \wedge S^0)^{\mathbb{T}} & i \leq 0 \\ (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}}_i)^{\mathbb{T}} & i > 0 \end{cases}$

$$F^i/F^{i-1} = \begin{cases} (THH(X)^{(\Sigma^{2i}\mathbb{T}_+)})^{\mathbb{T}} & i \leq 0 \\ (THH(X)^{E\mathbb{T}} \wedge \Sigma^{2i-1}\mathbb{T}_+)^{\mathbb{T}} & i > 0 \end{cases}$$

The Filtration



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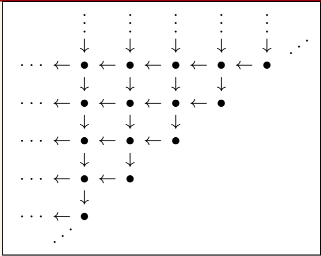

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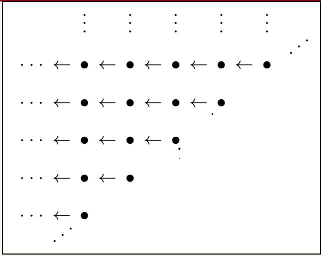
The Spectral Sequence

Filtration on $TP(X)$ with associated graded

$$F^i/F^{i-1} \simeq \Sigma^{2i} THH(X)$$

Spectral sequence

$$E^1_{i,j} = \pi_{i+j}\Sigma^{2i} THH(X) = \underline{THH_{j-i}(X)}$$



$(1,1)$ periodic on E^1

Renumber: Double filtration degree

$$E^{2r}_{2i,j} = (E^r_{i,i+j})^{\text{old}}, \qquad d_{2r} = (d_r)^{\text{old}}$$

Greenlees Tate Spectral Sequence

Conditionally convergent spectral sequence

$$E^2_{2i,j} = THH_j(X) \Longrightarrow TP_{2i+j}(X). \qquad (E^r_{2i+1,j} = 0)$$

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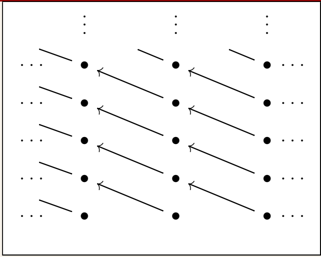

(2, 0) periodic on E^2

Renumber: Double filtration degree

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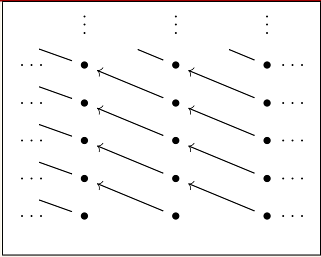

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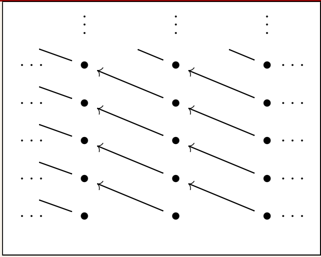

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Combining the Multiplication and Filtration

Multiplicative spectral sequence:

$TP(X) \wedge TP(Y) \to TP(X \wedge Y)$ a filtered map

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$$TP(X) \wedge TP(Y) \to TP(X \wedge Y) \text{ a filtered map}$$

Multiplication	Filtration
• Diagonal $E\mathbb{T} \to E\mathbb{T} \times E\mathbb{T}$ • Mult. $\widetilde{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \simeq \widetilde{E\mathbb{T}}$	• Simplicial/cellular filt. on $E\mathbb{T}$ • Filtration on $\widetilde{E\mathbb{T}}$

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In homotopy category, easy obstruction theory cellular approximation to diagonal & multiplication.

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What about $TP(X) \wedge_{TP(R)} TP(X) \to TP(X \wedge_R Y)$?
$\Longrightarrow$ map of spectral sequences

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Combining the Multiplication and Filtration

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Künneth Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$

$\implies$ map of spectral sequences

Righthand spectral sequence is Tate spectral sequence for

$THH(X \wedge_R Y) \cong THH(X) \wedge_{THH(R)} THH(Y)$

E^2 periodic with $\pi_*(THH(X) \wedge_{THH(R)} THH(Y))$ in each even column

Lefthand spectral sequence has (renumbered) E^2-term

$\pi_* \operatorname{Gr}(TP(X) \wedge_{TP(R)} TP(Y)) \cong \pi_*(\operatorname{Gr} TP(X) \wedge_{\operatorname{Gr} TP(R)} \operatorname{Gr} TP(Y))$

E^2-term is $\pi_* \operatorname{Gr} TP(R)$-module $\implies (2,0)$-periodic

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$$\pi_* \operatorname{Gr}(TP(X) \wedge_{TP(R)} TP(Y)) \cong \pi_*(\operatorname{Gr} TP(X) \wedge_{\operatorname{Gr} TP(R)} \operatorname{Gr} TP(Y))$$

E^2-term is $\pi_* \operatorname{Gr} TP(R)$-module $\implies (2,0)$-periodic

Künneth Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$
$\implies$ map of spectral sequences

Righthand spectral sequence is Tate spectral sequence for

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Proposition

Map of spectral sequences is an isomorphism on E^2

Outline of Proof of Künneth Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$
induces isomorphism of E^2-terms of spectral sequences

RHSS: Tate spectral sequence $\implies$ conditionally convergent.

Theorem

If $R = Hk$, k a perfect field of characteristic $p > 0$, and X and Y are smooth and proper over k, then the LHSS is conditionally convergent.

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Where do we use hypotheses?

- X smooth and proper $\Longrightarrow$ $THH(X)$ compact $THH(R)$-module.
- $TP_*(k) = \mathbb{W}k[v, v^{-1}]$ finite global dimension.
  $THH(X)$ compact $\Longrightarrow$ $TP(X)$ compact.
- Equivariantly, Hk is a compact $THH(Hk)$-module.

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