# A Strong Künneth Theorem for Topological Periodic Cyclic Homology

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Indiana University

MIT Topology Seminar

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# Overview

Topological periodic cyclic homology ($TP$) is the analogue of cyclic periodic homology ($HP$) using $THH$ in place of $HH$. If $k$ is a finite field, then smooth and proper $k$-categories satisfy a strong Künneth theorem

$$TP(X) \wedge^{L}_{TP(k)} TP(Y) \to TP(X \otimes_k Y)$$

is an isomorphism in the derived category of $TP(k)$-modules.

- Joint work with Andrew Blumberg
- Preprint Soon

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1. Introduction to $TP$

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1. Introduction to $TP$
2. Structure and properties of $TP$

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1. Introduction to $TP$
2. Structure and properties of $TP$
3. The Künneth theorem

# Hochschild Homology

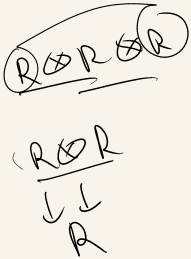
## Cyclic bar construction

$$N^{cy}_q R = \underbrace{R \otimes \cdots \otimes R}_{q \text{ factors}} \otimes R$$

$$R \otimes \cdots \otimes R$$

$$\otimes \qquad\qquad \otimes$$

$$R$$




Chain complex

Cyclic structure $\Longrightarrow$ Connes' $B$ operator $\quad B\colon N^{cy}R \to N^{cy}R[-1]$

# Hochschild Homology

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$$\begin{array}{ccc} & R \otimes \cdots \otimes R & \\ \otimes & & \otimes \\ & R & \end{array}$$

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# Hochschild Homology and Cyclic Homology

Cyclic bar construction

$$N^{cy}_q R = \underbrace{R \otimes \cdots \otimes R}_{q \text{ factors}} \otimes R$$

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Chain complex

Construct Double Complex:

$$\vdots$$
$$\downarrow$$
$$\bullet$$
$$\downarrow$$
$$\bullet$$
$$\downarrow$$
$$\bullet$$
$$\downarrow$$
$$\bullet$$

$HH$

Cyclic structure $\Longrightarrow$ Connes' $B$ operator $\quad B\colon N^{cy}R \to N^{cy}R[-1]$

$B^2 = 0$

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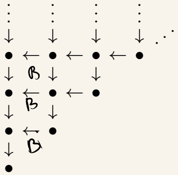

*HC*

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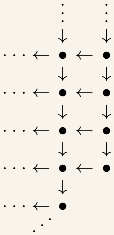

$HN$

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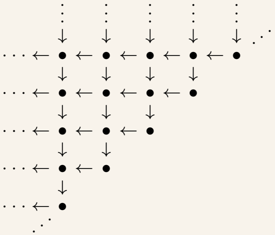

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Cyclic bar construction (Bökstedt)

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Spectrum

Cyclic structure $\Longrightarrow$ circle group action

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Cyclic structure $\Longrightarrow$ circle group action

$\Sigma N^{cy} R \to N^{cy} R$

$N^{cy} R \to \Omega N^{cy} R$

$\mathbb{T}_+ \wedge N^{cy} R \to N^{cy} R$

$S^0 \vee S^1$

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*HH* corresponds to *THH*

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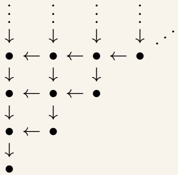

*HH* corresponds to *THH*
*HC* corresponds to $THH_{h\mathbb{T}}$

$(THH \wedge E\mathbb{T}_+)/\mathbb{T}$

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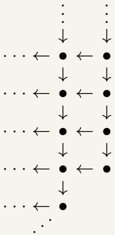

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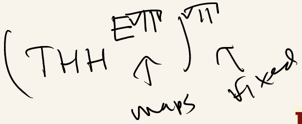

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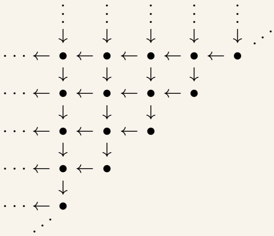

*HH* corresponds to *THH*
*HP* corresponds to $THH^{t\mathbb{T}}$

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For a ring spectrum $R$, define the Topological Periodic Cyclic Homology of $R$ by $TP(R) = THH(R)^{t\mathbb{T}}$.

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- Characteristic $p$ replacement for $HP$ (?)

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# Künneth Theorem

## Theorem

*Lax symmetric monoidal functor*

$$TP(X) \wedge^{L}_{TP(R)} TP(Y) \to TP(X \otimes^{L}_{R} Y)$$

## Definition

A $k$-algebra $X$ is smooth when it is compact as an $X \otimes_k X^{\mathrm{op}}$-module, i.e., when $R\,\mathrm{Hom}^{X \otimes_k X^{\mathrm{op}}}(X, -)$ commutes with direct sums.

## Definition

A $k$-algebra $X$ is proper when it is compact as a $k$-module.

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*Let k be finite field. The lax symmetric monoidal functor*

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in derived cat TP(k)-modules (in stable cat)

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# Review of Tate Construction

$E\mathbb{T}$ $\quad E\mathbb{T}_+ \to S^0 \to \widetilde{E\mathbb{T}}$

Smash with $Z^{E\mathbb{T}}$ and take fixed points

$$(Z^{E\mathbb{T}} \wedge E\mathbb{T}_+)^{E\mathbb{T}} \to (Z^{E\mathbb{T}})^{\mathbb{T}} \to (Z^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$

$(X^{E\mathbb{T}} \wedge E\mathbb{T}_+)^{E\mathbb{T}} \simeq \Sigma(X^{E\mathbb{T}})_{h\mathbb{T}} \simeq \Sigma X \wedge_{h\mathbb{T}}$ (Adams Isomorphism)

**Definition**

For $Z$ a $\mathbb{T}$-equivariant spectrum $Z^{t\mathbb{T}} = (Z^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$.
(Composite of derived functors.)

$$\Sigma Z_{h\mathbb{T}} \to Z^{h\mathbb{T}} \to Z^{t\mathbb{T}} \to \Sigma^2 Z_{h\mathbb{T}}$$

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$(E\mathbb{T} \to *)_+$

$E\mathbb{T}$ $\qquad E\mathbb{T}_+ \to S^0 \to \widetilde{E\mathbb{T}}$ ↖ based base pt is $\mathbb{T}$ fixed-pt.

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$(X^{E\mathbb{T}} \wedge E\mathbb{T}_+)^{E\mathbb{T}} \simeq \Sigma(X^{E\mathbb{T}})_{h\mathbb{T}} \simeq \Sigma X \wedge_{h\mathbb{T}}$ (Adams Isomorphism)

## Definition

For $Z$ a $\mathbb{T}$-equivariant spectrum $Z^{t\mathbb{T}} = (Z^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$.
(Composite of derived functors.)

$$\Sigma Z_{h\mathbb{T}} \to Z^{h\mathbb{T}} \to Z^{t\mathbb{T}} \to \Sigma^2 Z_{h\mathbb{T}}$$

$TP(X) = THH(X)^{t\mathbb{T}}$

# Review of Tate Construction

$E\mathbb{T}$ $\qquad E\mathbb{T}_+ \to S^0 \to \widetilde{E\mathbb{T}}$

Smash with $Z^{E\mathbb{T}}$ and take fixed points

$$(Z^{E\mathbb{T}} \wedge E\mathbb{T}_+)^{E\mathbb{T}} \to (Z^{E\mathbb{T}})^{\mathbb{T}} \to (Z^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$

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# The Multiplication

$$TP(X) \wedge TP(Y) \cong (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \wedge (THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$
$$\to (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \wedge THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$
$$\to (THH(X)^{E\mathbb{T}} \wedge THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$
$$\to ((THH(X) \wedge THH(Y))^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$
$$\to ((THH(X \wedge Y))^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \cong TP(X \wedge Y)$$

- $\widetilde{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \simeq \widetilde{E\mathbb{T}}$
- Use diagonal map $E\mathbb{T} \to E\mathbb{T} \times E\mathbb{T}$
- $THH(X) \wedge THH(Y) \cong THH(X \wedge Y)$

# The Multiplication

$$
\begin{aligned}
TP(X) \wedge TP(Y) &\cong (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \wedge (THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \\
&\to (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \wedge THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \\
&\to (THH(X)^{E\mathbb{T}} \wedge THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \\
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\end{aligned}
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- $\widetilde{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \simeq \widetilde{E\mathbb{T}}$
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$(-)^{\mathbb{T}}$ is lax symmetric monoidal

# The Multiplication

$$\begin{aligned} TP(X) \wedge TP(Y) &\cong (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \wedge (THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \\ &\to (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \wedge THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \\ &\to (THH(X)^{E\mathbb{T}} \wedge THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \\ &\to ((THH(X) \wedge THH(Y))^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \\ &\to ((THH(X \wedge Y))^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \cong TP(X \wedge Y) \end{aligned}$$

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&\to (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \wedge THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \\
&\xrightarrow{\simeq} (THH(X)^{E\mathbb{T}} \wedge THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \\
&\to ((THH(X) \wedge THH(Y))^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \\
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- $\widetilde{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \simeq \widetilde{E\mathbb{T}}$
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- $\widetilde{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \simeq \widetilde{E\mathbb{T}}$
- Use diagonal map $E\mathbb{T} \to E\mathbb{T} \times E\mathbb{T}$ $\longleftarrow$ This is coherent
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\begin{aligned}
TP(X) \wedge TP(Y) &\cong (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \wedge (THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \\
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- $\widetilde{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \simeq \widetilde{E\mathbb{T}}$ $\longleftarrow$ This can be made coherent!
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- Use diagonal map $E\mathbb{T} \to E\mathbb{T} \times E\mathbb{T}$ $\longleftarrow$ This is coherent
- $THH(X) \wedge THH(Y) \cong THH(X \wedge Y)$

$$TP(X) \wedge TP(R) \wedge TP(Y) \to TP(X \wedge R \wedge Y) \to TP(X \wedge Y)$$

$$\downarrow$$

$$TP(X \wedge_R Y)$$

# The Multiplication

$$
\begin{aligned}
TP(X) \wedge TP(Y) &\cong (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \wedge (THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \\
&\to (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \wedge THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \\
&\to (THH(X)^{E\mathbb{T}} \wedge THH(Y)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}} \\
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- $\widetilde{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \simeq \widetilde{E\mathbb{T}}$ $\longleftarrow$ This can be made coherent!
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- $THH(X) \wedge THH(Y) \cong THH(X \wedge Y)$

$$TP(X) \wedge TP(R) \wedge TP(Y) \to TP(X \wedge R \wedge Y)$$

$$TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$$

# The Filtration

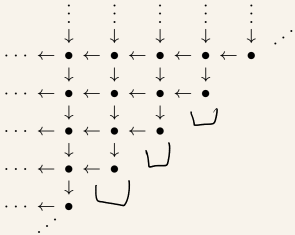
Filtration on $TP(X)$ with associated graded

$$F^i/F^{i-1} \simeq \Sigma^{2i} THH(X)$$

$TP(X) = (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$

Simplicial filtration on $E\mathbb{T}$

$\mathbb{T}_+, \Sigma^2\mathbb{T}_+, \Sigma^4\mathbb{T}_+, \ldots$



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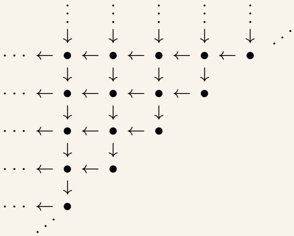

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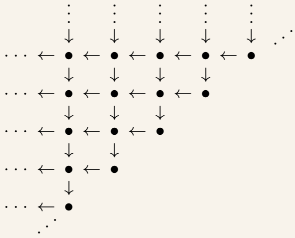

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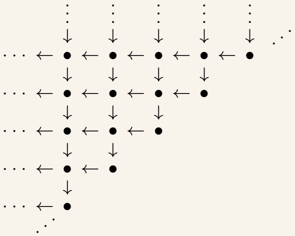

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Simplicial filtration on $E\mathbb{T}$

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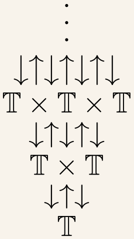

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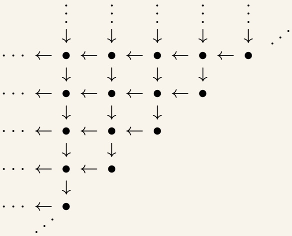
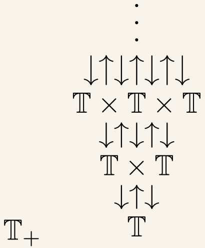


$$TP(X) = (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$

Simplicial filtration on $E\mathbb{T}$

$\mathbb{T}_+$, $\Sigma^2\mathbb{T}_+$, $\Sigma^4\mathbb{T}_+$, . . .



# The Filtration

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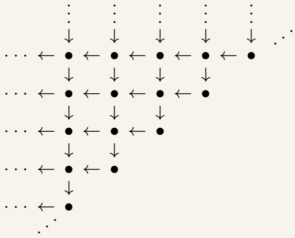

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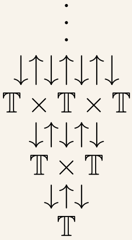
Simplicial filtration on $E\mathbb{T}$

$\mathbb{T}_+,\ \Sigma^2\mathbb{T}_+,\ \Sigma^4\mathbb{T}_+, \ldots$

$$\mathbb{T}_+ \wedge (\mathbb{T}/\{1\}) \wedge \Delta[1]/\partial\Delta[1]$$



# The Filtration

Filtration on $TP(X)$ with associated graded

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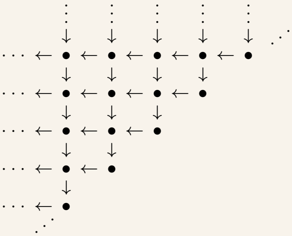
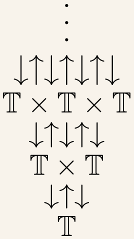


$$TP(X) = (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$

Simplicial filtration on $E\mathbb{T}$

$\mathbb{T}_+, \Sigma^2\mathbb{T}_+, \Sigma^4\mathbb{T}_+, \ldots$

$$\mathbb{T}_+ \wedge (\mathbb{T} \times \mathbb{T}/(\mathbb{T} \vee \mathbb{T})) \wedge \Delta[2]/\partial\Delta[2]$$



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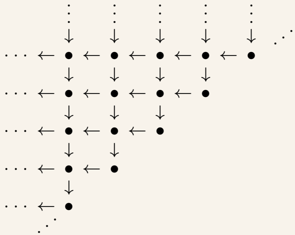
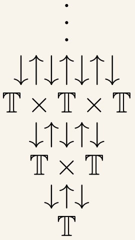
Simplicial filtration on $E\mathbb{T}$

$$\mathbb{T}_+,\ \Sigma^2\mathbb{T}_+,\ \Sigma^4\mathbb{T}_+, \ldots$$





# The Filtration

Filtration on $TP(X)$ with associated graded

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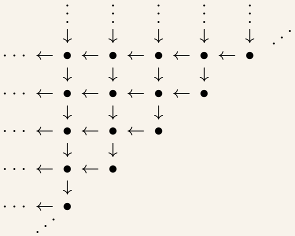

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Simplicial filtration on $E\mathbb{T}$ / on $\widetilde{E\mathbb{T}}$

$\mathbb{T}_+, \Sigma^2\mathbb{T}_+, \Sigma^4\mathbb{T}_+, \ldots$ / $S^0, \Sigma\mathbb{T}_+, \Sigma^3\mathbb{T}_+, \ldots$

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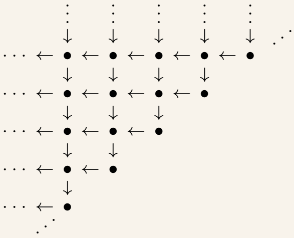

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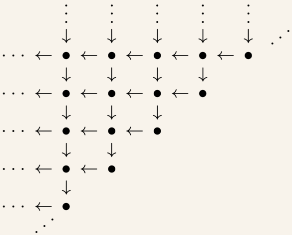
Filtration on $TP(X)$: $F^i TP(X) = \begin{cases} (THH(X)^{(E\mathbb{T}, E\mathbb{T}_{-i-1})} \wedge S^0)^{\mathbb{T}} & i \leq 0 \\ (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}}_i)^{\mathbb{T}} & i > 0 \end{cases}$

$$F^i/F^{i-1} = \begin{cases} (THH(X)^{(\Sigma^{2i}\mathbb{T}_+)})^{\mathbb{T}} & i \leq 0 \\ (THH(X)^{E\mathbb{T}} \wedge \Sigma^{2i-1}\mathbb{T}_+)^{\mathbb{T}} & i > 0 \end{cases}$$

# The Filtration

Filtration on $TP(X)$ with associated graded

$$F^i/F^{i-1} \simeq \Sigma^{2i} THH(X)$$



$$TP(X) = (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$

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# The Filtration

Filtration on $TP(X)$ with associated graded

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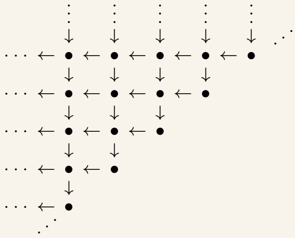

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$\mathbb{T}_+, \Sigma^2\mathbb{T}_+, \Sigma^4\mathbb{T}_+, \ldots$ / $S^0, \Sigma\mathbb{T}_+, \Sigma^3\mathbb{T}_+, \ldots$

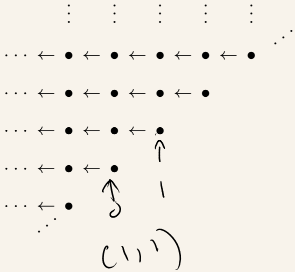
Filtration on $TP(X)$: $F^i TP(X) = \begin{cases} (THH(X)^{(E\mathbb{T}, E\mathbb{T}_{-i-1})} \wedge S^0)^{\mathbb{T}} & i \leq 0 \\ (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}}_i)^{\mathbb{T}} & i > 0 \end{cases}$

$$F^i/F^{i-1} = \begin{cases} (THH(X)^{(\Sigma^{2i}\mathbb{T}_+)})^{\mathbb{T}} & i \leq 0 \\ (THH(X)^{E\mathbb{T}} \wedge \Sigma^{2i-1}\mathbb{T}_+)^{\mathbb{T}} & i > 0 \end{cases}$$

# The Spectral Sequence

Filtration on $TP(X)$ with associated graded

$$F^i/F^{i-1} \simeq \Sigma^{2i} THH(X)$$



Spectral sequence

$$E^1_{i,j} = \pi_{i+j}\Sigma^{2i} THH(X) = THH_{j-i}(X)$$

Renumber: Double filtration degree

$$E^{2r}_{2i,j} = (E^r_{i,i+j})^{\text{old}}, \qquad d_{2r} = (d_r)^{\text{old}}$$

## Spectral Sequence

Conditionally convergent spectral sequence

$$E^2_{2i,j} = THH_j(X) \Longrightarrow TP_{2i+j}(X). \qquad (E^r_{2i+1,j} = 0)$$

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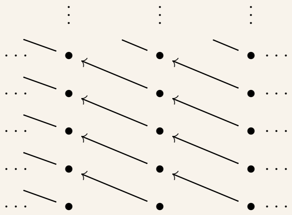

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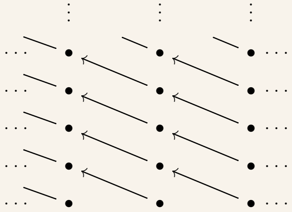

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(2,0)

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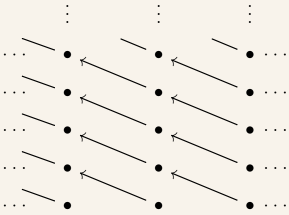

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# Combining the Multiplication and Filtration

| Multiplication | Filtration |
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| • Diagonal $E\mathbb{T} \to E\mathbb{T} \times E\mathbb{T}$<br>• Mult. $\widetilde{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \simeq \widetilde{E\mathbb{T}}$ | • Simplicial/cellular filt. on $E\mathbb{T}$<br>• Filtration on $\widetilde{E\mathbb{T}}$ |

Can we make $TP(X) \wedge TP(Y) \to TP(X \wedge Y)$ a filtered map?

In homotopy category, easy obstruction theory cellular approximation to diagonal & multiplication $\Longrightarrow$ multiplicative spectral sequence.

$$TP(X) = (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$

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$X = H\mathbb{Z}$ $\quad H\mathbb{F}_p$ $\quad \mathbb{Z}$ $\quad \mathbb{F}_p$

$TP_{-n}(X) \cong H^n(\mathbb{Q}P^\infty)$

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# Approximation of the diagonal for simplicial spaces

Let $X_\bullet$ be a simplicial space, $|X_\bullet|$ its geometric realization. $|X^n_\bullet|$ vs $|X_\bullet|^n$

**Problem**

Parametrize a contractible spaces of filtered approximations of the diagonal maps $|X_\bullet| \to |X_\bullet|^n$ for all $n$ that compose appropriately.

Find an $A_\infty$ operad $\mathcal{A}$ and a map of operads

$$\mathcal{A}(n) \to \mathrm{Filt}(|X_\bullet|, |X_\bullet|^n) \subset \mathscr{T}(|X_\bullet|, |X_\bullet|^n).$$

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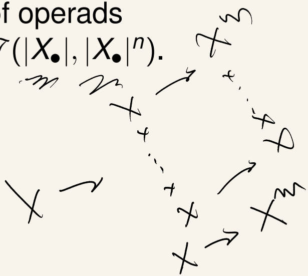

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## Barycentric Coordinates and Milnor Coordinates on $\Delta[m]$

Barycentric $t_0, \ldots, t_m$, $\sum t_i = 1 \leftrightarrow$ Milnor $0 \leq u_0 \leq u_1 \leq \cdots \leq u_{m-1} \leq 1$

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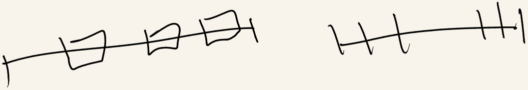
## Solution

1. The overlapping little 1-cubes operad $\mathcal{C}_1^{\equiv}$



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# Filtered Monoidal Structure

A filtered approximation of the diagonal gives a map

$$E\mathbb{T}_{i+j-1} \to (E\mathbb{T}_{i-1} \times E\mathbb{T}) \cup (E\mathbb{T} \times E\mathbb{T}_{j-1}) \subset E\mathbb{T} \times E\mathbb{T}$$

Hence a map $(E\mathbb{T}, E\mathbb{T}_{i+j-1}) \to (E\mathbb{T}, E\mathbb{T}_{i-1}) \times (E\mathbb{T}, E\mathbb{T}_{j-1})$

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Parametrized

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Little 1-cubes: Moore construction (Moore loop space)

Use length parameter to make fully associative

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# Künneth Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$

$\implies$ map of spectral sequences

Righthand spectral sequence is Tate spectral sequence for

$THH(X \wedge_R Y) \cong THH(X) \wedge_{THH(R)} THH(Y)$

$E^2$ periodic with $\pi_*(THH(X) \wedge_{THH(R)} THH(Y))$ in each even column

Lefthand spectral sequence has (renumbered) $E^2$-term

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**Proposition**

Map of spectral sequences is an isomorphism on $E^2$

# Outline of Proof of Künneth Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$
induces isomorphism of $E^2$-terms of spectral sequences

RHSS: Tate spectral sequence $\Longrightarrow$ conditionally convergent.

**Theorem**

*If $R = Hk$ and $X$ and $Y$ are smooth and proper, then the LHSS is conditionally convergent.*

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