Introduction to *THH* and Related Theories

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Indiana University

Conference on Floer Homology and Homotopy Theory

July 17, 2017

Overview

THH is a refinement of Hochschild homology (or Hochschild-Mitchell homology) that captures torsion information better. This will be an expository talk about *THH* and related theories like *TC* and *TP*.

Slides available at

`http://pages.iu.edu/~mmandell/talks/UCLA.pdf`

References and examples are not comprehensive!
["Also" = Algebraic precursor, see internal reference]

Outline

- Background and Examples
- Basic Properties
- Recent work on *TP*

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- Background and Examples
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What kind of thing is *THH*?

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Input

Output

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Input

A ring

Output

What kind of thing is *THH*?

Input

A ring, or d.g. ring

Output

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Input

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A spectrum in particular gives:

- A homology and cohomology theory
- A graded abelian group

$THH(\mathcal{C})$

$THH_*(\mathcal{C})$

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A spectrum, genuine $\mathbb{T}$-equivariant

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- TP, analogue of HP ($= HC^{per}$); analogues of HC & HN ($= HC^{-}$)

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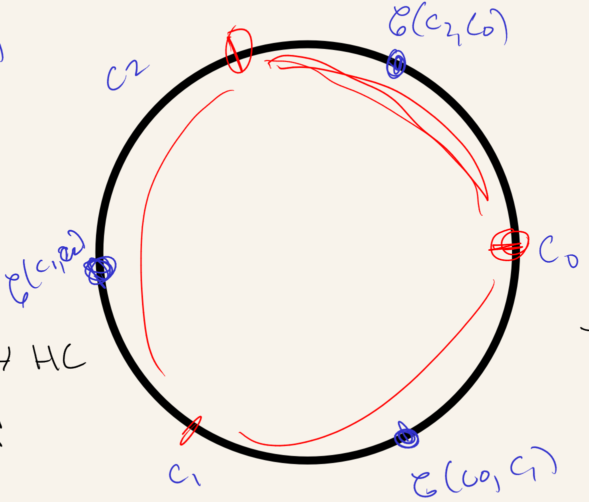
- *TC*, no analogue for *HH*

Construction

$THH(\mathcal{C})$

$C_3 < \mathbb{T}$



$THC \leftarrow HC$

$THH_{h\mathbb{T}}$

$TN \hookrightarrow HN$

$THH^{h\mathbb{T}}$

$\Sigma THH_{h\mathbb{T}} \to THH^{h\mathbb{T}} \to TP$

How does *THH* arise?

- $THH(\Sigma^\infty_+ \Omega X) \simeq \Sigma^\infty_+ \Lambda X$
  - Algebraic K-theory of Spaces: $A(X) \simeq \Sigma^\infty_+ X \times Wh^{\text{Diff}}(X)$ [1]
  - K-theoretic Novikov conjecture (TC) [2]
- String topology category / Fukaya category [3]
  - Floer homotopy type and ΛX [4]
  - Wrapped Fukaya category and ΩX [5]
- Proposed knot invariant based on Khovanov homotopy type [6]

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- Algebraic K-theory computations (selected examples) (TC)
  - Quotient fields of CDVRs with perfect residue fields [1]
  - The sphere spectrum [2],[3]
- Replacement for HH in characteristic p
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Morita Invariance

Definition

- Dwyer-Kan equivalence (DK-equivalence)
- Module category $\mathfrak{Mod}_{\mathscr{A}}$
- Morita equivalence (following [1]): A map $\mathscr{A} \to \mathscr{B}$ that induces a DK-equivalence $\mathfrak{Mod}_{\mathscr{A}} \to \mathfrak{Mod}_{B}$

$\mathscr{C}$ $\quad \pi_0\mathscr{C}(a,b)$

$\pi_0\mathscr{C} \leftarrow$ homotopy cat

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Theorem

THH takes Morita equivalences to weak equivalences of cyclotomic spectra. [2]

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Thick Closure

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$\mathcal{C}(-,x)$ $\quad$ $\mathcal{C} \to \mathrm{Mod}_{\mathcal{C}}$ $\quad$ $\pi_0\,\mathrm{Mod}_{\mathcal{C}}$

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- $\pi_0\mathfrak{Mod}_{\mathscr{A}}$ is a triangulated category.
- $\mathscr{A} \to \widehat{\mathscr{A}}_{\mathrm{perf}}$ is a Morita equivalence
- A map $\mathscr{A} \to \mathscr{B}$ is a Morita equivalence if and only if it induces a DK-equivalence $\widehat{\mathscr{A}}_{\mathrm{perf}} \to \widehat{\mathscr{B}}_{\mathrm{perf}}$

Example: Tilting

A map $\mathscr{A} \to \mathscr{B}$ is a Morita equivalence if and only if it induces a DK-equivalence on thick closures $\widehat{\mathscr{A}}_{\mathrm{perf}} \to \widehat{\mathscr{B}}_{\mathrm{perf}}$.

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- Let A and B be A_∞ ring spectra (or d.g. rings)
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Then ${}_BM$ is in $\widehat{B}_{\text{perf}}$ and ${}_AN$ is in $\widehat{A}_{\text{perf}}$, and DK-equiv. $\widehat{A}_{\text{perf}} \to \widehat{B}_{\text{perf}}$,
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Then $M \wedge^L_A (-)$ and $B \wedge^L_B (-)$ "are" inverse DK-equivs. $\mathfrak{Mod}_A \leftrightarrows \mathfrak{Mod}_B$

Then ${}_BM$ is in $\widehat{B}_{\mathrm{perf}}$ and ${}_AN$ is in $\widehat{A}_{\mathrm{perf}}$, and DK-equiv. $\widehat{A}_{\mathrm{perf}} \to \widehat{B}_{\mathrm{perf}}$, Morita equivalences

$$A \xrightarrow{\sim} \widehat{A}_{\mathrm{perf}} \xrightarrow{\simeq} \widehat{B}_{\mathrm{perf}} \xleftarrow{\sim} B$$

The Dennis-Waldhausen Morita Argument

Example: $M \wedge^L_A N \simeq B$ and $N \wedge^L_B M \simeq A$ as bimodules

The Localization Sequence

Definition

- Triangulated quotient
- Localization pair of spectral categories $\mathscr{A} \subseteq \mathscr{B}$
- Map/equivalence of localization pairs
- $THH(\mathscr{B}, \mathscr{A}) := \mathrm{Cof}(THH(\mathscr{A}) \to THH(\mathscr{B}))$

Cofiber sequence

$$THH(\mathscr{A}) \to THH(\mathscr{B}) \to THH(\mathscr{B}, \mathscr{A}) \to \Sigma THH(\mathscr{A})$$

$a \quad \mathcal{B}$

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Theorem

THH takes equivalences of localization pairs to weak equivalences of cyclotomic spectra [1]

[1] A. Blumberg, M. Mandell, “Localization theorems in topological Hochschild homology and topological cyclic homology”: Paper, MR (Also, for exact categories, B. Keller [25])

Example: *THH* of a scheme

Let X be a quasi-projective scheme over a field or over $\mathbb{Z}$

Definition

$THH(X) = THH(\mathscr{D}^{\mathbb{S}}_{\mathrm{perf}}(X))$

For U an open subscheme of X, $\mathscr{D}^{\mathbb{S}}_{\mathrm{perf}}(X \text{ on } (X-U))$ denotes the full spectral subcategory of $\mathscr{D}^{\mathbb{S}}_{\mathrm{perf}}(X)$ of objects supported on $X-U$.

Theorem

$(\mathscr{D}^{\mathbb{S}}_{\mathrm{perf}}(X), \mathscr{D}^{\mathbb{S}}_{\mathrm{perf}}(X \text{ on } (X-U))) \to (\mathscr{D}^{\mathbb{S}}_{\mathrm{perf}}(U), 0)$ is an equivalence of localization pairs [1]

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[1] R. Thomason, T. Trobaugh, "Higher algebraic *K*-theory of schemes and of derived categories": Paper, MR (§5)

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Example: *THH* of a scheme and Mayer-Vietoris

Let $X = U \cup V, \quad Z = X - U = V - (V \cap U)$

Map of cofiber sequences

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Corollary (Mayer-Vietoris)

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In fact, *THH* satisfies étale hypercover descent [1]

[1] T Geisser, L Hesselholt, "Topological cyclic homology of schemes": Paper, MR

Symmetric Monoidality

Smash product of spectral categories

- $\mathcal{O}(\mathscr{X} \wedge \mathscr{Y}) = \mathcal{O}(\mathscr{X}) \times \mathcal{O}(\mathscr{Y})$
- $(\mathscr{X} \wedge \mathscr{Y})((x, y), (x', y')) = \mathscr{X}(x, x') \wedge \mathscr{X}(y, y')$

Observation (Strong Symmetric Monoidality)

$$THH(\mathscr{X}) \wedge THH(\mathscr{Y}) \cong THH(\mathscr{X} \wedge \mathscr{Y})$$

Corollary (Lax Symmetric Monoidality)

$$TP(\mathscr{X}) \wedge TP(\mathscr{Y}) \to TP(\mathscr{X} \wedge \mathscr{Y})$$
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Künneth Theorems

Let R be a commutative ring or E_∞ ring spectrum

k-linear d.g. category, R-linear spectral category

Corollary (Strong/Lax Künneth Theorems)

$$THH(\mathscr{X}) \wedge_{THH(R)} THH(\mathscr{Y}) \xrightarrow{\cong} THH(\mathscr{X} \wedge_R \mathscr{Y})$$

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Example: $THH_*(\mathbb{F}_p) \cong \mathbb{F}_p[x]$, $\deg(x) = 2$

$$0 \to THH_*(\mathscr{X}) \otimes_{THH_*(\mathbb{F}_p)} THH_*(\mathscr{Y}) \to THH_*(\mathscr{X} \otimes_{\mathbb{F}_p} \mathscr{Y}) \to \mathrm{Tor}^{\mathbb{F}_p[x]}_{1,*-1}(THH_*(\mathscr{X}), THH_*(\mathscr{Y})) \to 0$$

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Introduction: *TP*, *HP*, and De Rham Cohomology

Let X be a smooth quasi-projective variety over $\mathbb{Q}$

Then $TP(X) \simeq HP(X)$

Let $TP^{\mathbb{C}}(X) = TP(X) \wedge_{H\mathbb{Q}} H\mathbb{C} \simeq HP^{\mathbb{C}}(X/\mathbb{C})$

Hochschild-Kostant-Rosenberg + Connes B + Zariski descent + Grothendieck algebraic De Rham:

$$TP_n^{\mathbb{C}}(X) \cong \bigoplus_{i \in \mathbb{Z}} H^{2i-n}(X(\mathbb{C}); \mathbb{C})$$

If X is a quasi-projective scheme over $\mathbb{F}_q$ (following Hesselholt) $TP(X)$ is a "higher De Rham" theory, e.g.,

$$TP_{2n}(\mathbb{P}^m/\mathbb{F}_p) \cong \mathbb{Z}_p^\wedge[x]/x^{m+1} \qquad \text{(and 0 in odd degrees)}$$

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$$TP(\mathscr{X}) \wedge^{L}_{TP(k)} TP(\mathscr{Y}) \to TP(\mathscr{X} \otimes_k \mathscr{Y})$$

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Let k be a perfect field of positive characteristic. For smooth and proper k-linear d.g. categories the map

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$TP_*(k) \cong \mathbb{W}_k[v, v^{-1}]$, $\deg(v) = 2$. ($\mathbb{W}_{\mathbb{F}_p} = \mathbb{Z}_p^{\wedge}$)

Also Tensor-Tor short exact sequence, non-naturally split

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Tabuada (arXiv:1707.04248) shows that using the theory of non-commutative motives, these two properties plus an easy calculation gives a new proof of two of the classical Weil conjectures (and non-commutative generalizations):

- Rationality of the zeta function (Dwork 1960)
- Functional equation for the zeta function (Grothendieck 1965)

(Replaces p-adic analysis and étale cohomology arguments with topology and triangulated/spectral category arguments)

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- $TP^Q_*(\mathscr{X} \otimes_{\mathbb{F}_q} \mathscr{Y}) \cong TP^Q_*(\mathscr{X}) \overset{\text{super}}{\otimes}_Q TP^Q_*(\mathscr{Y})$

Tabuada (arXiv:1707.04248) shows that using the theory of non-commutative motives, these two properties plus an easy calculation gives a new proof of two of the classical Weil conjectures (and non-commutative generalizations):

- Rationality of the zeta function (Dwork 1960)
- Functional equation for the zeta function (Grothendieck 1965)

(Replaces p-adic analysis and étale cohomology arguments with topology and triangulated/spectral category arguments)

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But $TP(C_*(\Omega M; \mathbb{Z}/2))$? $TP(M)$????

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