E_2 Structures and Derived Koszul Duality in String Topology

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Special Session on Structured Homotopy Theory

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Overview

Goal: Construct an equivalence of E_2 algebras between two models for the Thom spectrum of the free loop space that are related by derived Koszul duality.

Method: Study the functoriality and invariance properties of topological Hochschild cohomology.

- Joint work with Andrew Blumberg
- Preprint arXiv:1801.03549, to appear in *Alg. & Geo. Top.*

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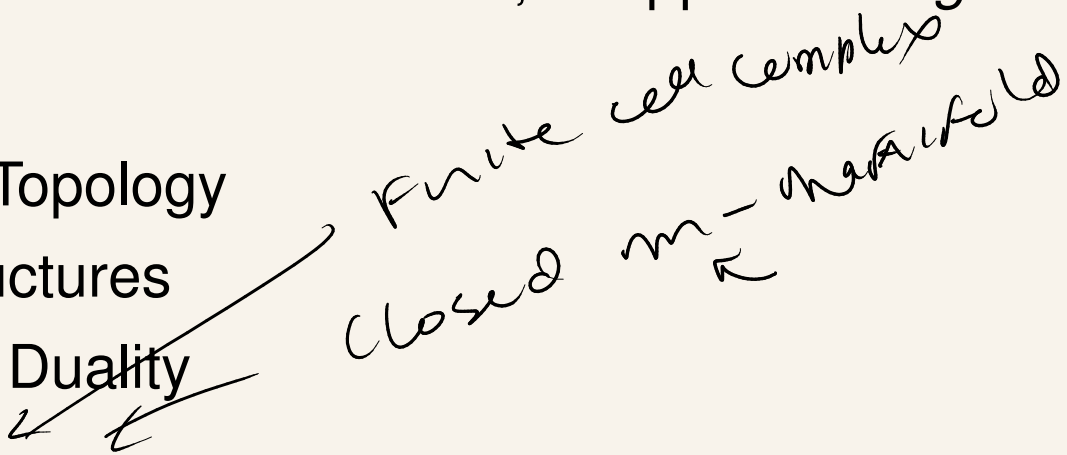

Spaces X, M are always simply connected in what follows

Chas-Sullivan String Product

Product on the free loop space

$$H_p(\Lambda M) \otimes H_q(\Lambda M) \to H_{p+q-m}(\Lambda M)$$

Intersect chains, multiply loops

Shifted associative commutative algebra structure

$$H_{p+m}(\Lambda M) \otimes H_{q+m}(\Lambda M) \to H_{p+q+m}(\Lambda M)$$

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Bracket $\longrightarrow$ **Gerstenhaber algebra**

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(Batalin-Vilkovisky structure)

Deligne Conjecture

Prototypical Gerstenhaber algebra: Hochschild cohomology $HH^{-*}(A)$

Another set of examples:

Homology of E_2-algebras / E_2 ring spectra

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E_2-space (double loop space)

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The Gerstenhaber algebra structure on $HH^{-*}(A)$ comes from an E_2-algebra structure on the Hochschild cochains.

Theorem: Kontsevich-Soibelman, McClure-Smith, Tamarkin, Voronov, Berger-Fresse, etc.(?)

McClure-Smith: Even true for A an A_∞ ring spectrum

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Berger Fresse

String Topology and the Deligne Conjecture

Anderson(?) / Jones(?)

$$H_*(\Lambda X) \cong HH^{-*}(C^*(X), C_*(X))$$
$$H_{*+m}(\Lambda M) \cong HH^{-*}(C^*(M))$$

In 1999, “everyone” “knew” this was a Gerstenhaber isomorphism.

Cohen-Jones / Tradler proved this is an algebra isomorphism

Complete proof for Gerstanhaber isomorphism still not published, but finished in thesis of Eric Malm (2010).

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String Topology and Duality

C^*X

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C^*X $\qquad\qquad$ $C_*(\Omega X)$

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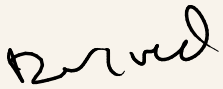

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Proof of Gerstenhaber isomorphism goes through "Koszul duality" (Buchweitz, Keller, Felix-Menichi-Thomas)

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Blumberg-Cohen-Teleman on $HH^{-*}(C^*M) \cong HH^{-*}(C_*(\Omega M))$: a connection between string topology and topological field theory

Spectral Models for String Topology

DX $\qquad$ $\Sigma^\infty_+ \Omega X$

On homology:

$C^*(X)$ $\qquad$ $C_*(\Omega X)$

Spectral Models for String Topology

$$THC(DX) \qquad THC(\Sigma^{\infty}_{+}\Omega X)$$

On homology:

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E_2 ring spectra

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Gerstenhaber algebras

Spectral Models for String Topology

Main Theorem

Koszul duality weak equivalence (zigzag)

$$THC(DX) \simeq THC(\Sigma^\infty_+ \Omega X)$$

of E_2 ring spectra

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Argument: Study functoriality of *THC*