Selfsimilar groups and conformal dynamics
–in preparation–

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1 Introduction

This is a brief survey of some connections between conformal dynamics and selfsimilar groups.

2 Rational maps

Complex dynamics. Let $\widehat{\mathbb{C}}$ denote the Riemann sphere and let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. Iteration of f yields a dynamical system on the sphere which is conformal in the sense of Riemannian geometry: away from the $2d - 2$ branch points, f sends infinitesimal circles to infinitesimal circles. There is tension: since f has degree larger than one, f wants to be expanding. But, since there are branch points, f wants to be contracting. Thus, the behavior of branch points under iteration plays a crucial role.

Postcritically finite maps. The simplest possible behavior is when these branch points are all eventually periodic, i.e. when the set

$$P_f = \cup_{n>0} f^n\{\text{branch points}\}$$

is finite. In this case, one says that f is *postcritically finite.* [1] Note that P_f contains the set of branch values (images of branch points) of every iterate of f. Thus, to say that f is postcritically finite is equivalent to saying that the collection of iterates of f yield branched coverings of the sphere which are all unramified above the complement of finitely many points.

[1] The term "postcritically finite" is also applied as an adjective to describe certain iterated function systems on $\mathbb{R}^n$.

Expansion. The set P_f is a disjoint union of P_f^a, the subset consisting of points in P_f which land on branch points under iteration, and its complement P_f^r. For example, if $f(z) = z^2 + i$, then $P_f = \{i, i+1, -i, \infty\}$ and $P_f^a = \{\infty\}$. and In this case, it is possible to show that with respect to a suitable metric, f is uniformly expanding off a neighborhood of P_f^a. There is a general philosophy in dynamics that expanding dynamical systems on compact spaces are completely determined up to topological conjugacy by a finite amount of homotopy-theoretic data. For example, if $g : S^1 \to S^1$ is an expanding covering map of the circle, then the degree of g is a complete invariant of the topological conjugacy class. It is therefore natural to consider the problem of classifying postcritically finite rational maps by homotopy-theoretic data.

Thurston's classification. In principle, such a classification has been given by Thurston. Consider more general objects, coined *Thurston maps*, which are orientation-preserving branched covering maps F of the two-sphere S^2 to itself, for which the analogous set P_F is finite. Two such maps F, G are called *equivalent* if they are conjugate up to isotopy relative to P_F. For example: conjugating F by an arbitrary orientation-preserving homeomorphism of the sphere, and pre- or post-composing F by a homeomorphism isotopic to the identity relative to P_F, preserves the equivalence class of F. Thurston gave a characterization theorem which gives necessary and sufficient conditions for a such a map F to be equivalent to a rational map f. He also gave a rigidity theorem which asserts that in all but one family of well-known exceptions, f is unique up to conjugacy by linear fractional transformations.

In practice, however, the problem of deciding when two Thurston maps F, G are equivalent, or when a given map F is equivalent to a rational map f, is extremely difficult. To apply Thurston's characterization, one must rule out the existence of certain topological obstructions which a priori can have arbitrarily complicated complexity. Finding a conjugacy up to isotopy amounts to locating the right mapping class element (or elements) from among an a priori countably infinite set of choices, and there is not a well-developed theory of characteristic topological invariants which one might exploit to narrow down the list of such choices.

Algebraic invariants. Since the definition of equivalence involves homotopy-theoretic conditions, it is natural to seek an algebraic formulation of this equivalence. The formulations in [Pil1] and [Kam], although algebraic, did not lend themselves to computations.

In [Nek], a much more elegant definition is given. Let $M_0 = S^2 - P_F$ and $M_1 = S^2 - F^{-1}(P_F)$, so that $F : M_1 \to M_0$ is an unramified covering and $M_1 \subset M_0$. Let $t \in M_0$ be a basepoint and $\mathfrak{M}(F)$ be the set of homotopy classes of paths in M_0 joining t to an element of $F^{-1}(t)$. There are natural left and right actions of the fundamental group $\pi_1(M_0, t)$, giving $\mathfrak{M}$ the structure of a *permutational bimodule* over the fundamental group $\pi_1(M_0, t)$. A basic result of the ensuing theory is that two Thurston maps F, G are equivalent if and only if their associated bimodules are isomorphic via an isomorphism of fundamental groups induced by an orientation-preserving homeomorphism of the plane. Bartholdi and Nekrashevych demonstrate the power of these methods by solving, in a definitive way, some old, explicit, and simple questions of Douady and Hubbard on the classification of quadratic polynomial branched coverings [BN]. The answers turn out to be surprisingly subtle, delicate, and complex.

A key role is played by the *iterated monodromy group* (IMG) of a Thurston map F. Let $\mathcal{T}$ denote the set of iterated inverse images of t under $F^{-n}, n = 0, 1, 2, \ldots$. Then $\pi_1(M_0, t)$ acts on $\mathcal{T}$ as usual by path-lifting; the IMG of F is the quotient of $\pi_1(M_0, t)$ by the kernel of this action.

The IMG of F can be regarded as a group of automorphisms of an infinite rooted tree with uniform d-fold branching whose vertices are identified with finite words in an alphabet X where $\#X = d = \deg F$. Fix an identification of the set $F^{-1}(t)$ with an alphabet X and for each $x \in X$ choose an arc λ_x joining t to x in M_0. Lifting these paths under iterates of f and concatenating gives an identification of the set of iterated preimages of t with the set of finite words in the alphabet x. The action of a group element g on a word xw beginning with x can, in principle, be computed via the recursive formula

$$(xw)^g = x^g w^{g|_x}$$

where x^g is the image of x under the action of g by path-lifting and $g|_x$, the *restriction of g to x*, is the element of $\pi_1(M_0, t)$ defined by running from t via λ_x to x, over via the lift of a loop representing g to x^g, and then back to t via the reverse of $\lambda_{x^g}^{-1}$. A priori the complexity of this calculation may explode, since the element $g|_x$ might be more complicated than g itself. In the case when F is suitably expanding, however, this action is *contracting* and the process of restriction contracts sufficiently large word lengths uniformly with respect to any generating set. Thus, these computations are possible in linear time.

Questions.

- Is there an algorithm which takes as input two Thurston maps F, G and decides whether F, G are equivalent? Can this algorithm be made so as to produce an equivalence if it exists? This problem is at least as hard as the conjugacy problem for mapping class elements of punctured spheres. The answer is more likely yes when F, G are suitably expanding.

- Is there an algorithm which takes as input a Thurston map and decides whether F is obstructed? Can this algorithm be made so as to produce the obstruction if it exists? Again, the answers are likely easier when the map is suitably expanding.

It has been announced by Bartholdi, Kaimanovich, Nekrashevych, and Virag [**?**] that the IMGs associated to postcritically finite polynomials are amenable.

3 Riemann surface laminations

The IMGs of postcritically finite rational maps arise naturally in other constructions in complex dynamics [LM], [Su]. Let f be such a map, and let $\mathcal{L}_f = \{(z_0 z_1 z_2 \ldots | f(z_{i+1}) = z_i \forall i\}$ be the space of backwards orbits of points in the product topology, minus those backwards orbits which consist of cycles lying in P_f^a. Then $\mathcal{L}_f$ is a so-called *Riemann surface lamination*: a topological space foliated by Riemann surfaces, called leaves; the local structure is that of a product of an open disk in the complex plane with a Cantor set. It turns out that in this case all leaves are conformally isomorphic to complex planes.

Let $\pi : \mathcal{L}_f \to \widehat{\mathbb{C}}$ be given by projection onto the zeroth coordinate. Fix a basepoint $t \in \widehat{\mathbb{C}} - P_f$, an element $\hat{t}$ with $\pi(\hat{t}) = t$, and let L be the leaf containing $\hat{t}$. Then path-lifting defines a monodromy action of $\pi_1(M_0, t)$ on the set $\pi^{-1}(t)$, i.e. an action of $\pi_1(M_0, t)$ on a Cantor set. It is easy to see that this action is isomorphic to that of the IMG of f, and that the orbit of $\hat{t}$ under this action is precisely the set $L \cap \pi^{-1}(t)$. If we choose a set S of generators s for $\pi_1(M_0, t)$ represented by loops γ_s, then $G = (\pi|_L)^{-1}(\cup_{s \in S} \gamma_s)$ is the Schreier graph of the action of the IMG of f on the orbit of $\hat{t}$. If $\hat{t} = (t, t, t, \ldots)$ where t is a fixed point with multiplier λ, then $\pi|_L$ is essentially the classical Königs linearization function which gives a semiconjugacy from $w \mapsto \lambda w$ on the complex plane L to the map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Cabrera [Cab] has recently shown that for the polynomials $f(z) = z^2 + c$ for which the critical point is periodic, the homeomorphism type of $\mathcal{L}_f$ determines c.

Questions.

- What are the implications for $\mathcal{L}_f$ of the amenability of the IMG of f?

Hénon maps. In certain cases, the inverse limit of the Julia set of a polynomial admits another description. Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial all of whose critical points converge to attracting cycles (i.e. is *hyperbolic.* For example, $p(z) = z^2 + c$ where 0 is periodic will do. The map $H : \mathbb{C}^2 \to \mathbb{C}^2$ given by $H(x, y) = (p(x) - ay, x)$ is an automorphism, called a *Hénon map.* By a theorem of Hubbard and Oberste-Vorth [HOV], if a is sufficiently small, the inverse limit $\hat{J}_p$ of the Julia set of p is homeomorphic to the Julia set J of H, where

$$J = J^+ \cap J^, -$$

$$J^\pm = \partial K^\pm,$$

and $K^\pm$ are the loci of points which remain bounded under forward/backward iteration, respectively. It would be good to have combinatorial models for J. Work on this was done by Ricardo Oliva in his thesis [Oli].

4 Combinatorial models

For postcritically finite polynomials, there are good combinatorial models for the dynamics with which one can compute information about the topological dynamics. For postcritically finite rational functions, however, this is more difficult. Constructions of Nekrashevych [Nek] give combinatorial models for such maps and, indeed, for far more general kinds of expanding dynamical systems.

Selfsimilarity complexes. Here, briefly, is the construction. Let f be a postcritically finite rational map and S a set of generators for $\pi_1(M_0, t)$ where $M_0 = \widehat{\mathbb{C}} - P_f$ as above. Represent each $s \in S$ by a loop γ_s. Fix an identification of the set $f^{-1}(t)$ with an alphabet X and for each $x \in X$ choose an arc λ_x joining t to x in M_0. By lifting these arcs and loops under iterates of f, one obtains an infinite abstract one-complex Σ whose 0-cells are identified with finite words in X and whose edges are of two types: *horizontal edges* which are lifts of the loops γ_s, $s \in S$, and *vertical edges* which are lifts of the arcs λ_x, $x \in X$. The so-called *selfsimilarity complex* Σ carries a natural structure as a complete geodesic metric space in which every embedded 1-cell is isometric to the Euclidean unit interval; this is like the word metric on the Cayley graph of a finitely generated group. With this metric, Σ is a complete geodesic metric space. The map f induces a cellular map $f_\Sigma : \Sigma_{\geq 1} \to \Sigma$ where $\Sigma_{\geq 1}$ is the subcomplex consisting of points at distance ≥ 1 from the basepoint corresponding to the empty word.

The fact that f is expanding near its Julia set has the following geometric implications. Consider a loop in Σ starting at the basepoint, running vertically away from the basepoint, going over horizontally one unit, and then returning vertically to the basepoint. Expansion implies that the image under π of this loop has a length which is uniformly bounded. This fact eventually implies that the IMG of f is contracting and that Σ is hyperbolic. There is a natural projection map $\pi : \partial_\infty \Sigma \to J_f$ where J_f is the Julia set of f, and this gives a conjugacy from the map on the boundary of Σ induced by f_Σ to the restriction of the rational map f on J_f. Thus, at least in principle, this provides a combinatorial model for the dynamics.

Questions.

- Can one detect when J_f is a Sierpinski carpet using this model? In the "dictionary" between rational maps and Kleinian groups, such f are somewhat analogous to convex compact Kleinian groups with incompressible ideal boundary; see [Pil3], [McM].

- There is a plethora of combination procedures which take as input Thurston maps and some gluing or collapsing data and return as output another Thurston map: mating, tuning, renormalization, captures, etc. There are also some corresponding decomposition procedures; see [Pil2]. If a Thurston map F is obstructed, there is a canonical such obstruction with optimal properties. How is this manifested algebraically? Is there an analog, for IMGs of Thurston maps, of the free product and JSJ decompositions for three-manifold groups?

Finiteness principles. There are many analogies between the selfimilarity complex Σ and the Cayley graph of a Gromov hyperbolic groups. Given a group G generated by a finite set S, its *Cayley graph* is the graph Γ whose vertices are elements of the group and whose edges join g and sg where $s \in S$, regarded as a geodesic metric space in the obvious way. Edges are naturally labelled by elements of S. Loosely, the *cone* C_g determined by $v \in G$ is the set of all $g \in G$ for which v is on a geodesic joining g and the identity, equipped with the induced metric. A fundamental result is the *finiteness of cone types*: as v varies in G, there are only finitely many isometry types of cones–any cone is sent by some group element into one of finitely many *model cones*.

A similar result holds for selfsimilarity complexes [HP].

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