Bifurcations in Euclidean Spaces
Summary of Local Codimension-1 Bifurcations

**Saddle-Node**

**Hopf**
Saddle-Node Bifurcation

A saddle-node bifurcation occurs when a **single** real eigenvalue changes sign

$$(\mu, x)$$ such that

$$f(x; \mu) = 0$$

**EP conditions** ($n$)

$$\text{One } \lambda_i \text{ of } (D_x f)(x; \mu) = 0$$

$$\text{Nonhyperbolicity condition}$$

$$\det(D_x f)(x; \mu) = 0$$

(but must check that only one eigenvalue is 0)

Nondegeneracy and transversality conditions still exist, but their expression requires normal form theory
Lorenz Example

\[
\begin{align*}
\dot{x} &= 10(y - x) \\
\dot{y} &= Rx - y - xz \\
\dot{z} &= xy - \frac{8}{3}z \\
D_x f &= \begin{pmatrix}
-10 & 10 & 0 \\
R - z & -1 & -x \\
y & x & -\frac{8}{3}
\end{pmatrix}
\end{align*}
\]

\[
\det(D_x f) = -\frac{80}{3} + \frac{80R}{3} - 10x^2 - 10xy - \frac{80z}{3}
\]

\[
\begin{align*}
10(y - x) &= 0 \\
Rx - y - xz &= 0 \\
x y - \frac{8}{3}z &= 0 \\
-\frac{80}{3} + \frac{80R}{3} - 10x^2 - 10xy - \frac{80z}{3} &= 0
\end{align*}
\]

\[
(R^*, x, y, z) = (1,0,0,0)
\]

\[
\lambda_{1,2,3} = -11, -\frac{8}{3}, 0
\]
Lorenz Example
Hopf Bifurcation

A Hopf bifurcation occurs when the real parts of a single pair of complex eigenvalues changes sign

\[(\mu, x) \text{ such that } f(x; \mu) = 0\]

**EP conditions** \((n)\)

\[\Re \lambda_i, \lambda_j \text{ of } (D_y f)(x; \mu) = 0, \quad \text{but } \Im \lambda_i, \lambda_j \neq 0\]

\[\det(2(D_x f)(x; \mu) \otimes 1) = 0\]

(must check for neutral saddles)

A trick from linear algebra

If \(\lambda_i\) are the eigenvalues of \(M\), then the eigenvalues of \(2M \otimes 1 = \lambda_i + \lambda_j\)

Nondegeneracy and transversality conditions still exist, but their expression requires normal form theory
The Bialternate Matrix Product

If \( A \) and \( B \) are \( n \times n \) matrices, then \( A \odot B \) is the \( \frac{n(n-1)}{2} \times \frac{n(n-1)}{2} \) matrix

\[
\begin{align*}
A \odot B = & \frac{1}{2} \left( \begin{array}{cc} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{array} \right) + \left( \begin{array}{cc} b_{qq} & b_{qs} \\ a_{qq} & a_{qs} \end{array} \right) \\
& \frac{1}{2} \left( \begin{array}{cc} b_{pr} & b_{ps} \\ a_{qr} & a_{qs} \end{array} \right) + \left( \begin{array}{cc} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{array} \right)
\end{align*}
\]

\[(p,q),(r,s)\]

\((2,1)(2,1) \quad (2,1)(3,1) \quad (2,1)(3,2) \quad (2,1)(3,3) \quad \ldots \quad (2,1)(n,n)\)

\((3,1)(2,1) \quad (3,1)(3,1) \quad (3,1)(3,2) \quad (3,1)(3,3) \quad \ldots \quad (3,1)(n,n)\)

\((3,2)(2,1) \quad (3,2)(3,1) \quad (3,2)(3,2) \quad (3,2)(3,3) \quad \ldots \quad (3,2)(n,n)\)
The Hopf Condition Matrix in 3D

\[ 2\mathbf{M} \odot \mathbf{1}_{(p,q)(r,s)} = \begin{cases} 
-m_{ps} & r = q \\
 m_{pr} & r \neq p \text{ and } s = q \\
 m_{pp} + m_{qq} & r = p \text{ and } s = q \\
 m_{qs} & r = p \text{ and } s \neq q \\
 -m_{qr} & s = p \\
 0 & \text{otherwise}
\end{cases} \]

(2,1)(2,1)  (2,1)(3,1)  (2,1)(3,2)  
(3,1)(2,1)  (3,1)(3,1)  (3,1)(3,2)  
(3,2)(2,1)  (3,2)(3,1)  (3,2)(3,2)  

\[ 2\mathbf{M} \odot \mathbf{1} = \begin{pmatrix} 
 m_{11} + m_{22} & m_{23} & -m_{13} \\
 m_{32} & m_{11} + m_{33} & m_{12} \\
 -m_{31} & m_{21} & m_{22} + m_{33}
\end{pmatrix} \]
Lorenz Example

\[
\begin{align*}
\dot{x} &= 10(y - x) \\
\dot{y} &= Rx - y - xz \\
\dot{z} &= xy - \frac{8}{3}z
\end{align*}
\]

\[
D_x f = \begin{pmatrix}
-10 & 10 & 0 \\
R - z & -1 & -x \\
y & x & -\frac{8}{3}
\end{pmatrix}
\]

\[
2D_x f \otimes 1 = \begin{pmatrix}
-11 & -x & 0 \\
x & -\frac{38}{3} & 10 \\
-y & R - z & -\frac{11}{3}
\end{pmatrix}
\]

\[
\text{det}(2D_x f \otimes 1) = -\frac{4598}{9} + 110R - \frac{11x^2}{3} + 10xy - 110z
\]
Lorenz Example

\[10(y - x) = 0\]
\[Rx - y - xz = 0\]
\[xy - \frac{8}{3}z = 0\]
\[-\frac{4598}{9} + 110R - \frac{11x^2}{3} + 10xy - 110z = 0\]

\[\begin{cases} 
(R^*, \bar{x}, \bar{y}, \bar{z}) = \\
(4.64, 0, 0, 0) \\
(24.74, \pm 7.96, \pm 7.96, 23.74)
\end{cases}\]

At \((4.64, 0, 0, 0)\)
\[\lambda_{1,2,3} = -13.6667, -2.66667, 2.66667\]
Neutral saddle

At \((24.74, \pm 7.96, \pm 7.96, 23.74)\)
\[\lambda_{1,2,3} = -13.6667, 0 + 9.62453i, 0 - 9.62453i\]
Hopf Bifurcation
Lorenz Example
Summary of Global Codimension-1 Bifurcations

Saddle-Node on a Loop

Homoclinic

Saddle-Cycle

Heteroclinic
Summary of Unfolded Codimension-2 Points

- **Cusp**

- **Bogdanov-Takens**

- **Generalized Hopf**
Zero-Hopf Point

A Zero-Hopf point is a point where a saddle-node and a Hopf bifurcation occur simultaneously at the same equilibrium point.

At a Zero-Hopf point:
1) A real eigenvalue changes sign and
2) the real parts of a pair of complex eigenvalues changes sign

\[ \lambda_1 = 0, \lambda_{2,3} = \pm ki \]
A Double-Hopf point is a point where two distinct Hopf bifurcations occur simultaneously at the same equilibrium point.

At a Double-Hopf point
1) The real parts of a pair of complex eigenvalues changes sign and
2) The real parts of a second pair of complex eigenvalues changes sign

\[ \lambda_{1,2} = \pm k_1 i, \lambda_{3,4} = \pm k_2 i \]
Saddle-Focus Homoclinic Bifurcation

\[ \beta < 0 \]

\[ \beta = 0 \]

\[ \beta > 0 \]
Flip Bifurcation of Limit Cycles

\[ \alpha < 0 \quad \alpha = 0 \quad \alpha > 0 \]
Neimark-Sacker Bifurcation

\[ L_0 \]

\( \alpha > 0 \)

\( \alpha = 0 \)

\( \alpha > 0 \)
Bifurcations on Invariant Tori

Figure 6.33: Some co-dimension-1 bifurcations of limit cycles in three-dimensional phase space (modified from Izhikevich 2000).
Bifurcations of the Lorenz System

The diagram illustrates the bifurcations of the Lorenz system in a 3D coordinate system. The system's parameters are visualized with labels for the axes and a specific point labeled as Hom.
The Double Homoclinic Bifurcation

- R = 13
- R = 13.9265
- R = 15
The Saddle Cycles
More Saddle Cycles
The Basins of Attraction

\( R = 13 \)

\( R = 15 \)

\( R = 20 \)

Homoclinic Explosion
Transient Chaos

$R = 21$
The Heteroclinic Bifurcation

\( R = 24.06 \)
The Hopf Bifurcations

\[ R = 24.7368 \]
The “Full” Picture

The diagram illustrates the evolution of a system through various dynamical regimes, starting from simple dynamics, through transient chaos, and ending with a chaotic attractor. Key features include:

- **Homoclinic** and **Heteroclinic** bifurcations
- **Maxima** and **Minima** of certain functions
- **Critical Points** labeled as **P** and **H**
- **Parameter Values** indicated by **\(p^\pm\)**

The diagram transitions through different stages:
- **Simple Dynamics**
- **Transient Chaos**
- **Chaotic Attractor**

The diagram is a visual representation of how the system's behavior changes as parameters are varied, highlighting the complex interplay of attractors and bifurcations in the Lorenz system.
The “Full” Picture