Bifurcations on a Line

Reading: Chapter 3
Parameter Dependence

\[ \dot{x} = f(x; c) = -x^3 + 5x + c \]
A Family of Dynamical Systems

\[ \langle S, T, P, \phi_t (x; p) \rangle \]

\[ \phi_t : S \times P \rightarrow S \]
(c, x) such that \( f(x; c) = -x^3 + 5x + c = 0 \)
Geometric Picture of a Bifurcation

\[
\dot{x} = f(x; c) = -x^3 + 5x + c
\]

Saddle-Node Bifurcation (Fold Bifurcation)

Structurally Unstable (Nonhyperbolic)
Analytical Picture of a Bifurcation

\[(c, x)\) such that

\[f(x, c) = -x^3 + 5x + c = 0\]
\[f'(x) = -3x^2 + 5 = 0\]

\[
(c, x) = \left( \pm \frac{10}{3} \sqrt{\frac{5}{3}}, \pm \sqrt{\frac{5}{3}} \right)
\]

EP condition

Nonhyperbolicity condition
Hysteresis
Potential Functions and Bifurcation

\[ V(x) = -\int -x^3 + 5x + c \, dx = \frac{1}{4} x^4 - \frac{5}{2} x^2 - c x \]
More Parameter Dependence

\[ \dot{x} = f(x; d) = -x^3 + dx \]

\[ d = -5 \]
\[ d = 0 \]
\[ d = 5 \]
(d, x) such that  

\[ f(x; d) = -x^3 + dx = 0 \]
Even More Parameter Dependence

\[ \dot{x} = f(x; c) = -x^3 + (1 - c)x + c \]

Transcritical Bifurcation
The General Cubic System

\[ \dot{x} = f(x; d, c) = -x^3 + dx + c \]
(c, d, x) such that \( \dot{x} = f(x; d, c) = -x^3 + dx + c = 0 \)
\((c, d)\) Parameter Chart

\((c, d)\) such that

\[
\begin{align*}
  f(x; d, c) &= -x^3 + dx + c = 0 \\
  f'(x; d, c) &= -3x^2 + d = 0
\end{align*}
\]

\[4d^3 = 27c^2\]

EP condition

Nonhyperbolicity condition
Slices of the Equilibrium Surface
Transcritical Bifurcation in the Cubic System

\[ F \]

\[ F \]

\[ F \]

\[ F \]
Putting It All Together

\[ f(x; d, c) = 0 \]

\[ \dot{x} = f(x; d, c) = -x^3 + dx + c \]

\[
\begin{aligned}
    f(x; d, c) &= 0 \\
    f'(x; d, c) &= 0
\end{aligned}
\]
Combining Local Bifurcations

\[ \dot{x} = -x^5 + x^3 + r x \]
Summary of Dynamics on a Line

- Limit Sets
  - Equilibrium points only \((f(x) = 0)\)

- Stability
  - Stable \((f'(x) < 0)\)
  - Unstable \((f'(x) > 0)\)
  - Nonhyperbolic \((f'(x) = 0)\)

- Phase Portraits

- Bifurcation Diagrams (equilibrium curves, \(f(x; p) = 0\))
  - Saddle-node \((f(x; p) = f'(x; p) = 0, \ldots)\)
  - Transcritical \((f(x; p) = f'(x; p) = 0, \ldots)\)
  - Pitchfork \((f(x; p) = f'(x; p) = 0, \ldots)\)
  - …

- Equilibrium Surfaces \((f(x; p_1, p_2) = 0)\)

- Parameter Charts (bifurcation curves, \(f(x; p_1, p_2) = f'(x; p_1, p_2) = 0)\)
Another Kind of Change with Parameters