A Gentle Introduction to Dynamical Systems Theory

Reading: Chapter 1
Isaac Newton and His 2 Bodies

\[ F = ma \]

Given a differential equation, find its solutions

\[ r(\theta) = \frac{a(1 - e^2)}{1 + e \cos \theta} \]
Henri Poincaré and His 3 Bodies

Problem: The solution to almost all differential equations cannot be expressed as an explicit formula.

Given a differential equation you cannot solve, merely try to say something useful about its solutions.

Universality

Stephen Smale

René Thom

You are not given a differential equation but you are still asked to say something useful about its solutions!
A dynamical system is one whose future state depends on its current state and inputs in some principled way.
A Dynamical System

\[ \langle S, T, \phi_t \rangle \]

State Space

Ordered Time Set

Evolution Operator
**A State Space**

\[ \langle S, T, \phi_t \rangle \]

- The state space $S$ defines the space of possible states of the system.
- Must contain enough information to uniquely determine the future state of the system.
- May be
  - symbolic or numerical
  - discrete or continuous, or a hybrid of the two
  - any dimension or topology
An Ordered Time Set

\[ \langle S, T, \phi_t \rangle \]

- An ordered time set $T$ specifies the possible times at which the state of the system is defined.
- May be
  - discrete (map): $\mathbb{Z}$
  - continuous (flow): $\mathbb{R}$
An Evolution Operator

\[ \langle S, T, \phi_t \rangle \]

- An evolution operator \( \phi_t : S \rightarrow S \)
  - transforms an initial state \( x_0 \in S \) at time \( t = 0 \)
  - to another state \( x_t \in S \) at time \( t \in T \)
- May be
  - given explicitly or implicitly
  - deterministic or stochastic
- Two key properties
  - \( \phi_0(x) = x \)
  - \( \phi_{s+t}(x) = \phi_t(\phi_s(x)) \)
A Finite State Machine

\[ \langle S, \mathbb{Z}^*, f(s) \rangle \]

\[ S = \{ s_0, s_1, s_2, s_3 \} \]

\[ \mathbb{Z}^* = \{ 0 \} \cup \mathbb{Z}^+ \]

\[ = \{ 0, 1, 2, \ldots \} \]

<table>
<thead>
<tr>
<th>s</th>
<th>f(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>s_0</td>
<td>s_2</td>
</tr>
<tr>
<td>s_1</td>
<td>s_2</td>
</tr>
<tr>
<td>s_2</td>
<td>s_3</td>
</tr>
<tr>
<td>s_3</td>
<td>s_2</td>
</tr>
</tbody>
</table>
Evolution Operator

Evolution Operator

\[ x_k \rightarrow x_{k+1} = f(x_k) \]

Next-State Function

\[ \phi_t(x) = f(f(\ldots f(x)))) = f^t(x) \]

Evolution Operator

\[ \begin{array}{c|cccccc}
    \text{Next State} & 0 & 1 & 2 & 3 & \ldots \\
    \hline
    s_0 & s_0 & s_2 & s_3 & s_2 & \ldots \\
    s_1 & s_1 & s_2 & s_3 & s_2 & \ldots \\
    s_2 & s_2 & s_3 & s_2 & s_3 & \ldots \\
    s_3 & s_3 & s_2 & s_3 & s_2 & \ldots \\
\end{array} \]
A Periodic Orbit
A Fixed Point
Bifurcation
**Autonomous vs. Nonautonomous**

\[ x_{k+1} = f(x_k) \]

\[ x_{k+1} = f(x_k, in_k) \]
\[ x_{k+1} = f_t(x_k) \]
FSM as a Deterministic Finite Automata

\[ \langle S, \mathbb{Z}^*, f(s;i), I \rangle \]

\[ \langle S, \mathbb{Z}^*, f_t(s) \rangle \]

\[ \langle Q, \Sigma, \delta, q_0, F \rangle \]
Another Iterated Map

Continuous State: $S = \mathbb{R}$

Discrete Time: $T = \{0, 1, 2, 3, 4, \ldots\}$

Implicit Evolution Operator:

\[ x_{k+1} = f(x_k) \quad \text{(a next-state function)} \]

\[ \phi_t(x) = f(f(\ldots f(x))) = f^t(x) \]

Example: $x_{k+1} = 3x_k(1-x_k)$
More Examples

- Finite State Machine
- Iterated Maps
- Ordinary Differential Equations
- Cellular Automata
- Turing Machines
- Hybrid Systems
- Stochastic Differential Equations
- Partial Differential Equations
- Delay-Differential Equations
- Integro-Differential Equations
- Differential-Algebraic Equations
Basic Concepts

Vector Field

Initial State
Solution Trajectory

Flow

Limit Set
Stability

Phase Portrait

Bifurcation Diagram

Parameter Chart
Syllabus

• 1-D dynamical systems
  - Limit sets
  - Stability
  - Phase portraits
  - Bifurcations

• 2-D dynamical systems
  - New limit sets, stabilities, phase portraits and bifurcations
  - Coupled oscillators

• n-D dynamical systems
  - General limit sets, stabilities, phase portraits and bifurcations
  - Chaotic dynamics

• Advanced topics
  - Nonautonomous dynamical systems
  - Infinite-dimensional dynamical systems
  - Hybrid dynamical systems