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A transient Markov chain with finitely many cutpoints

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Dedicated to David Freedman with admiration

Abstract: We give an example of a transient reversible Markov chain that almost surely has only a finite number of cutpoints. We explain how this is relevant to a conjecture of Diaconis and Freedman and a question of Kaimanovich. We also answer Kaimanovich's question when the Markov chain is a nearest-neighbor random walk on a tree.

1. Introduction

While studying extensions of De Finetti's theorem to Markov chains, Diaconis and Freedman [3] stated a general conjecture for transient Markov chains $\{S_n\}$. We give a result on cutpoints that is relevant to their conjecture. We begin with some background.

We say that an event A in the space of trajectories of the Markov chain is **exchangeable** if it is invariant under finite permutations, i.e., if $(S_0, S_1, \ldots) \in A$, then so is $(S_{\pi(0)}, \ldots, S_{\pi(n)}, S_{n+1}, \ldots)$ for any n and any permutation π of $\{0, \ldots, n\}$. The σ-field of exchangeable events, $\mathcal{E}$, is called the exchangeable σ-field. Let $\overline{\mathcal{E}}$ be the completion of $\mathcal{E}$. A transient process visits each state only finitely often, and so for each state x in the state space X there is a random variable $V(x)$ that counts the number of visits, $V(x) := \#\{n \geq 0\,;\ S_n = x\}$. We call the collection $V := \{V(x)\}_{x \in X}$ the **occupation numbers** of the process. Clearly, V is $\mathcal{E}$-measurable. A natural question, posed by Kaimanovich [6], is to determine under what conditions the exchangeable σ-field is generated by V. This was motivated by similar issues arising in the study [7] of random walks on lamplighter groups.

Write $V_n(x) := \#\{k \in [0, n]\,;\ S_k = x\}$. Note that an event $A \in \sigma(S_j\,;\ j \geq 0)$ is invariant under permutations of $S_0, \ldots, S_n$ if and only if $A \in \sigma(V_n, S_{n+1}, S_{n+2}, \ldots)$. Therefore

$$\mathcal{E} = \bigcap_n \sigma(V_n, S_{n+1}, S_{n+2}, \ldots)\,. \tag{1}$$

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For any Markov chain $\{S_n\}$, the sequence of transitions $\{(S_n, S_{n+1})\}$ is also a Markov chain; for such chains of transitions, Kaimanovich's question was posed earlier as a conjecture by Diaconis and Freedman in [3]. To be precise, let $M_n(x, y)$ be the number of transitions made from x to y up to time n, so that $M_n(x, y)$ increases to a finite limit $M(x, y)$ as $n \to \infty$. They made the following conjecture in [3]:

Conjecture 1.1. *The intersection of the σ-fields*

$$\bigcap_n \sigma(M_n, S_{n+1}, S_{n+2}, \ldots) \tag{2}$$

is always generated (up to completion) by M.

By comparing (2) to (1), we see that (2) is just the exchangeable σ-field for the chain of transitions $\{(S_n, S_{n+1})\}$.

James and Peres [5] related the questions above to *cutpoints* of the Markov chain trajectory. Call x a **cutpoint** if for some k, we have $S_k = x$ and the future of the chain, $\{S_{k+1}, S_{k+2}, \ldots\}$, is disjoint from its past $\{S_0, S_1, \ldots, S_k\}$. Call S_k a **strong cutpoint** if the probability of a transition from S_i to S_j is 0 whenever $i < k < j$. In [5], Conjecture 1.1 was proved under the condition

(3) the Markov chain $\{S_n\}$ has infinitely many cutpoints almost surely.

We give a brief outline to illustrate the connection; see [5] for more details. Under the assumption (3), the portions $\psi_1, \psi_2, \ldots$ of the space-time path (n, S_n) between successive cutpoints are conditionally independent given M, and the intersection (2) is contained in the tail σ-field of the $\{\psi_j\}_{j \geq 1}$, which is trivial (given M) by Kolmogorov's zero-one law. Conditional triviality of a σ-field given M means that the σ-field is generated by M up to completion.

James and Peres [5] also showed that if $\{S_n\}$ almost surely has infinitely many strong cutpoints, then $\overline{\mathcal{E}}$ is generated by the occupation numbers. Thus, if every transient Markov chain had infinitely many strong cutpoints a.s., then Kaimanovich's question would be resolved.

In general, one expects that a random walk that is "very transient" will have infinitely many strong cutpoints. As shown in [1, 5, 8], transient random walks on Cayley graphs have infinitely many strong cutpoints a.s. More precisely, Lawler [8] proved (3) for simple random walk on the lattices $\mathbf{Z}^d$ for $d \geq 4$ and his argument applies to strong cutpoints and to any Cayley graph with volume growth at least polynomial of degree 5. This was extended, using a different argument, to $\mathbf{Z}^3$ in [5]. Blachère [1] extended the argument of [5] and showed that simple random walks on all transient Cayley graphs of groups have infinitely many strong cutpoints.

This raises the natural question of whether *every* transient Markov chain has infinitely many cutpoints a.s.; a positive answer would establish the conjecture of Diaconis and Freedman. In Section 3 we show, however, that this is not true, even for birth-and-death chains.

2. Exchangeability, transition counts and trees

In this section, we show that for transient nearest-neighbor walks on trees, the exchangeable σ-field is generated by the occupation numbers. This result was established in the thesis [4] of the first author, but was never published; the proof here is shorter than in [4], but relies on the same ideas. Note that the example in Section 3 is a nearest-neighbor random walk on a special tree (a halfline) such that the walk a.s. has finitely many cutpoints, so the proof cannot rely on cutpoints.

Consider a transient Markov chain as in the introduction. If $V(x) > 0$, let $U(x)$ be the state visited by the Markov chain immediately after its last visit to x. For completeness, define $U(x) := x$ when $V(x) = 0$. Let $\overline{\sigma}$ denote the completion of a σ-field.

Theorem 2.1. *Let $\{S_n\}$ be a transient Markov chain starting at a fixed state, x_0. Then $\mathcal{E} \subseteq \overline{\sigma}\big(\{M(x,y), U(x)\,;\ x, y \in X\}\big)$.*

Proof. As in Wilson [9], we imagine running the Markov chain by using infinite stacks under each of the states. The stack under a state x consists of possible successors to x and is generated independently of all other stacks by using the transition probabilities from x repeatedly for independent successors. Once the stacks are generated, the chain moves by moving to the state given at the top of the stack under x_0 and removing ("popping") the top state under x_0. This is repeated from the current state, and so on. The number of states under x that are eventually popped equals $V(x)$ and the last one is $U(x)$. Let $W(x)$ be the ordered list of states under x that are popped, *excluding* the last one. Write $[W(x)]$ for multi-set of states in $W(x)$, i.e., the unordered list of states (with repetition) in $W(x)$. Note that $\sigma\big(M(x,y), U(x)\,;\ x, y \in X\big) = \sigma\big([W(x)], U(x)\,;\ x \in X\big)$.

We first claim that if $W(x)$ is re-ordered for x in some finite set of states A, then the resulting chain $\{S'_n\}$ starting at x_0 will have the same counts $M(x,y)$ and same final exits $U(x)$. It suffices to prove this when A is a singleton. Moreover, if A is not x_0, then we may simply begin the chain when it first reaches A and pop the states that are used before then, reducing the situation to $A = \{x_0\}$. Thus, let $A = \{x_0\}$. The transitions of the chain $(S_0, S_1, \dots)$ describe an Eulerian circuit of a directed multi-graph, G. That is, G consists of directed edges (S_k, S_{k+1}) connecting vertices $\{S_k\}$ and each vertex has the same number of edges leading to it as leading away from it, except that x_0 has one more edge leading away. When $W(x_0)$ is re-ordered, the sequence $(S'_0, S'_1, \dots)$ does not leave G (while using each edge at most once) since the number of possible arrivals to a vertex via an edge of G is at most the number of possible departures. Thus, $(S'_0, S'_1, \dots)$ traverses a subgraph G' of G. If we re-order again to the original order, then this argument shows that the resulting graph covered, G, is a subgraph of G'. Thus, $G' = G$. Therefore, the final transition counts are the same, as claimed. In addition, the stacks were popped in the same order at all vertices other than x_0, so their final exits are unchanged, as is $U(x_0)$.

We next claim that the distribution of $\{S_n\}$ given $[W(x)]$ and $U(x)$ for all $x \in X$ can be represented as follows: Choose randomly and uniformly an ordering $W(x)$ for each $[W(x)]$, independently for each $x \in X$. Then the resulting walk starting from x_0 and determined by these stacks has the same law as the Markov chain. To see this, consider the set B of trajectories that correspond to a given collection of $[W(x)]$ and $U(x)$. Let $\{S_n\} \in B$ be one such trajectory. Since re-ordering any finite set of the corresponding $W(x)$ gives a finite permutation of $\{S_n\}$ with the same counts and final exits, B and the conditional Markov chain measure on B are preserved. Therefore the Markov chain measure is preserved under re-ordering every $W(x)$. The only such invariant measure is the one described, so the claim is proved.

Finally, let $C \in \mathcal{E}$. Let B be the set of trajectories that correspond to a given collection of $[W(x)]$ and $U(x)$. Since both C and B are invariant under re-ordering any finite $W(x)$, so is $C \cap B$. In addition, the orderings $W(x)$ are independent given all $[W(x)]$ (and $U(x)$), so the conditional probability of C given B is 0 or 1 by Kolmogorov's 0-1 law. Let D_0 be the union of those B for which the conditional probability of C given B is 0 and D_1 be the union of the other B. Then $P[C \cap D_0] = 0$, so $P[C \triangle D_1] = 0$. Since $D_1 \in \overline{\sigma}\big(M(x,y), U(x)\,;\ x, y \in X\big)$, the theorem is proved. □

Corollary 2.1. *For a transient nearest-neighbor random walk on a tree (with arbitrary transition probabilities), we have* $\overline{\mathcal{E}} = \overline{\sigma}(V)$.

Proof. Since a transient random walk on a tree T must tend to some end of T, it follows that the pointers $U(x)$ are determined by the occupation field V. In view of the preceding theorem, it suffices to show that the transition numbers $M(x,y)$ are also determined by V. Write $L_0 = S_0 = x_0$, and for $j \geq 1$ define $L_j = U(L_{j-1})$. The sequence $L = \{L_j\,;\ j \geq 0\}$ is known as the *loop-erasure* of the trajectory $\{S_k\,;\ k \geq 0\}$. Consider the finite tree $T_F = T_F(L_k)$ that is spanned by L_k and all vertices x with $V(x) > 0$ and that can be reached from x_0 without visiting L_k. The proof will now follow from the following **claim:** *Given a finite walk from x_0 to y on a finite tree T_F, the edge transition numbers M_F of the walk are determined by the occupation numbers V_F of all vertices except y.* The claim is proved by induction on the number N of vertices in T_F. The base case $N \leq 2$ is clear. For $N > 2$, the tree T_F has some leaf z that is different from y. Let z_* denote the neighbor of z. Clearly $M(z, z_*) = V(z)$ and $M(z_*, z) = V(z) - \mathbf{1}_{z=x_0}$. Removing z from the tree and subtracting $V_F(z)$ from $V_F(z_*)$ reduces the problem to a tree with $N-1$ vertices and completes the induction step. To apply the claim to our situation, take $y = L_k$ and observe that for all vertices $w \in T_F(L_k)$ except possibly L_k itself, the occupation number $V(w)$ determined by the infinite random walk path coincides with $V_F(w)$, the occupation number determined by the portion of that path in $T_F(L_k)$. (It is certainly possible that $V(L_k) > V_F(L_k)$, due to excursions of the random walk from L_k to the complement of T_F.) □

3. A transient birth-and-death chain with finitely many cutpoints

We shall exhibit a birth-and-death chain, i.e., a nearest-neighbor random walk on $\mathbf{N}$, which is transient but has only finitely many cutpoints a.s. We shall use the following basic fact about random walks and electrical networks. Let $r_k > 0$ be given for $k \geq 1$. (Interpret r_k as the resistance of the edge between k and $k+1$.) Consider the birth-and-death chain on $\{1, 2, \ldots, n\}$ where the transition probability from 1 to 2 is 1, and for $k > 1$, the transition probability from k to $k+1$ is $r_{k-1}/(r_{k-1}+r_k)$ and the transition probability from k to $k-1$ is $r_k/(r_{k-1}+r_k)$. Then the probability that the chain reaches n before 1 when starting from k equals $\sum_{j=1}^{k-1} r_j / \sum_{j=1}^{n-1} r_j$. See [2], §§II.1 and IX.2. Of course, this can also be phrased as a standard gambler's ruin calculation. In particular, taking a limit as $n \to \infty$ shows that transience is equivalent to $\sum_{j=1}^{\infty} r_j < \infty$.

Theorem 3.1. *Fix $\beta > 1$. Let $r_k > 0$ have the property that $r_k \asymp k^{-1}(\log k)^{-\beta}$ for all $k \geq 2$, where the symbol $\asymp$ means that the ratio of the two sides is bounded above and below by positive constants that do not depend on k. Consider the birth-and-death chain on $\mathbf{N} = \{1, 2, \ldots\}$ with transition probability $r_{k-1}/(r_{k-1}+r_k)$ from k to $k+1$ and transition probability $r_k/(r_{k-1}+r_k)$ from k to $k-1$ for all $k \geq 2$. (The transition probability from 1 to 2 is 1.) Then this chain is transient and has only finitely many cutpoints a.s.*

Proof. We may assume the chain starts at 1. Since $\sum_k r_k < \infty$, the walk is transient.

Denote $t_k := \sum_{j \geq k} r_j$. The usual gambler's ruin calculation shows that the probability that the walk will have k as a cutpoint is $p_k = r_k / t_k$.

Let $j < k$. Given that k is a cutpoint, let $Q_k(j)$ be the conditional probability that j is a cutpoint. Then $Q_k(j)$ is the probability that a walk starting at $j+1$ visits $k+1$ before visiting j, i.e.,

$$Q_k(j) = \frac{r_j}{(t_j - t_{k+1})} \,. \tag{4}$$

This is also the conditional probability

$$\mathbf{P}[j \text{ is a cutpoint} \mid \ k \text{ is a cutpoint}, F_{k+1}] \,,$$

where F_{k+1} is any event determined by the future of the walk after it reaches $k+1$ for the first time.

Let $C_{j,k}$ be the set of cutpoints in $(2^j, 2^k]$ and $A_{j,k} := |C_{j,k}|$. Write $a_m := P[A_{m,m+1} > 0]$ and

$$b_m := \min \Big\{ \sum_{i=1}^{2^{m-1}} Q_k(k-i) \, ; \ k \in (2^m, 2^{m+1}] \Big\} \,.$$

On the event that $A_{m,m+1} > 0$, let ℓ_m be the largest cutpoint in $C_{m,m+1}$. Bound below the expected number of cutpoints in $(2^{m-1}, 2^{m+1}]$ by conditioning on the last cutpoint in $(2^m, 2^{m+1}]$, if there is one:

$$\begin{aligned} \sum_{j=2^{m-1}+1}^{2^{m+1}} p_j &= \mathbf{E}[A_{m-1,m+1}] \\ &\geq a_m \mathbf{E}[A_{m-1,m+1} \mid A_{m,m+1} > 0] \\ &= a_m \mathbf{E}\big[\mathbf{E}[A_{m-1,m+1} \mid A_{m,m+1} > 0, \ \ell_m]\big] \\ &\geq a_m b_m \,. \end{aligned} \tag{5}$$

Now $t_j \asymp (\log j)^{-\beta+1}$, whence $p_j \asymp (j \log j)^{-1}$ for $j \geq 2$. Furthermore, we have $t_{k-i} - t_{k+1} \asymp i r_k \asymp i r_{k-i}$ for $1 \leq i \leq 2^{m-1}$ and $2^m < k \leq 2^{m+1}$. By (4), this means that $Q_k(k-i) \geq c/i$ for some constant $c > 0$ and i, k in those ranges, which gives in turn that $b_m \geq c' m$ for some constant $c' > 0$. On the other hand, the left-hand side of (5) is at most $c''(\log\log 2^{m+1} - \log\log 2^m) \leq c'''/m$ for some $c'', c''' < \infty$. It follows that $a_m = O(1/m^2)$ is summable, so that there are a.s. only finitely many cutpoints by the Borel-Cantelli lemma. It also follows that with positive probability, there are no cutpoints at all. □

4. Concluding remarks

Given a transient Markov chain $\{S_j\}$ with a fixed starting state, it is easy to see that for any n, the event A_n that $S_0, S_1 \ldots, S_n$ are all cutpoints has positive probability. Indeed, starting from a trajectory $S_0, S_1, S_2 \ldots$, consider the corresponding loop-erased path $\{L_j\}$ obtained by erasing cycles in the path as they are created. More precisely, $L_0 = x_0$ and $L_j = U(L_{j-1})$ for $j > 0$, where $U(\cdot)$ is the ultimate successor function defined in Section 2. Fix a sequence of vertices $(x_1, \ldots x_n)$ such that the event $B_n = \{(L_0, \ldots, L_n) = (x_0, \ldots, x_n)\}$ has $P(B_n) > 0$. If B_n holds for the trajectory $\{S_j^*\}$, then $x_j = L_j = S_{k_j}$ for some random sequence $\{k_j\}$, and we define a new trajectory $\{S_j^*\}$ with $S_j^* = L_j$ for $j = 0, \ldots, n$ and $S_{n+i}^* = S_{k_n+i}$ for $i > 0$. For this new trajectory $x_0, \ldots, x_n$ are all cutpoints. We conclude that $P(A_n) \geq P(B_n) \prod_{j=1}^n p(x_{j-1}, x_j) > 0$.

We do not know whether every transient Markov chain has an infinite expected number of cutpoints. For any birth-and-death chain, this does hold since (in the notation of the preceding proof) $\sum_{k \geq m} p_k \geq \sum_{k \geq m} r_k/t_m = 1$ for every m, whence the series $\sum_k p_k$ diverges.

Another natural question that we cannot answer is whether a *simple* random walk on any transient graph of bounded degree must have infinitely many cutpoints a.s.

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