Stationary Determinantal Processes
(Fermionic Lattice Gases)

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distribution of eigenvalues of random matrices
(Wigner 1950s, ...)

distribution of zeroes of Riemann zeta function
(Montgomery 1973, 1975, ...)

Macchi (1972–5)

Soshnikov (survey 2000, other papers)

Shirai and Takahashi (2003)

Shirai and Yoo (Glauber dynamics 2002)

Lyons (2003)

Borodin, Okounkov, Olshanki, Johansson, Reshetikhin, ...

Let $\mathbb{T}^d$ be the d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ and λ_d be unit Lebesgue measure on $\mathbb{T}^d$. For integrable f on $\mathbb{T}^d$, define its **Fourier coefficients**

$$\forall k \in \mathbb{Z}^d \qquad \widehat{f}(k) := \int_{\mathbb{T}^d} f(x) e^{-2\pi i k \cdot x} \, d\lambda_d(x) \, .$$

Suppose that $f : \mathbb{T}^d \to [0, 1]$. Define a $\mathbb{Z}^d$-invariant probability measure $\mathbf{P}^f$ on the Borel sets of $\{0, 1\}^{\mathbb{Z}^d}$ by defining the probabilities of the cylinder sets

$$\mathbf{P}^f[\eta(e_1) = 1, \ldots, \eta(e_n) = 1] := \det \left[\widehat{f}(e_j - e_i)\right]_{1 \le i,j \le n}$$

for all $e_1, \ldots, e_n \in \mathbb{Z}^d$. If $d = 1$, this is a minor of the Toeplitz matrix

$$\begin{bmatrix} & & \ddots & & & & \\ \cdots & \widehat{f}(-1) & \widehat{f}(0) & \widehat{f}(1) & \cdots & & \\ \cdots & \widehat{f}(-2) & \widehat{f}(-1) & \widehat{f}(0) & \widehat{f}(1) & \cdots & \\ & \cdots & \widehat{f}(-2) & \widehat{f}(-1) & \widehat{f}(0) & \widehat{f}(1) & \cdots \\ & & & & & \ddots & \end{bmatrix} .$$

EXAMPLE: $\mathbf{P}^f[\eta(e)=1] = \widehat{f}(\mathbf{0}) = \int_{\mathbb{T}^d} f\,d\lambda_d$. All $\mathbf{P}^f$ are ergodic, so the density of 1s is a.s. the average of f. How much of f can we recover from a sample of $\mathbf{P}^f$ — or, equivalently, from $\mathbf{P}^f$ itself?

If $\eta(e)$ are independent, then there is no more information than $\widehat{f}(\mathbf{0})$. Fortunately, f is then constant and conversely:

EXAMPLE: If $f \equiv p$, then $\mathbf{P}^f$ is Bernoulli(p) measure and conversely.

EXAMPLE: The covariance of $\eta(\mathbf{0})$ and $\eta(e)$ is

$$\begin{aligned}\mathbf{P}^f[\eta(\mathbf{0})=1, \eta(e)=1] - \mathbf{P}^f[\eta(\mathbf{0})=1]\mathbf{P}^f[\eta(e)=1] \\ = \begin{vmatrix} \widehat{f}(\mathbf{0}) & \widehat{f}(e) \\ \widehat{f}(-e) & \widehat{f}(\mathbf{0}) \end{vmatrix} - \widehat{f}(\mathbf{0})^2 = -|\widehat{f}(e)|^2\end{aligned}$$

Thus, we can recover $|\widehat{f}|$ from $\mathbf{P}^f$. Can we recover f up to rotation and flips in the variables? How are properties of f reflected in properties of $\mathbf{P}^f$?

Example: If $d = 1$ and

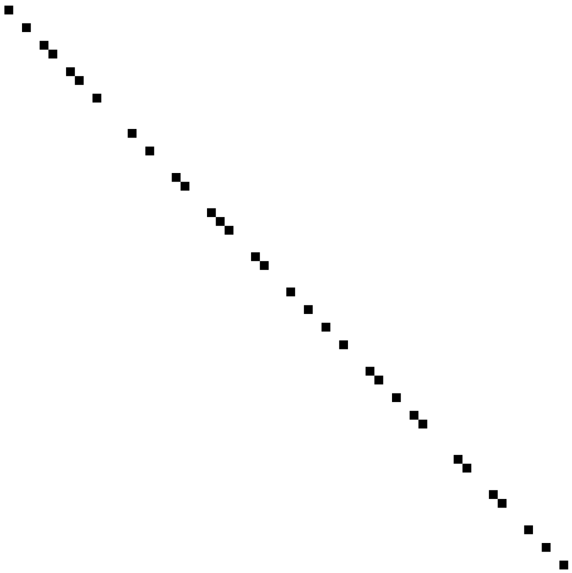
$$f(x) := \sin^2 \pi x = \frac{1}{2} - \frac{1}{4} e^{2\pi i x} - \frac{1}{4} e^{-2\pi i x},$$

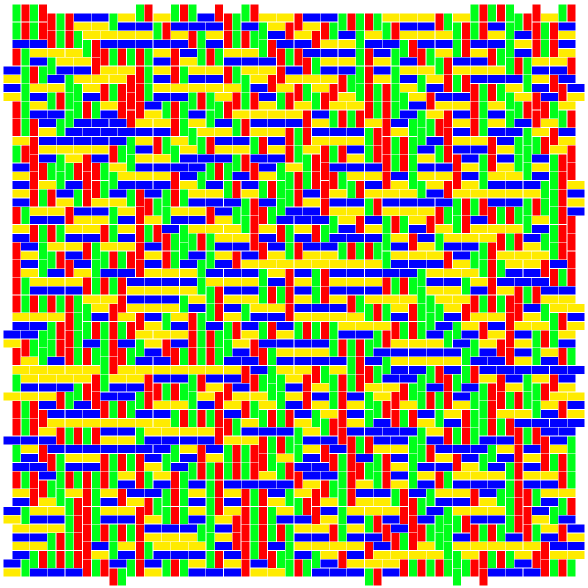
then $\mathbf{P}^f$ is 1-dependent. In particular, the restriction to $2\mathbb{Z}$ is Bernoulli(1/2).

Example: If $d = 1$ and $f := \mathbf{1}_{[0,1/2]}$, then

$$\widehat{f}(n) = \begin{cases} 1/2 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0 \text{ is even}, \\ 1/(\pi i n) & \text{if } n \text{ is odd}. \end{cases}$$

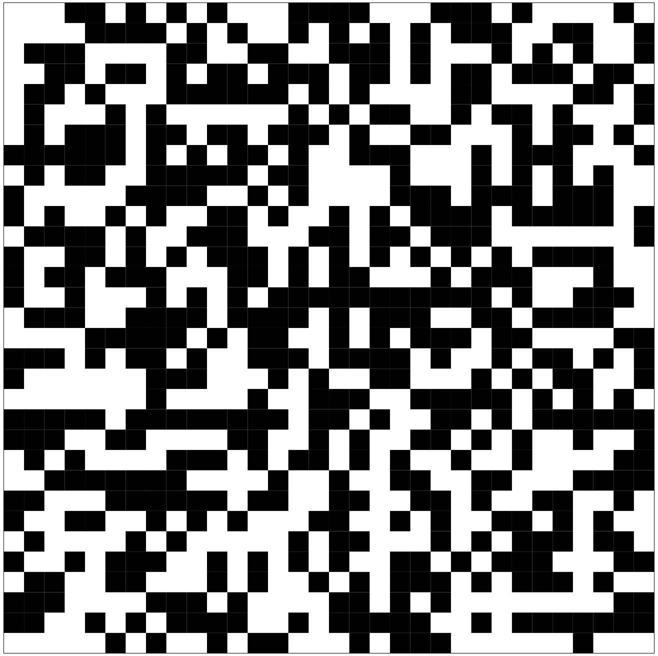
Again the restriction to $2\mathbb{Z}$ is Bernoulli(1/2). $\mathbf{P}^f$ is the zig-zag process of Johansson (2000) derived from uniform domino tilings in the plane (Burton-Pemantle 1993). The process shows the dominos that go up or to the right from the diagonal.



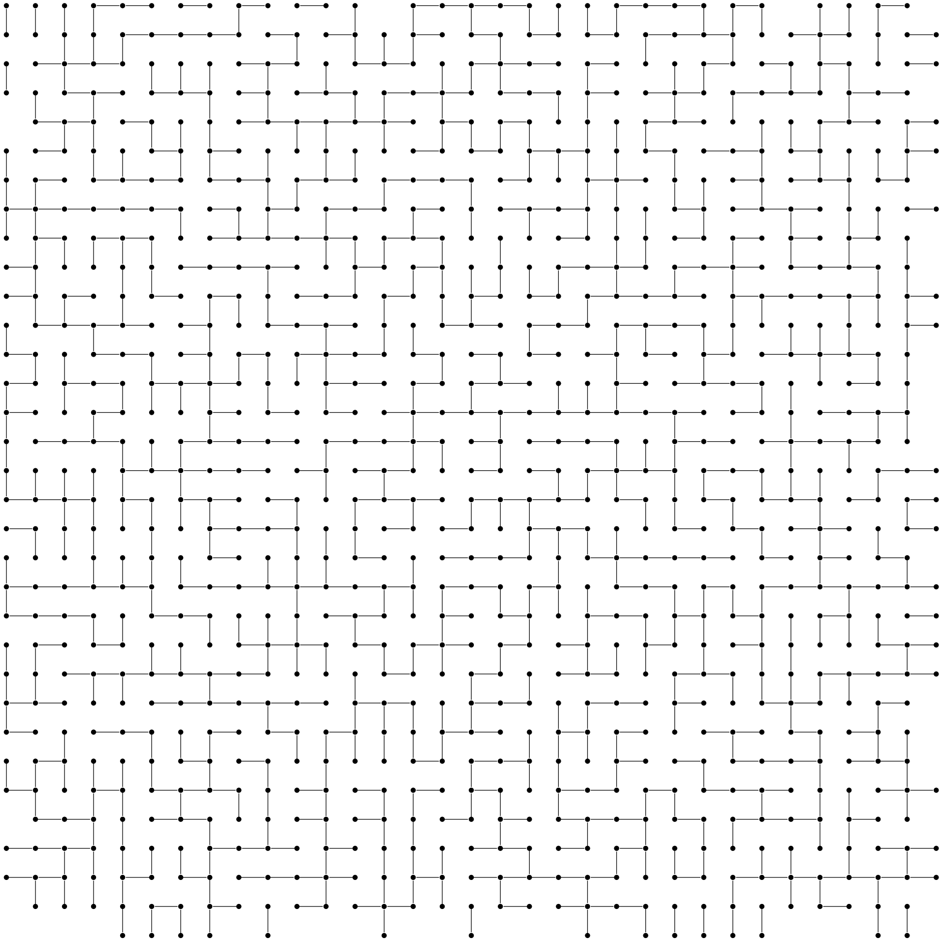


(D.B. Wilson)

EXAMPLE: $f(x, y) := \sin^2 \pi x / (\sin^2 \pi x + \sin^2 \pi y)$.

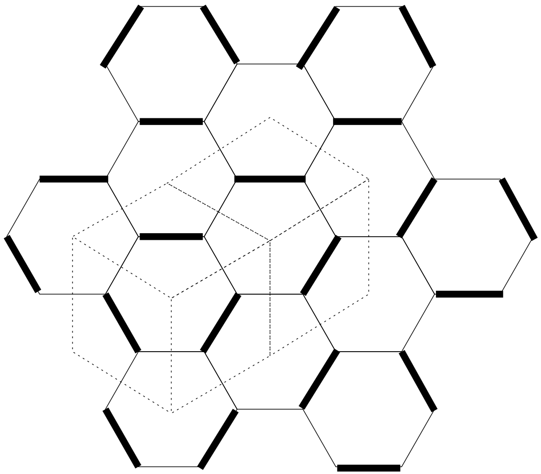


These are the horizontal edges of a uniform spanning tree in the square lattice (Pemantle 1991).



(D.B. Wilson)

EXAMPLE: $d = 1$ and $f := \mathbf{1}_{[0,1/3]}$ corresponds to a slice of the dimer model on the honeycomb lattice (Kenyon 1997).



(R. Kenyon)

EXAMPLE: Let $0 < a < 1$ and $d = 1$. If $f(x) := (1-a)^2/|e^{2\pi i x} - a|^2$, then $\mathbf{P}^f$ is a renewal process (Soshnikov 2000). The number of 0s between successive 1s has the same distribution as the number of tails until 2 heads appear for a coin that has probability a of coming up tails. Here,

$$\widehat{f}(k) = \frac{1-a}{1+a} a^{|k|} \,.$$

EXAMPLE: If $d = 1$ and f is the reciprocal of a trigonometric polynomial of degree n, then $\mathbf{P}^f$ is a regenerative process, regenerating after n successive 1s appear.

EXAMPLE: If $d = 1$ and $g(x) = f(2x)$, then $\mathbf{P}^g \restriction 2\mathbb{Z}$ looks like $\mathbf{P}^f$ (on $\mathbb{Z}$), as does $\mathbf{P}^g \restriction (2\mathbb{Z}+1)$. These two restrictions are independent of each other since $\widehat{g}(2k+1) = 0$ for all $k \in \mathbb{Z}$.

Why are these processes interesting?

- Relations with the theory of Toeplitz determinants.
- Connections to certain combinatorial models.
- Parameter space is infinite dimensional and rich.
- Unusual property of negative association.
- All are Bernoulli, i.e., isomorphic to i.i.d. processes.
- Interesting phase diagram in all dimensions.

An event $\mathcal{A}$ is called **increasing** if whenever $A \in \mathcal{A}$ and $e \in \mathbb{Z}^d$, we have also $A \cup \{e\} \in \mathcal{A}$. From a general study of determinantal probability measures by L., we deduce the following for our processes:

- If η has the distribution $\mathbf{P}^f$, then $\mathbf{1} - \eta$ has the distribution $\mathbf{P}^{\mathbf{1}-f}$. I.e., $\eta \sim \mathbf{P}^f \iff \mathbf{1} - \eta \sim \mathbf{P}^{\mathbf{1}-f}$.

- If $f \le g$, then $\mathbf{P}^f \preccurlyeq \mathbf{P}^g$, i.e., for all increasing events $\mathcal{A}$, we have $\mathbf{P}^f[\mathcal{A}] \le \mathbf{P}^g[\mathcal{A}]$. Alternatively, $\exists \eta_f, \eta_g$ on a common probability space with $\eta_f \sim \mathbf{P}^f$, $\eta_g \sim \mathbf{P}^g$, and $\eta_f(e) \le \eta_g(e)$ for all $e \in \mathbb{Z}^d$.

- If A is a finite subset of $\mathbb{Z}^d$ and $\eta_0 \in 2^A$, then $\mathbf{P} := \mathbf{P}^f[\,\bullet \mid \eta{\restriction}A = \eta_0]$ has **negative associations**, i.e., if $\mathcal{A}_1$ and $\mathcal{A}_2$ are increasing events that depend on disjoint subsets of $\mathbb{Z}^d$, then
$$\mathbf{P}[\mathcal{A}_1 \cap \mathcal{A}_2] \le \mathbf{P}[\mathcal{A}_1]\mathbf{P}[\mathcal{A}_2]\,.$$
I.e., if $\mathcal{A}$ is any increasing event, then $\mathbf{P}[\,\bullet \mid \mathcal{A}] \preccurlyeq \mathbf{P}$ off the "support" of $\mathcal{A}$.

Example: If $p \le f \le q$, then

$$\text{Bernoulli}(p) \preccurlyeq \mathbf{P}^f \preccurlyeq \text{Bernoulli}(q).$$

This means that there are random fields η_p, η, η_q on a common probability space with $\eta_p \sim \text{Bernoulli}(p)$, $\eta_q \sim \text{Bernoulli}(q)$, and $\eta \sim \mathbf{P}^f$ that satisfy

$$\forall e \in \mathbb{Z}^d \qquad \eta_p(e) \le \eta(e) \le \eta_q(e) \quad \text{a.s.}$$

Can we improve this? Let μ_p **be Bernoulli(p)** and

$$\mathsf{GM}(f) := \exp \int_{\mathbb{T}^d} \log f \, d\lambda_d \,.$$

Theorem (L. & Steif). *For any* $f : \mathbb{T}^d \to [0,1]$, *we have*

$$\mu_p \preccurlyeq \mathbf{P}^f \iff p \le \mathsf{GM}(f)$$
$$\mathbf{P}^f \preccurlyeq \mu_q \iff q \ge 1 - \mathsf{GM}(\mathbf{1} - f)\,.$$

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Proof. We do $d := 1$. Let r_n be the probability of having n consecutive 1's. By Szegő's limit theorem,

$$r_{n+1}/r_n \downarrow \mathsf{GM}(f) = \lim_{n\to\infty} r_n^{1/n}\,.$$

Because of negative associations, for any fixed n and any $a_1, \ldots, a_n \in \{0,1\}$,

$$\mathbf{P}^f[\eta_0 = 1 \mid \eta_i = a_i, i = 1, \ldots, n]$$

is minimized when all the a_i's are 1. In this case, the value is r_{n+1}/r_n. Since this is at least $\mathsf{GM}(f)$, we deduce that $\mu_p \preccurlyeq \mathbf{P}^f$ for $p \le \mathsf{GM}(f)$.

Conversely, if $\mathbf{P}^f \succcurlyeq \mu_p$, then certainly $r_n \ge p^n$ for all n. Hence $\mathsf{GM}(f) \ge p$. ∎

EXAMPLE: Let $f(x, y) := \sin^2 \pi x / (\sin^2 \pi x + \sin^2 \pi y)$, so that $\mathbf{P}^f$ is the law of the horizontal edges of a uniform spanning tree in the square lattice $\mathbb{Z}^2$. Then $\mathsf{GM}(f) = e^{-4\mathrm{G}/\pi}$, where

$$\mathrm{G} := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.9160^-$$

is **Catalan's constant**. Therefore $\mu_p \preccurlyeq \mathbf{P}^f \preccurlyeq \mu_{1-p}$ for $p := e^{-4\mathrm{G}/\pi} = 0.3115^+$.

What is the entropy of $\mathbf{P}^f$?

Conjecture: Entropy is a concave functional of f.

EXAMPLE: If $p := \widehat{f}(0)$ is fixed, then the maximum entropy among such $\mathbf{P}^f$ is when $f \equiv p$, in which case the entropy is $-p \log p - (1-p) \log(1-p)$. Which f give minimum entropy? Conjecture: some $\mathbf{1}_A$.

EXAMPLE: Consider all $f := \mathbf{1}_A$ with $p := |A|$ fixed. Which A give the largest and smallest entropies?

Phase Multiplicity

For the ferromagnetic Ising model, it suffices to consider the plus and minus states in order to determine whether there is a phase transition. Likewise, it suffices for determinantal processes to consider analogous conditionings.

Let $B_n^d := [-n, n]^d \cap \mathbb{Z}^d$.

For $f : \mathbb{T}^d \to [0, 1]$, we define a probability measure $(\mathbf{P}^f)^+$ on $2^{\mathbb{Z}^d}$ that will be "$\mathbf{P}^f$ conditioned on all 1's at ∞". We want to define $(\mathbf{P}^f)^+$ by

$$(\mathbf{P}^f)^+ := \lim_{n\to\infty} \mathbf{P}^f\left[\,\bullet \mid \eta \equiv 1 \text{ on } (B_n^d)^c\right].$$

We proceed in stages. Assume throughout that $f \neq \mathbf{0}$ and $f \neq \mathbf{1}$.

Let

$$(\mathbf{P}^f)_n^+ := \lim_{k\to\infty} \mathbf{P}^f \left[\,\bullet \mid \eta \equiv 1 \text{ on } B_{n+k}^d \backslash B_n^d \right].$$

Negative associations implies that, when restricted to B_n^d, the sequence on the right is stochastically decreasing and hence necessarily converges. Therefore $(\mathbf{P}^f)_n^+$ is well defined. Next let

$$(\mathbf{P}^f)^+ := \lim_{n\to\infty} (\mathbf{P}^f)_n^+ .$$

For fixed r, $(\mathbf{P}^f)_n^+$ restricted to B_r^d, is, for $n > r$, stochastically increasing in n and hence converges. This implies that its limit $(\mathbf{P}^f)^+$ is well defined and completes the definition of $(\mathbf{P}^f)^+$. The stochastic monotonicity results also imply that

$$(\mathbf{P}^f)^+ \preccurlyeq \mathbf{P}^f .$$

Using 0 instead of 1 boundary conditions, one defines $(\mathbf{P}^f)^-$, which satisfies

$$\mathbf{P}^f \preccurlyeq (\mathbf{P}^f)^- .$$

The probability measure $\mathbf{P}^f$ has **phase multiplicity** if $(\mathbf{P}^f)^- \neq (\mathbf{P}^f)^+$. (The most extreme boundary conditions are when all 1's or all 0's are used and the measures corresponding to any other limiting boundary conditions are "stochastically trapped" between these two special cases.)

THEOREM (L. & STEIF). *$\mathbf{P}^f$ has phase uniqueness if and only if there is a nonzero trigonometric polynomial T such that*

$$\frac{|T|^2}{f(\mathbf{1}-f)} \in L^1(\mathbb{T}^d)\,.$$

This follows from

THEOREM (L. & STEIF). *$(\mathbf{P}^f)^+ = \mathbf{P}^f$ iff there exists a nonzero trigonometric polynomial T such that*

$$\frac{|T|^2}{f} \in L^1(\mathbb{T}^d)\,.$$

Moreover, if $(\mathbf{P}^f)^+ \neq \mathbf{P}^f$, then $(\mathbf{P}^f)^+ = \delta_{\mathbf{0}}$.

One-Dimensional Examples: $|T|^2/f \in L^1(\mathbb{T}^d)$?

EXAMPLE: If $f : \mathbb{T} \to [0,1]$ is continuous and has a finite number of 0's with f approaching each of these 0's at most polynomially quickly, then $(\mathbf{P}^f)^+ = \mathbf{P}^f$. If the 0's are x_i and have order at most n, then take $T := \prod_{i=1}^k \sin^n 2\pi(x - x_i)$.

EXAMPLE: Suppose $f : \mathbb{T} \to [0,1]$ is continuous, vanishes at a single point x_0, and equals $e^{-1/|x-x_0|}$ in a neighborhood of x_0. Since the rate at which a trigonometric polynomial approaches 0 is at most polynomially quickly, there is no nonzero trigonometric polynomial T with $|T|^2/f \in L^1(\mathbb{T})$. Hence $(\mathbf{P}^f)^+ = \delta_{\mathbf{0}}$.

EXAMPLE: Let $f(x) := \sqrt{x}|\sin(\pi/x)|$ on $[0,1]$. Then $(\mathbf{P}^f)^+ \neq \mathbf{P}^f$ since there are an infinite number of first-order zeroes. But if g is the increasing rearrangement of f on $[0,1]$, then $(\mathbf{P}^g)^+ = \mathbf{P}^g$.

Two-Dimensional Examples: $|T|^2/f \in L^1(\mathbb{T}^d)$?

EXAMPLE: In 2 dimensions, the function

$$f(x,y) := \sin^2(2\pi y - \cos 2\pi x)$$

generates a process for which $(\mathbf{P}^f)^+ \neq \mathbf{P}^f$. This is because f vanishes to 2nd order on a curve

$$2\pi y = \cos 2\pi x$$

that is not in the zero set of any trigonometric polynomial in two variables. However, f has the same distribution as $g(x,y) := \sin^2 2\pi y$ and $(\mathbf{P}^g)^+ = \mathbf{P}^g$.

EXAMPLE: Suppose $f : \mathbb{T}^2 \to [0,1]$ is real analytic on a neighborhood of its zero set. Then $(\mathbf{P}^f)^+ = \mathbf{P}^f$ iff $f^{-1}(0)$ is contained in a (nontrivial) algebraic variety, where we view $\mathbb{T}^2$ as $\{(z_1, z_2) \in \mathbb{C}^2;\ |z_1| = |z_2| = 1\}$. This is because the slowest f can vanish at a point is of order $x^2 + y^2$ and $1/(x^2 + y^2)$ is not integrable. So all the zeroes of f must be cancelled by those of T.

The key to analysis is the following representation of conditional probabilities. For any set $B \subset \mathbb{Z}^d$, write $[B]_f$ for the closure in $L^2(f)$ of **the linear span of the complex exponentials** $\{\mathbf{e}_k\,;\ k \in B\}$, where $\mathbf{e}_k(x) := e^{2\pi i k \cdot x}$. Let $(\bullet, \bullet)_f$ denote the **inner product** in $L^2(f)$. Then

$$\mathbf{P}^f\Big[\eta(\mathbf{0}) = 1 \mid \eta\restriction B \equiv 1\Big] = \|P^{\perp}_{[B]_f}\mathbf{1}\|^2_f\,,$$

where $P^{\perp}_{[B]_f}$ denotes **orthogonal projection onto the orthocomplement** of $[B]_f$ in $L^2(f)$.

Proof. Assume B is finite. Note that $\widehat{f}(k-j) = (\mathbf{e}_j, \mathbf{e}_k)_f$, so that

$$\mathbf{P}^f[\eta\restriction B \equiv 1] = \det[(\mathbf{e}_j, \mathbf{e}_k)_f]_{j,k\in B}\,,$$

and similarly for $\mathbf{P}^f[\eta\restriction(B \cup \{\mathbf{0}\}) \equiv 1]$. Thus, the conditional probability is the quotient of the squares of the volumes of two parallelepipeds, so equal to the square of the length of the altitude on the base. This gives the equation. ∎

We use this to prove that if $(\mathbf{P}^f)^+ \neq \delta_{\mathbf{0}}$, then there exists a nonzero trigonometric polynomial T such that

$$\frac{|T|^2}{f} \in L^1(\mathbb{T}^d)\,.$$

If $(\mathbf{P}^f)^+ \neq \delta_{\mathbf{0}}$, then for large n,

$$\mathbf{P}^f[\eta(\mathbf{0}) = 1 \mid \eta{\restriction}B \equiv 1] > 0\,,$$

where $B := (B_n^d)^c$. Thus, $u := P^{\perp}_{[B]_f}\mathbf{1} \neq \mathbf{0}$. Now $u \in L^2(f)$ and for all $k \in B$, we have

$$0 = (u, \mathbf{e}_k)_f = \widehat{uf}(k)\,.$$

That is, $T := uf$ is a nonzero trigonometric polynomial with $\widehat{T}$ supported in $B^c = B_n^d$. So

$$\infty > \int |u|^2 f = \int \left|\frac{T}{f}\right|^2 f = \int \frac{|T|^2}{f}\,. \qquad \blacksquare$$

Now consider 1-sided phase multiplicity. As before, we want to define $(\mathbf{P}^f)^{+,1}$ on $2^{\mathbb{Z}}$, which will be "$\mathbf{P}^f$ conditioned on all 1's at $-\infty$" (the superscript "1" refers to the fact that we are doing this on 1 side). We define $(\mathbf{P}^f)^{+,1}$ by

$$(\mathbf{P}^f)^{+,1} := \lim_{n\to\infty} \lim_{k\to\infty} \mathbf{P}^f\big[\,\bullet \mid \eta \equiv 1 \text{ on } [-n-k, -n]\big]\,.$$

We say the probability measure $\mathbf{P}^f$ has **1-sided phase multiplicity** if

$$(\mathbf{P}^f)^{-,1} \neq (\mathbf{P}^f)^{+,1}\,.$$

THEOREM (L. & STEIF). *For $f : \mathbb{T} \to [0,1]$, the measure $\mathbf{P}^f$ has 1-sided phase uniqueness if and only if $\mathsf{GM}(f)\mathsf{GM}(\mathbf{1}-f) > 0$. In particular, this depends only on the distribution of f.*

Example: Suppose f is continuous, bounded away from 1, vanishes at a single point x_0, and equals

$$e^{-1/|x-x_0|^{1/2}}$$

in a neighborhood of x_0. Since a trigonometric polynomial vanishes at its zeroes at most polynomially quickly, there cannot exist a trigonometric polynomial T such that $|T|^2/f \in L^1(\mathbb{T})$. Hence $\mathbf{P}^f$ has phase multiplicity. On the other hand,

$$\mathsf{GM}(f)\mathsf{GM}(\mathbf{1}-f) > 0\,,$$

so $\mathbf{P}^f$ has 1-sided phase uniqueness.

To prove the theorem, we need some background. If $\mathsf{GM}(f) > 0$, define the analytic function

$$\Phi_f(z) := \exp \frac{1}{2} \int_{\mathbb{T}} \frac{e^{2\pi i t} + z}{e^{2\pi i t} - z} \log f(t)\, dt$$

for $|z| < 1$. The **outer** function

$$\varphi_f(t) := \lim_{r \uparrow 1} \Phi_f(r e^{2\pi i t})$$

exists for λ_1-a.e. $t \in \mathbb{T}$ and satisfies

$$|\varphi_f|^2 = f \qquad \lambda_1\text{-a.e.}$$

If $\mathsf{GM}(f) = 0$, then define $\varphi_f := \mathbf{0}$.

Using a formula of Kolmogorov and Wiener for prediction of stationary processes, we get

Theorem. *For any measurable* $f : \mathbb{T} \to [0, 1]$,

$$\mathbf{P}^f \Big[\eta(0) = 1 \mid \eta \restriction (-\infty, -n] \equiv 1 \Big] = \sum_{j=0}^{n-1} |\widehat{\varphi_f}(j)|^2 \,.$$

Taking $n \to \infty$, we find that

$$(\mathbf{P}^f)^{+,1}\big[\eta(0) = 1\big] = \|\widehat{\varphi_f}\|_2^2 = \|\varphi_f\|_2^2 \,.$$

If $\mathsf{GM}(f) > 0$, then we get

$$(\mathbf{P}^f)^{+,1}\big[\eta(0) = 1\big] = \|f\|_1 = \widehat{f}(0) = \mathbf{P}^f\big[\eta(0) = 1\big] \,.$$

Since $(\mathbf{P}^f)^{+,1} \preccurlyeq \mathbf{P}^f$, it follows that $(\mathbf{P}^f)^{+,1} = \mathbf{P}^f$. However, if $\mathsf{GM}(f) = 0$, then

$$(\mathbf{P}^f)^{+,1}\big[\eta(0) = 1\big] = 0 \,,$$

so $(\mathbf{P}^f)^{+,1} = \delta_{\mathbf{0}}$.

Remark. The above asymptotics for $n = 1$ give

$$\begin{aligned}\lim_{k\to\infty} r_{k+1}/r_k &= \lim_{k\to\infty} \mathbf{P}^f[\eta(0) = 1 \mid \eta_{-1} = \eta_{-2} = \cdots = \eta_{-k} = 1] \\ &= |\widehat{\varphi_f}(0)|^2 = |\Phi_f(0)|^2 = \mathsf{GM}(f) \,.\end{aligned}$$

This is the Szegő limit theorem we used earlier.

But why does the formula define a probability measure?

Proof of Shirai-Takahashi: The Toeplitz matrix T_f is the matrix of the multiplication operator $M_f : g \mapsto f \cdot g$ on $L^2(\mathbb{T}^d)$. This is a positive contraction. For $A \subseteq \mathbb{Z}^d$, write $P_A : \ell^2(\mathbb{Z}^d) \to \ell^2(A)$ for the orthogonal projection and $(T_f)_A := P_A T_f P_A$ for the compression of T_f to $\ell^2(A)$. **Claim:** Given any finite $A \subseteq \mathbb{Z}^d$ partitioned as $A_0 \cup A_1$, then

$$\mu_A\big(\eta^{-1}(0) \cap A = A_0\big) := \det\Big(P_{A_0}\big(I_A - (T_f)_A\big) + P_{A_1}(T_f)_A\Big)$$

defines a consistent family of probability measures that are the restrictions of $\mathbf{P}^f$ to A. This follows from

Lemma. *If E is finite and S and T are E by E matrices, then*

$$\det(S+T) = \sum_{A \subseteq E} \det\left(P_A S + P_{E\setminus A} T\right).$$

If also S and T are nonnegative definite and commute, then each of these terms is ≥ 0.

LEMMA. *If E is finite and S and T are E by E matrices, then*

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Proof. If S and T are positive definite and commute, then $ST^{-1} = T^{-1/2} S T^{-1/2}$ is also positive definite, so

$$\begin{aligned} \det\left(P_A S + P_{E \setminus A} T\right) &= \det T \det\left(P_A S T^{-1} + P_{E \setminus A}\right) \\ &= \det T \det\left(S T^{-1}\right)_A \geq 0 . \end{aligned}$$

∎