Determinantal Probability Measures: Appendix

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Abstract. We present the appendix of the original submission, removed at the request of the referee from the published article. In it, we show how to prove known theorems counting bases of regular matroids. This includes the famous Matrix-Tree Theorem. We have changed the equation numbers to match the published article.

Appendix: Counting Bases in Regular Matroids

If $\mathcal{M}$ is regular, or more generally, complex unimodular, then any complex unimodular representation M of $\mathcal{M}$ has the well-known property that

$$|\mathcal{B}| = \det(M_{(r)} M_{(r)}^*)\,,$$

which is a consequence of (2.1). This formula can provide a useful way to count $\mathcal{B}$ when a useful complex unimodular representation can be found. A more general form holds as well: If M and M' are both complex unimodular representations of $\mathcal{B}$, then

$$|\mathcal{B}| = |\det(M'_{(r)} M_{(r)}^*)|\,.$$

This is because, by (5.7), the right-hand side is just $|(c'\xi_H, c\xi_H)|$ for some constants c, c' of absolute value $\sqrt{|\mathcal{B}|}$.

We give short proofs in this section of two known useful (real) unimodular representations. Our first is the Matrix-Tree Theorem. As we noted in Section 2, the entries of the combinatorial Laplacian of G are $(\star_x,\, \star_y)$ $(x, y \in \mathsf{V})$.

The Matrix-Tree Theorem. *Let $G = (\mathsf{V}, \mathsf{E})$ be a finite connected graph. Let $x_0, y_0 \in \mathsf{V}$. Then the number of spanning trees of G equals*

$$\left|\det\left[(\star_x,\, \star_y)\right]_{x \ne x_0, y \ne y_0}\right|.$$

Proof. Any set of all the stars but one is a basis for $\bigstar$. The discussion at the beginning of this section shows that we need merely prove that if $\mathbf{u}$ is the wedge product (in some order) of the stars at all the vertices other than x_0, then $\mathbf{u}$ has all its coefficients belonging to $\{0, \pm 1\}$ (in the standard orthonormal basis of $\Lambda^{|\mathsf{V}|-1}\mathsf{E}$). Since $\bigstar$ is a subspace representation of the graphic matroid, as observed in Section 2, the only non-zero coefficients of $\mathbf{u}$ are those in which choosing one edge in each $\star_x$ for $x \ne x_0$ yields a spanning tree; moreover, each spanning tree occurs exactly once since there is exactly one way to choose an edge incident to each $x \ne x_0$ to get a given spanning tree. This means that its coefficient is ± 1. ∎

Our proof of the Matrix-Tree Theorem also shows that $\mathbf{P}^\bigstar$ is uniform. In combination with Theorem 5.1, we thus deduce not only the Transfer Current Theorem of Burton and Pemantle (1993), but also its special case, Kirchhoff's Theorem, which was an ingredient in all prior proofs of the Transfer Current Theorem.

We next consider general regular matroids and give a short proof of a theorem of Maurer (1976). First, however, we review some additional facts about regular matroids. A **hyperplane** of a matroid on E is a maximal set in E that does not contain a base. A **cocircuit** of a matroid on E is the complement of a hyperplane. Let $\mathcal{M}$ be a regular matroid of rank r. Then there is an $(r \times E)$ coordinatization matrix M of $\mathcal{M}$ that is **totally unimodular**, i.e., such that every $(k \times k)$-submatrix of M has determinant equal to 0 or ± 1 ($k \le r$) (see Theorem 3.1.1 of White (1987)). Let $H \subseteq \ell^2(E)$ be the row space of M. If C^* is a cocircuit of $\mathcal{M}$, then the matrix $M_{E \setminus C^*}$ has rank $r-1$. Pivot operations can therefore transform $M_{E\setminus C^*}$ to have its last row identically 0. These same row operations on M lead to a last row $u_{C^*} \in H$ all of whose elements are 0 or ± 1 since M is totally unimodular (compare Lemma 2.2.20 of Oxley (1992)). Because the complement of C^* is a hyperplane, the 0's in the last row occur only for those $e \notin C^*$. If B is a base and $e \in B$, let $C^*(e, B) := \{e' \in E\,;\ B \setminus \{e\} \cup \{e'\} \in \mathcal{B}\}$. The set $C^*(e,B)$ is a **fundamental** cocircuit. Write $u(e, B) := u_{C^*(e,B)}$. The vectors $u(e,B)$ $(e \in B)$ of H corresponding to all fundamental cocircuits form a basis of H (compare Welsh (1976), p. 170).

Theorem. *Let $\mathcal{B}$ be the set of bases of a regular matroid on E. Let $B, B' \in \mathcal{B}$. Then*

$$|\mathcal{B}| = \left| \det \left[\big(u(e,\ B), u(e', B') \big) \right]_{e \in B, e' \in B'} \right| .$$

Proof. Let $\mathbf{u}(B) := \bigwedge_{e \in B} u(e, B)$. Since the vectors $\langle u(e,B)\,;\ e \in B \rangle$ form a basis of H, there is a constant $c(B)$ such that $\mathbf{u}(B) = c(B)\xi_H$. All the coefficients of $\mathbf{u}(B)$ are integers since the coefficients of each $u(e,B)$ are 0 or ± 1. Since $B \cap C^*(e,B) = \{e\}$ for all $e \in B$, the coefficient $\big(\mathbf{u}(B),\ \theta_B\big) = \pm 1$. Therefore, the smallest (in absolute value) non-0 coefficient of ξ_H occurs for B. Since B is arbitrary, it follows that all non-0 coefficients of ξ_H have the same magnitude. Therefore, all non-0 coefficients of $\mathbf{u}(B)$ are ± 1. Since the same holds for $\mathbf{u}(B')$, it follows that $|c(B)| = |c(B')|$. Hence $\left| \big(\mathbf{u}(B),\ \mathbf{u}(B')\big) \right| = \|\mathbf{u}(B)\|^2 = |\mathcal{B}|$. ∎

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