Random Complexes and ℓ^2-Betti Numbers

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Abstract. Uniform spanning trees on finite graphs and their analogues on infinite graphs are a well-studied area. On a Cayley graph of a group, we show that they are related to the first ℓ^2-Betti number of the group. Our main aim, however, is to present the basic elements of a higher-dimensional analogue on finite and infinite CW-complexes, which relate to the higher ℓ^2-Betti numbers. One consequence is a uniform isoperimetric inequality extending work of Lyons, Pichot, and Vassout. We also present an enumeration similar to recent work of Duval, Klivans, and Martin.

§1. Introduction.

Enumeration of spanning trees in graphs began with Kirchhoff (1847). Cayley (1889) evaluated this number in the special case of a complete graph. Cayley's theorem was extended to higher dimensions by Kalai (1983), who showed that a certain enumeration of k-dimensional complexes in an $(n-1)$-dimensional simplex resulted in n to the power $\binom{n-2}{k}$. An extension of Kalai's result to general simplicial complexes was given by Duval, Klivans, and Martin (2008); an extension in a different direction was given by Adin (1992). There is more than one natural way to extend the notion of spanning tree to higher dimensions; we choose a slightly different one than the choice of Duval, Klivans, and Martin (2008). The fact that both choices agree in the case of a simplex follows from the remark on p. 341 of Kalai (1983). Our choice is more closely related to matroids and this makes such objects exist in greater generality than those of Duval, Klivans, and Martin (2008). We give an enumeration result similar to that of Duval, Klivans, and Martin (2008).

Although Kirchhoff did not state any of his results using the language of probability, they can easily and fruitfully be stated that way. The theory of random spanning trees, chosen uniformly from among all of them in a given graph, began again with the papers

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of Broder (1989) and Aldous (1990). The theory was extended to infinite graphs by Pemantle (1991) in response to questions of the present author. Since then, the theory on infinite graphs has developed significantly and led to the discovery by Schramm (2000) of the SLE processes, a major development in contemporary probability theory. On general infinite graphs, there are two natural and important extensions of the uniform spanning tree measures, called free and wired uniform spanning forest measures. See Lyons (1998) for a survey and Benjamini, Lyons, Peres, and Schramm (2001) for details.

The enumerations in higher dimensions alluded to above give different weights to different subcomplexes, depending on the torsion of their homology groups. Correspondingly, the probability measures we consider are not necessarily the uniform measures, but rather, are proportional to these same weights. In fact, on a finite CW-complex X and in a given dimension k, we define two probability measures on k-dimensional subcomplexes of X; their difference depends on the k-th Betti number of X. Each of these measures has free and wired extensions to infinite CW-complexes, X. This gives four measures in all. Differences among the four depend on (ℓ^2-)homology of X. In particular, all four coincide iff the reduced kth ℓ^2-homology group of X vanishes. In case X admits an action by a group Γ with compact quotient X/Γ, a difference among the measures can be measured by the kth ℓ^2-Betti number of X with respect to Γ. This leads to a uniform isoperimetric inequality. All our measures will be determinantal, whence they satisfy various strong properties such as negative associations (Lyons (2003), Borcea, Brändén, and Liggett (2009)).

Unfortunately, we are not able to answer analogues of some of the basic questions answered by Pemantle (1991), as we lack analogues of the algorithms of Broder (1989), Aldous (1990), or Wilson (1996).

We also give a suggestive analogy to percolation theory for the case of dimension 1. If it could be extended, one could resolve an important question of Gaboriau (2002) relating cost to the first ℓ^2-Betti number. Again, an algorithm that extended one known for uniform spanning trees would be of use; alternatively, a way to deduce topology from the definition of a determinantal probability measure would help in this case and that of the previous paragraph.

Most of our results were announced in Lyons (2003), Sec. 12.

§2. Determinantal Measures.

We begin with a review of the definitions and basic properties of determinantal probability measures that we shall use. In fact, we restrict ourselves to determinantal measures arising from orthogonal projections. See Lyons (2003) for more details and proofs.

Let E be a finite set and let $\mathscr{B}$ be a nonempty collection of subsets of E. Recall that the pair $\mathscr{M} := (E, \mathscr{B})$ is a **matroid** with **bases** $\mathscr{B}$ if the following exchange property is satisfied:

$$\forall T, T' \in \mathscr{B} \;\; \forall e \in T \setminus T' \;\; \exists e' \in T' \setminus T \quad (T \setminus \{e\}) \cup \{e'\} \in \mathscr{B}\,.$$

All bases have the same cardinality, called the **rank** of the matroid. In our case, E will be a set of vectors in a complex vector space and $\mathscr{B}$ will be the collection of maximal linearly independent subsets of E, where "maximal" means with respect to inclusion. Matroids of this type are called **vectorial** (though in general, one allows any field to underlie the vector space, not merely the complex numbers). The **dual** of a matroid $\mathscr{M} = (E, \mathscr{B})$ is the matroid $\mathscr{M}^\perp := (E, \mathscr{B}^\perp)$, where $\mathscr{B}^\perp := \{E \setminus T\,;\; T \in \mathscr{B}\}$.

If $E \subset \mathbb{C}^s$, the usual way of representing the corresponding vectorial matroid $\mathscr{M}$ is by an $(s \times E)$-matrix M whose columns are the vectors in E with respect to the usual basis of $\mathbb{C}^s$. One calls M a **coordinatization matrix** of $\mathscr{M}$. In this case, the rank of the matrix M equals the rank of the matroid and a base of $\mathscr{M}$ is a set of columns forming a basis of the column space of M.

For subsets $A \subseteq [1, s]$, $B \subseteq E$, let $M_{A,B}$ denote the matrix determined by the rows of M indexed by A and the columns of M indexed by B. Let $P_H : \ell^2(E) \to \ell^2(E)$ be the orthogonal projection onto the row space H of M. One definition of the determinantal probability measure $\mathbf{P}^H$ on $\mathscr{B}$ corresponding to M is

$$\mathbf{P}^H(T) = |\det M_{A,T}|^2 / \det\big(M_{A,E}(M_{A,E})^*\big) \tag{2.1}$$

for $T \in \mathscr{B}$ whenever the rows indexed by A form a basis of H, where the superscript $*$ denotes adjoint. (One way to see that this defines a probability measure is to use the Cauchy-Binet formula.) As indicated by the notation, this depends on M only through H; this is not hard to verify by considering a change of basis, but is immediate from another formula,

$$\mathbf{P}^H(T) = \det[Q_H]_{T,T} \tag{2.2}$$

for $T \in \mathscr{B}$, where Q_H is the matrix of P_H . The representation (2.2) has a useful extension, namely, for every $D \subseteq E$,

$$\mathbf{P}^H[D \subseteq T] = \det[Q_H]_{D,D}\,. \tag{2.3}$$

In case E is infinite and H is a closed subspace of $\ell^2(E)$, the determinantal probability measure $\mathbf{P}^H$ is defined via the requirement that (2.3) hold for all finite $D \subset E$.

We shall use the following theorems from Lyons (2003).

PROPOSITION 2.1. *Let E be a finite set and H be a subspace of $\ell^2(E)$. Then $\mathbf{P}^H$ is supported on the subsets of E whose cardinality equals the dimension of H.*

PROPOSITION 2.2. *Let E be a finite set. For a subspace $H \subseteq \ell^2(E)$ and its orthogonal complement $H^\perp$, we have*

$$\forall T \in 2^E \qquad \mathbf{P}^{H^\perp}(E \setminus T) = \mathbf{P}^H(T)\,.$$

Given two probability measures $\mathbf{P}^1$, $\mathbf{P}^2$ on 2^E, we say that $\mathbf{P}^2$ **stochastically dominates** $\mathbf{P}^1$ and write $\mathbf{P}^1 \preccurlyeq \mathbf{P}^2$ if there is a random pair $(T^1, T^2) \in 2^E \times 2^E$ with $T^i \sim \mathbf{P}^i$ (meaning the law of T^i is $\mathbf{P}^i$) and such that $T^1 \subseteq T^2$. We call such a random pair a **monotone coupling** of $\mathbf{P}^1$ and $\mathbf{P}^2$. (For convenience, we are mixing the definition of stochastic domination with a theorem of Strassen (1965).)

THEOREM 2.3. *Let E be finite or infinite and let $H \subseteq H'$ be closed subspaces of $\ell^2(E)$. Then $\mathbf{P}^H \preccurlyeq \mathbf{P}^{H'}$, with equality iff $H = H'$.*

Proof. The last clause about equality was not stated in Lyons (2003), so we prove it here. If $\mathbf{P}^H = \mathbf{P}^{H'}$, then for all $e \in E$, we have $\mathbf{P}^H[e \in T] = \mathbf{P}^{H'}[e \in T]$, i.e., $\|P_H \mathbf{1}_e\| = \|P_{H'} \mathbf{1}_e\|$. Combining this with the assumption that $H \subseteq H'$ yields $H = H'$. ∎

For a set $D \subseteq E$, recall that $\mathscr{F}(D)$ denotes the σ-field of events generated by the random variable $T \cap D$. Define the **tail** σ-field to be the intersection of $\mathscr{F}(E \setminus D)$ over all finite D. We say that a measure $\mathbf{P}$ on 2^E has **trivial tail** if every event in the tail σ-field has measure either 0 or 1.

THEOREM 2.4. *Let E be infinite and let H be a closed subspace of $\ell^2(E)$. The measure $\mathbf{P}^H$ has trivial tail.*

§3. Finite CW-Complexes.

We consider each cell of a CW-complex X to be oriented (except, of course, the 0-cells). Write $\Xi_k X$ for the set of k-cells of X. We identify cells with the corresponding basis elements of the chain and cochain groups, so that $\Xi_k X$ forms a basis of $C_k(X;\mathbb{C})$ and $C^k(X;\mathbb{C})$. The matrix (in this basis) of the boundary map $\partial_k = \partial_{k,X} : C_k(X;\mathbb{C}) \to C_{k-1}(X;\mathbb{C})$ is the matrix of incidence numbers. In the sequel, we shall not write the coefficient group $\mathbb{C}$. Recall that $Z_k(X) := \ker \partial_k$, $B_k(X) := \operatorname{im} \partial_{k+1}$, and $H_k(X) := Z_k(X)/B_k(X)$. We also have the coboundary map $\delta_k = \delta_{k,X} := \partial^*_{k+1}$ with its corresponding groups, $Z^k(X) := \ker \delta_k$, $B^k(X) := \operatorname{im} \delta_{k-1}$, and $H^k(X) := Z^k(X)/B^k(X)$.

Given a finite CW-complex X and a subset $T \subseteq \Xi_k X$ of its k-cells, write X_T for the subcomplex $T \cup \bigcup_{j=0}^{k-1} \Xi_j X$. We call T a **k-base** if it is a base of the matroid defined by the matrix of the boundary map ∂_k, i.e., if it is maximal with $Z_k(X_T) = 0$, while we call T a **k-cobase** if it is a base of the matroid defined by the matrix of the coboundary map δ_k, i.e., if it is maximal with the property that the kernel of $\delta_k : C^k(X_T) \to C^{k+1}(X)$ is trivial. We remark that since X_T is k-dimensional, T is a k-base iff $H_k(X_T) = 0$.

In a moment, we shall define a probability measure on the set of k-bases; later, we shall define another probability measure on the set of complements of k-cobases. Before giving these probability measures, we give some examples of k-bases and k-cobases. If G is a connected graph, then the empty set is the only 0-base, while the complement of each vertex is a 0-cobase. The 1-bases are the spanning trees. If G and $G^\dagger$ form a pair of dual graphs embedded in an orientable surface with all faces contractible, then consider the 2-complex X whose 1-skeleton is G and whose 2-cells are the faces of G. The 1-cobases of X are the sets T of edges such that for some spanning tree T' of $G^\dagger$, each edge in T crosses an edge of T' and vice versa. The complement of each face is a 2-base of X, while the empty set is the only 2-cobase. For another example noted by Kalai (1983), let X be the 5-simplex. Its 2-bases consist of 10 triangles. Some of these 2-bases form the usual triangulation of the projective plane using 6 vertices and 10 triangles. (This triangulation arises from the regular icosahedron by identifying antipodal points.)

Given a set T of k-cells and S of $(k-1)$-cells, we write $\partial_{S,T}$ for the submatrix of ∂_k whose rows are indexed by S and columns by T. The matrix of ∂_k defines a determinantal probability measure on the set of k-bases as in (2.1):

$$\mathbf{P}_k(T) := \mathbf{P}_{k,X}(T) := \frac{\det \partial_{S,T}\partial^*_{S,T}}{\det \partial_{S,\Xi_k X}\partial^*_{S,\Xi_k X}}$$

for any fixed $(k-1)$-cobase S. We call this measure the **kth lower matroidal measure on** X. Also, if we multiply this formula by $\det \partial_{S,\Xi_k X}\partial^*_{S,\Xi_k X}$ and sum over S, then the

Cauchy-Binet formula yields

$$\mathbf{P}_k(T) = \frac{\det \partial^*_{\Xi_{k-1}X,T} \partial_{\Xi_{k-1}X,T}}{\sum_S \det \partial_{S,\Xi_k X} \partial^*_{S,\Xi_k X}} \,. \tag{3.1}$$

Let $t_j(L)$ denote the order of the torsion subgroup of $H_j(X_L;\mathbb{Z})$. If we write $[G]$ for the torsion subgroup of an abelian group G, then in our notation, we have $t_j(L) = |[H_j(X_L;\mathbb{Z})]|$. We now show that the measure $\mathbf{P}_k$ is proportional to the square of the order of the torsion subgroup of the homology group of dimension $k-1$. Note that if X is connected and $k=1$, this shows that $\mathbf{P}_1$ is the uniform measure on spanning trees since 0-dimensional homology has no torsion; this gives a short proof of the Transfer Current Theorem of Burton and Pemantle (1993).

Proposition 3.1. *Let X be a finite CW-complex. For each k, there exists a_k such that for all k-bases T of X,*

$$\mathbf{P}_k(T) = a_k t_{k-1}(T)^2 \,.$$

To prove this, we use a presumably well-known lemma:

Lemma 3.2. *Let V be a subspace of $\mathbb{Q}^n$ of dimension r. Let $B_0 \subset V \cap \mathbb{Z}^n$ be a set of cardinality r that generates the group $V \cap \mathbb{Z}^n$. For any basis B of V that lies in $\mathbb{Z}^n$, identify B with the matrix whose columns are B in the standard basis of $\mathbb{Q}^n$ and write $\langle B \rangle$ for the subgroup of $\mathbb{Z}^n$ generated by B. Then for all such B, we have*

$$\det B^* B = |[\mathbb{Z}^n / \langle B \rangle]|^2 \det B_0^* B_0 \,.$$

Proof. By hypothesis on B_0, there exists an $r \times r$ integer matrix A such that $B = B_0 A$. We have

$$\det B^* B = \det A^* B_0^* B_0 A = \det A^* \det B_0^* B_0 \det A = (\det A)^2 \det B_0^* B_0 \,.$$

Also, $\mathbb{Z}^n / \langle B_0 \rangle = \mathbb{Z}^n / (V \cap \mathbb{Z}^n)$ is torsion free and $[\langle B_0 \rangle : \langle B \rangle] = |\det A|$, whence

$$|[\mathbb{Z}^n / \langle B \rangle]| = |[\mathbb{Z}^n / \langle B_0 \rangle]| \cdot [\langle B_0 \rangle : \langle B \rangle] = [\langle B_0 \rangle : \langle B \rangle] = |\det A| \,.$$

Comparing these identities gives the result. ∎

Proof of Proposition 3.1. Chain groups have integral coefficients for the duration of this proof. By (3.1), $\mathbf{P}_k(T)$ is proportional to $\det \partial^*_{\Xi_{k-1}X,T} \partial_{\Xi_{k-1}X,T}$. The columns of $\partial_{\Xi_{k-1}X,T}$ generate the group $B_{k-1}(X_T)$. Thus, Lemma 3.2 shows that $\mathbf{P}_k(T)$ is proportional to $|[C_{k-1}(X_T)/B_{k-1}(X_T)]|^2$. Therefore, it suffices to show that

$$[C_{k-1}(X_T)/B_{k-1}(X_T)] = [Z_{k-1}(X_T)/B_{k-1}(X_T)]$$

in order to complete the proof. Let $u \in [C_{k-1}(X_T)/B_{k-1}(X_T)]$. Write $u = v + B_{k-1}(X_T)$ with $v \in C_{k-1}(X_T)$. Let $n \in \mathbb{Z}^+$ be such that $nu = 0$, i.e., $nv \in B_{k-1}(X_T)$. Since $B_{k-1}(X_T) \subseteq Z_{k-1}(X_T)$, we have $\partial(nv) = 0$, which implies that $\partial v = 0$, i.e., that $v \in Z_{k-1}(X_T)$. Therefore $u \in [Z_{k-1}(X_T)/B_{k-1}(X_T)]$. ∎

The theorem of Kalai (1983) referred to in the introduction is that when X is an $(n-1)$-simplex and $1 \le k \le n-1$,

$$\sum_T t_{k-1}(T)^2 = n^{\binom{n-2}{k}},$$

where the sum is over all k-bases of X. For example, the 2-bases in the 5-simplex mentioned earlier that correspond to the usual triangulation of the projective plane have weight 4. Since the projective plane can be embedded* in $\mathbb{R}^4$, one may encounter it when taking random 2-bases in natural 4-dimensional complexes. We shall return to enumeration in Section 6.

From now on (except in the section on enumeration or otherwise notated), our chain and cochain coefficients will be in $\mathbb{C}$. We use the usual inner-product on $C_k(X)$, which also allows us to identify $C_k(X)$ with $C^k(X)$.

As in (2.2), another form of $\mathbf{P}_k$ is expressed using the orthogonal projection Q_k of $C_k(X)$ onto the row space of ∂_k, i.e., onto the space of coboundaries $B^k(X)$. In this form, we have

$$\mathbf{P}_k(T) = \det[Q_k]_{T,T}\,. \tag{3.2}$$

Of course, $B^k(X) = Z_k(X)^\perp$.

Another natural probability measure $\tilde{\mathbf{P}}^k$ on subsets of $\Xi_k X$ is given by the matrix of the coboundary map δ_k, the determinantal probability measure corresponding to orthogonal projection on the row space of δ_k, i.e., the column space of ∂_{k+1}, which is the space of boundaries, $B_k(X)$. The probability measure $\mathbf{P}^k(T) := \mathbf{P}^{k,X}(T) := \tilde{\mathbf{P}}^k(\Xi_k X \setminus T)$ is the determinantal probability measure corresponding to the subspace of k-cocycles, $Z^k(X) = B_k(X)^\perp$ (see Proposition 2.2). We call this measure the **kth upper matroidal measure on** X. It is supported by sets of k-cells that are complements of k-cobases. Since $B^k(X) \subseteq Z^k(X)$, it follows from Theorem 2.3 that the upper measure $\mathbf{P}^k$ stochastically dominates the lower measure $\mathbf{P}_k$, with equality iff $H^k(X) = 0$. (Note that since X is finite, $H^k(X)$ is isomorphic to $H_k(X)$.) As usual, let $b_k(X)$ denote the kth Betti number of X, the dimension of $H_k(X)$. By Proposition 2.1, one can add $b_k(X)$ k-cells to a sample from

* For example, it lies in the 4-skeleton of the 5-simplex; this skeleton is compact and naturally embedded in the 4-sphere.

$\mathbf{P}_k$ to get a sample from $\mathbf{P}^k$. Occasionally, we shall use the reduced Betti numbers $\tilde{b}_k(X)$, where $\tilde{b}_k(X) = b_k(X)$ for $k > 0$, but $\tilde{b}_0(X) = b_0(X) - 1$ (as long as $X \neq \varnothing$).

Recall that for a subcomplex Y of X, one writes $C_k(X,Y) := C(X)/C(Y)$ and that ∂ is defined on the corresponding chain complex, with kernels $Z_k(X,Y)$, images $B_k(X,Y)$, and quotients $H_k(X,Y) := Z_k(X,Y)/B_k(X,Y)$. Recall also that $C^k(X,Y) = \{u \in C^k(X)\,;\ u\restriction C_k(Y) = 0\}$, that $Z^k(X,Y)$ is the kernel of δ_k on $C^k(X,Y)$, that $B^k(X,Y)$ is the image of δ_{k-1} on $C^{k-1}(X,Y)$, and that $H^k(X,Y) := Z^k(X,Y)/B^k(X,Y)$.

Thus, T is the complement in $\Xi_k X$ of a k-cobase iff T is minimal with $Z^k(X,X_T) = 0$; note that since $C^{k-1}(X,X_T) = 0$, the latter condition is equivalent to $H^k(X,X_T) = 0$, and thus to $H_k(X,X_T) = 0$. Because the homology sequence of the pair (X,X_T) is exact, this last condition is also equivalent to the conjunction of the surjectivity of the natural map $H_k(X_T) \to H_k(X)$ and the injectivity of the natural map $H_{k-1}(X_T) \to H_{k-1}(X)$.

PROPOSITION 3.3. *Let X be a finite CW-complex. For each k, there exists a_k such that if T is the complement of a k-cobase of X, then*

$$\mathbf{P}^k(T) = a_k |H_k(X,X_T;\mathbb{Z})|^2 .$$

Proof. Again, for this proof, all coefficient groups not explicitly given are $\mathbb{Z}$. An argument precisely parallel to that proving Proposition 3.1 shows that $\mathbf{P}^k(T)$ is proportional to the square of the order of the torsion subgroup of $Z^{k+1}(X)$ modulo the image under the map δ_k of the k-cochains vanishing on $C_k(X_T)$, i.e., modulo $B^{k+1}(X,X_T)$. Since X_T is k-dimensional, $C^{k+1}(X,X_T) = C^{k+1}(X)$ and $Z^{k+1}(X,X_T) = Z^{k+1}(X)$, whence $\mathbf{P}^k(T)$ is proportional to $|[H^{k+1}(X,X_T)]|^2$. It is well known that $|[H^{k+1}(X,X_T)]| = |[H_k(X,X_T)]|$ (e.g., see Corollary 3.3 of Hatcher (2002)). Since in the present case, $H_k(X,X_T;\mathbb{C}) = 0$, it follows that $H_k(X,X_T;\mathbb{Z}) = [H_k(X,X_T;\mathbb{Z})]$. ∎

Here are some simple examples. Suppose that X is the 2-complex defined by a connected graph G embedded in the 2-torus, all of whose faces and edges are contractible. Let $G^\dagger$ be the graph dual to G. Then $\mathbf{P}_0$ is concentrated on the empty set, while $\mathbf{P}^0$ is the law of a uniform random vertex of G. The uniform spanning tree of G has law $\mathbf{P}_1$, while the edges of G that do not cross a uniform spanning tree of $G^\dagger$ have law $\mathbf{P}^1$. If $T \sim \mathbf{P}^1$, then T has non-contractible cycles, but no contractible cycles. The edges of such a T generate the homology $\mathbb{Z}^2$ of the 2-torus. This duality is shown in the random sample of Figure 1, where the gray edges have law $\mathbf{P}_1$ on a 50×50 square lattice torus graph G, and those edges belonging to a cycle in $G^\dagger$ for $\mathbf{P}^1$ are shown in black, the other edges not being shown at all. Finally, $\mathbf{P}_2$ is the law of the complement of a uniform random face of G and $\mathbf{P}^2$ is concentrated on the full set of all 2-cells of X. We conjecture that the

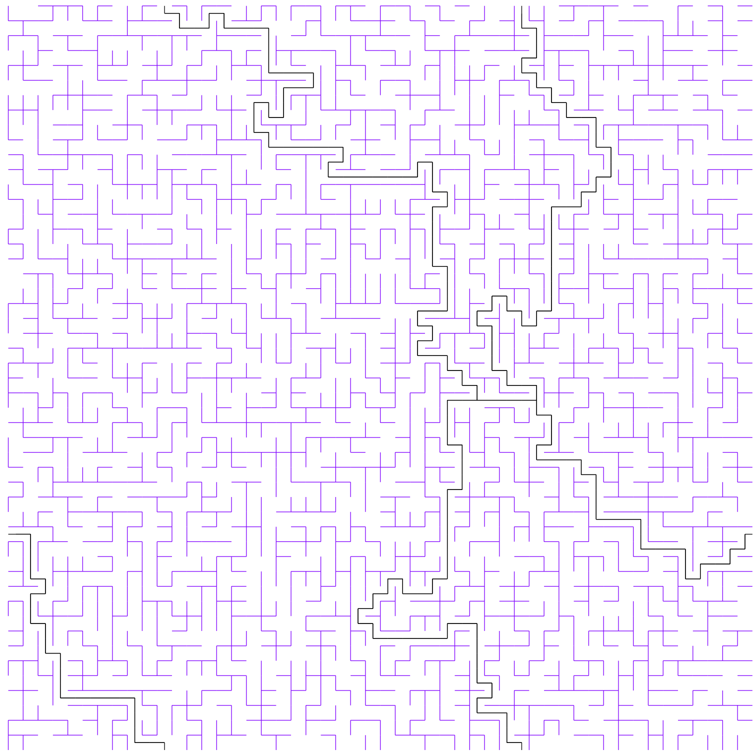

Figure 1.

expected number of edges that belong to a cycle for the law $\mathbf{P}^1$ on an $n \times n$ square torus graph is asymptotic to $Cn^{5/4}$ for some constant C; cf. Kenyon (2000).

In many circumstances such as the preceding paragraph, one has a pair (X, X^*) of dual cell structures on an orientable n-dimensional manifold; see, e.g., Chap. 10 of Seifert and Threlfall (1980), p. 84 of Rourke and Sanderson (1972), p. 59 of Matveev (2006), p. 228 of Bryant (2002), or p. 25 of Fenn (1983). In this case, there are bijections $\varphi_k : \Xi_k X \to \Xi_{n-k} X^*$ such that the matrix of δ_{n-k,X^*} equals that of $\partial_{k,X}$ or its negative. This implies that $\mathbf{P}_{k,X}$ and $\mathbf{P}^{n-k,X^*}$ have a coupling $\big(T, \varphi_k[\Xi_k X \setminus T]\big)$.

§4. Infinite CW-Complexes.

When X is infinite, there are natural extensions of the probability measures $\mathbf{P}_k$ and $\mathbf{P}^k$. We shall always assume that X is locally finite unless otherwise stated. In fact, the lower and upper measures each have two extensions, making four measures in all.

The k-cells form an orthonormal basis for the Hilbert space $C_k^{(2)}(X) := \ell^2(\Xi_k X)$, which is identified with its dual, the space of ℓ^2-cochains $C^k_{(2)}(X)$. As before, $C_k(X)$ denotes the space of k-chains (with complex coefficients and finite support). Let $Z_k(X) := \{u \in C_k(X)\,;\ \partial_k u = 0\}$ and $B_k(X) := \{\partial_{k+1} u\,;\ u \in C_{k+1}(X)\}$ be the usual cycle and boundary spaces. Let $C_c^k(X)$ denote the space of k-cochains that vanish off a finite set of k-cells. Let $Z_c^k(X) := \{u \in C_c^k(X)\,;\ \delta_k u = 0\}$ and $B_c^k(X) := \{\delta_{k-1} u\,;\ u \in C_c^{k-1}(X)\}$ be the cocycle and coboundary spaces that vanish off a finite set of k-cells. The measures $\mathbf{P}_k^{\mathrm{W}}$, $\mathbf{P}_k^{\mathrm{F}}$, $\mathbf{P}_{\mathrm{W}}^k$, $\mathbf{P}_{\mathrm{F}}^k$ can now be defined as the determinantal probability measures corresponding to orthogonal projections on, respectively, $\bar{B}_c^k(X)$, $Z_k(X)^\perp$, $\bar{Z}_c^k(X)$, or $B_k(X)^\perp$, as in (3.2), where the bars indicate closure in the ℓ^2-topology:

$$\mathbf{P}_{\mathrm{W}}^k \leftrightsquigarrow \bar{Z}_c^k(X), \quad \mathbf{P}_{\mathrm{F}}^k \leftrightsquigarrow B_k(X)^\perp$$
$$\mathbf{P}_k^{\mathrm{W}} \leftrightsquigarrow \bar{B}_c^k(X), \quad \mathbf{P}_k^{\mathrm{F}} \leftrightsquigarrow Z_k(X)^\perp$$

Those with the designation W are called **wired**, while the others are called **free**, by analogy with the case $k = 1$. In fact, $\mathbf{P}_1^{\mathrm{W}}$ is the wired (uniform) spanning forest measure, denoted WSF, while $\mathbf{P}_1^{\mathrm{F}}$ is the free (uniform) spanning forest measure, denoted FSF. For more about the terminology of free and wired, see below. Since $B_c^k(X) \perp Z_k(X)$, we have $\bar{B}_c^k(X) \subseteq Z_k(X)^\perp$, whence $\mathbf{P}_k^{\mathrm{W}} \preccurlyeq \mathbf{P}_k^{\mathrm{F}}$. Since $Z_c^k(X) \subseteq B_k(X)^\perp$, we also have $\mathbf{P}_{\mathrm{W}}^k \preccurlyeq \mathbf{P}_{\mathrm{F}}^k$. Similarly, since $B_c^k(X) \subseteq Z_c^k(X)$, we have $\mathbf{P}_k^{\mathrm{W}} \preccurlyeq \mathbf{P}_{\mathrm{W}}^k$ and since $B_k(X) \subseteq Z_k(X)$, we have $\mathbf{P}_k^{\mathrm{F}} \preccurlyeq \mathbf{P}_{\mathrm{F}}^k$. Thus, all measures stochastically dominate the wired lower measure $\mathbf{P}_k^{\mathrm{W}}$, while all are dominated by the free upper measure $\mathbf{P}_{\mathrm{F}}^k$. Hence, all four measures coincide iff $\mathbf{P}_k^{\mathrm{W}} = \mathbf{P}_{\mathrm{F}}^k$. We have $H_k(X) = 0$ iff $Z_k(X) = B_k(X)$ iff $Z_k(X)^\perp = B_k(X)^\perp$ iff $\mathbf{P}_k^{\mathrm{F}} = \mathbf{P}_{\mathrm{F}}^k$. Likewise, $\bar{Z}_c^k(X) = \bar{B}_c^k(X)$ iff $\mathbf{P}_k^{\mathrm{W}} = \mathbf{P}_{\mathrm{W}}^k$, which is implied by (but is not equivalent to) $H_c^k(X) = 0$.

When one has a pair (X, X^*) of dual cell structures on an orientable n-dimensional manifold, $\mathbf{P}_{k,X}^{\mathrm{F}}$ and $\mathbf{P}_{\mathrm{W}}^{n-k,X^*}$ have a coupling $\big(T, \varphi_k[\Xi_k X \setminus T]\big)$, as do $\mathbf{P}_{k,X}^{\mathrm{W}}$ and $\mathbf{P}_{\mathrm{F}}^{n-k,X^*}$.

Remark 4.1. All four kth matroidal measures are properly defined as long as the k-skeleton of X is locally finite; the $(k+1)$-skeleton of X need not be locally finite.

We now want to show that the free and wired measures are limits of the kinds of measures we considered on finite complexes.

Given a finite subcomplex $A \subset X$, write A° for the **combinatorial interior** of A, i.e., the set of all cells of A whose coboundary vanishes off of A. Although A° is not usually a subcomplex of X, we shall write $C^k(A^\circ)$ for the space of cochains vanishing off A°. Also, let $B^k(A^\circ)$ be the image of the restriction of δ_{k-1} to $C^{k-1}(A^\circ) \to C^k(A)$ and $Z^k(A^\circ)$ be the kernel of the restriction of δ_k to $C^k(A^\circ) \to C^{k+1}(A)$. For the determinantal probability measures corresponding to $B^k(A^\circ)$ and $Z^k(A^\circ)$, write $\mathbf{P}_{k,A^\circ}$ and $\mathbf{P}^{k,A^\circ}$, respectively. Note that both these measures give random subsets of $\Xi_k A$.

Remark 4.2. Instead of working with $B^k(A^\circ)$ and $Z^k(A^\circ)$ below, we could use instead the slightly different spaces $\bigcup Z_k(X, \overline{X \setminus A})^\perp$ and $\bigcup B_k(X, \overline{X \setminus A})^\perp$, respectively, where we regard elements of $C_k(X, \overline{X \setminus A})$ as subsets of $C_k(X)$. These may be somewhat more natural topologically, but are somewhat less explicit and yield slightly worse inequalities.

We call a sequence $\langle A_n \rangle$ of finite subcomplexes of X an **exhaustion** if $A_n \subseteq A_{n+1}$ for each n and $X = \bigcup_n A_n$. For probability measures Q_n on subsets of A_n, write $Q = \operatorname{wlim} Q_n$ if for all finite B, the restrictions of Q_n to B tend to the restriction of Q to B.

The following is straightforward to check.

LEMMA 4.3. *Let X be a locally finite complex with an exhaustion $\langle A_n \rangle$. We have $Z_k(X) = \bigcup_n Z_k(A_n)$, $B_k(X) = \bigcup_n B_k(A_n)$, $B_c^k(X) = \bigcup_n B^k(A_n^\circ)$, and $Z_c^k(X) = \bigcup_n Z^k(A_n^\circ)$, where all four unions are increasing.*

This gives

COROLLARY 4.4. *Let X be a locally finite complex with an exhaustion $\langle A_n \rangle$. Then $\mathbf{P}^{\mathrm{F}}_{k,X} = \operatorname{wlim}_n \mathbf{P}_{k,A_n}$, $\mathbf{P}_{\mathrm{F}}^{k,X} = \operatorname{wlim}_n \mathbf{P}^{k,A_n}$, $\mathbf{P}^{\mathrm{W}}_{k,X} = \operatorname{wlim}_n \mathbf{P}_{k,A_n^\circ}$, and $\mathbf{P}_{\mathrm{W}}^{k,X} = \operatorname{wlim}_n \mathbf{P}^{k,A_n^\circ}$.*

REMARK 4.5. We took a direct route to limits by using subspaces, rather than finite subcomplexes. But subcomplexes can also be used to complete the analogy to spanning forests: Let A be a finite subcomplex of X. Define a new complex A^* as follows. Let B be the set of cells of A whose closure intersects A° and B^c the rest. The cells of A^* are those in B plus one cell z_k of dimension k for each k with $\Xi_k B^c \neq \varnothing$. Every k-cell in B^c is identified with z_k in an orientation-preserving way. The attaching maps among the cells of A° are the same as in A, but the others are changed. This leads to $\delta_k z_k = -\sum_{e \in \Xi_k A^\circ} \delta_k e$, which implies that $B^k(A^\circ) = B^k(A^*)$ and $Z^k(A^\circ) = Z^k(A^*)$. Thus, one could use A_n^* in place of A_n° for the limits of Corollary 4.4, as is done traditionally in the case $k = 1$ to define the wired uniform spanning forest.

For a subcomplex $A \subseteq X$, define its ***k*th boundary** to be $\mathsf{bnd}_k(A) := \Xi_k A \setminus \Xi_k A^\circ$. Write $\operatorname{supp} u$ for the support of a chain, u. Our next proposition is an analogue of the fact that all the trees in the wired or free spanning forests of infinite connected graphs are infinite.

PROPOSITION 4.6. *Suppose that X is locally finite, $k \geq 1$, and $\tilde{b}_{k-1}(X) = 0$. If A is a finite subcomplex of X, then $b_{k-1}(X_{\mathfrak{F}} \cap A) \leq |\mathsf{bnd}_{k-1}(A)|$ a.s. when $\mathfrak{F}$ has any of the laws $\mathbf{P}_k^{\mathrm{W}}$, $\mathbf{P}_k^{\mathrm{F}}$, $\mathbf{P}^k_{\mathrm{W}}$, or $\mathbf{P}^k_{\mathrm{F}}$.*

Proof. Since $\mathbf{P}_k^{\mathrm{W}}$ is stochastically the smallest of the four measures, it suffices to prove the inequality for it. Let A be a finite subcomplex of X. Because of the hypothesis, there is a finite subcomplex $B \subset X$ such that every $(k-1)$-cycle of A is a $(k-1)$-boundary of B. In fact, we may ensure that $Z_{k-1}(A) \subseteq B_{k-1}(B^\circ)$ (in an extension of our earlier notation for cochains). By Corollary 4.4, it suffices to show for all such B that when $T \sim \mathbf{P}_{k,B^\circ}$, we have $b_{k-1}(B_T \cap A) \le |\mathsf{bnd}_{k-1}(A)|$. Let $u \in H_{k-1}(B_T \cap A)$. Since T forms a basis for the vector space $B_{k-1}(B^\circ) = \partial_k C_k(B^\circ)$, we have $B_{k-1}(B_T) = B_{k-1}(B^\circ) \supseteq Z_{k-1}(A)$. Therefore $u = \partial_k w + B_{k-1}(B_T \cap A)$ for some $w \in C_k(B_T)$. Let $y \in C_k(B_T \cap A)$ be the restriction of w to $B_T \cap A$. Then $\operatorname{supp}(\partial_k w - \partial_k y) \subseteq \mathsf{bnd}_{k-1}(A)$ and $u = \partial_k w - \partial_k y + B_{k-1}(B_T \cap A)$. This shows that every class in $H_{k-1}(B_T \cap A)$ is represented by an element of $Z_{k-1}(\mathsf{bnd}_{k-1}(A))$, whence $b_{k-1}(B_T \cap A) = \dim H_{k-1}(B_T \cap A) \le \dim Z_{k-1}(\mathsf{bnd}_{k-1}(A)) \le \dim C_{k-1}(\mathsf{bnd}_{k-1}(A)) = |\mathsf{bnd}_{k-1}(A)|$. ∎

We say an infinite CW-complex X has **bounded degree** if for every k the map ∂_k has bounded ℓ^2-norm. This guarantees that the four spaces $B^k_{(2)}(X) := \operatorname{im} \delta_{k-1}$, $B^{(2)}_k(X) := \operatorname{im} \partial_{k+1}$, $Z^k_{(2)}(X) := \ker \delta_k$ and $Z^{(2)}_k(X) := \ker \partial_k$ are well defined. Although the first two are not necessarily closed subspaces, we do have that $\bar{B}^k_c(X) = \bar{B}^k_{(2)}(X)$ and $\bar{B}_k(X) = \bar{B}^{(2)}_k(X)$, which is the same as $B_k(X)^\perp = \bar{B}^{(2)}_k(X)^\perp$. The corresponding statements for the kernels are not always true. We have

$$C^{(2)}_k(X) = \bar{B}^k_{(2)}(X) \oplus Z^{(2)}_k(X) = \bar{B}^{(2)}_k(X) \oplus Z^k_{(2)}(X)\,,$$

whence $\mathbf{P}^{\mathrm{W}}_k = \mathbf{P}^{\mathrm{F}}_k$ iff $\bar{Z}_k(X) = Z^{(2)}_k(X)$ and $\mathbf{P}^k_{\mathrm{W}} = \mathbf{P}^k_{\mathrm{F}}$ iff $\bar{Z}^k(X) = Z^k_{(2)}(X)$. We also deduce the ℓ^2-Hodge-de Rham decomposition

$$C^{(2)}_k(X) = \bar{B}^k_{(2)}(X) \oplus \bar{B}^{(2)}_k(X) \oplus \mathscr{H}^{(2)}_k(X)\,,$$

where $\mathscr{H}^{(2)}_k(X) := Z^{(2)}_k(X) \cap Z^k_{(2)}(X)$ is the space of harmonic ℓ^2-k-chains. Evidently, $\mathscr{H}^{(2)}_k(X)$ is isometrically isomorphic to $H^{(2)}_k(X) := Z^{(2)}_k(X)/\bar{B}^{(2)}_k(X)$, the reduced kth ℓ^2-homology group of X, which is also isometrically isomorphic to the reduced kth ℓ^2-cohomology group of X, $Z^k_{(2)}(X)/\bar{B}^k_{(2)}(X)$. All four matroidal measures coincide iff $\mathscr{H}^{(2)}_k(X) = 0$. In this case, we shall denote the common measure by simply $\mathbf{P}_k$.

In particular, suppose that Γ is a countable group acting freely on X by permutation of cells and the quotient X/Γ is compact. (Freeness here means that the stabilizer of each unoriented cell consists of only the identity of Γ.) In this case, we call X a **cocompact Γ-CW-complex**. Then X has bounded degree and all the above Hilbert spaces are Hilbert Γ-modules. The kth ℓ^2-Betti number of X is the von Neumann dimension of $\mathscr{H}^{(2)}_k(X)$ with respect to Γ: $\beta_k(X;\Gamma) := \dim_\Gamma \mathscr{H}^{(2)}_k(X)$. This is 0 iff $\mathscr{H}^{(2)}_k(X) = 0$. For more information about ℓ^2-homology, see Eckmann (2000). Note that the ℓ^2-Betti numbers of X are Γ-equivariant homotopy invariants of X: see Cheeger and Gromov (1986).

Recall that a countable group Γ is **amenable** if it has a **Følner exhaustion**, i.e., an increasing sequence of finite subsets F_n whose union is Γ such that for all finite $F \subset \Gamma$, we have $\lim_{n\to\infty} |FF_n \triangle F_n|/|F_n| = 0$. For $A \subseteq X$, write $\mathsf{bnd}\, A$ for the **topological boundary** of A in X. Suppose X is a Γ-CW-complex with finite fundamental domain D and Γ is amenable with Følner exhaustion $\langle F_n \rangle$. Set $A_n := F_n \bar{D}$. Then $\langle A_n \rangle$ is an exhaustion of X with $|\Xi_k \mathsf{bnd}\, A_n|/|F_n| \to 0$ as $n \to \infty$ for each k. By a theorem of Dodziuk and Mathai (1998), we have

$$\lim_{n\to\infty} b_k(A_n)/|F_n| = \beta_k(X;\Gamma) \tag{4.1}$$

for all k. Eckmann (1999) gave a simpler proof, and we shall give one that is even further streamlined, with an extension.

Fix k. Let $\Pi_n : C_k^{(2)}(X) \to C_k^{(2)}(X)$ denote the orthogonal projection onto $C_k(A_n)$ and $d_n(H)$ denote the ordinary trace of $\Pi_n P_H$ for a closed subspace H of $C_k^{(2)}(X)$. Eckmann (1999) noted the following:

$$0 \le d_n(H) \le \dim \Pi_n(H)\,,$$

with equality on the right if $H \subseteq C_k(A_n)$;

$$H = H_1 \oplus H_2 \quad \Longrightarrow \quad d_n(H) = d_n(H_1) + d_n(H_2)\,;$$

and

$$0 \le d_n(H) - |F_n| \dim_\Gamma H \le |\Xi_k \mathsf{bnd}\, A_n|$$

when H is Γ-invariant.

For example, we have that

$$\dim_\Gamma \bar{B}_c^k(X) = \lim_n \frac{d_n\big(\bar{B}_c^k(X)\big)}{|F_n|} \ge \limsup_n \frac{d_n\big(B^k(A_n)\big)}{|F_n|} = \limsup_n \frac{\dim B^k(A_n)}{|F_n|}\,.$$

Furthermore,

$$\begin{aligned}\dim_\Gamma Z_k(X)^\perp = \lim_n \frac{d_n(Z_k(X)^\perp)}{|F_n|} &\le \liminf_n \frac{\dim \Pi_n(Z_k(X)^\perp)}{|F_n|} \le \liminf_n \frac{\dim \Pi_n(Z_k(A_n)^\perp)}{|F_n|} \\ &= \liminf_n \frac{\dim C_k(A_n) \cap Z_k(A_n)^\perp}{|F_n|} = \liminf_n \frac{\dim B^k(A_n)}{|F_n|}\,.\end{aligned}$$

On the other hand, $\bar{B}_c^k(X) \subseteq Z_k(X)^\perp$, so that we have equalities everywhere and

$$\dim_\Gamma \bar{B}_c^k(X) = \dim_\Gamma Z_k(X)^\perp = \lim_n \frac{\dim B^k(A_n)}{|F_n|}\,,$$

which implies that

$$\bar{B}_c^k(X) = Z_k(X)^\perp\,. \tag{4.2}$$

An exactly parallel argument shows that

$$\bar{Z}_c^k(X) = B_k(X)^\perp \tag{4.3}$$

with

$$\dim_\Gamma B_k(X)^\perp = \lim_n \frac{\dim Z^k(A_n)}{|F_n|}\,.$$

Subtracting these identities, we obtain

$$\beta_k(X;\Gamma) = \dim_\Gamma B_k(X)^\perp - \dim_\Gamma \bar{B}_c^k(X) = \lim_n \frac{b_k(A_n)}{|F_n|}\,,$$

as desired.

Another consequence of (4.2) and (4.3) is the following:

PROPOSITION 4.7. *Suppose that Γ is a countable amenable group and X is a Γ-CW-complex whose k-skeleton is cocompact. Then $\mathbf{P}_k^{\mathrm{W}} = \mathbf{P}_k^{\mathrm{F}}$ and $\mathbf{P}^k_{\mathrm{W}} = \mathbf{P}^k_{\mathrm{F}}$.*

Of course, if $b_k(X) = 0$, then we also obtain that $\beta_k(X;\Gamma) = 0$, a result (essentially) of Cheeger and Gromov (1986).

REMARK 4.8. Since $Z_k^{(2)}(X) = B^k_{(2)}(X)^\perp$, it follows that we also have $\bar{Z}_k(X) = Z_k^{(2)}(X)$ in the amenable case, whence $H_k^{(2)}(X) = \bar{Z}_k(X)/\bar{B}_k(X)$. In the case that X does not have a locally finite k-skeleton, Cheeger and Gromov (1986) define $\beta_k(X;\Gamma)$ as follows. Consider an exhaustion of X by cocompact subcomplexes X_n. The inclusion of X_m in X_n for $m < n$ induces a homomorphism $j_{m,n} : H_k^{(2)}(X_m) \to H_k^{(2)}(X_n)$. Clearly $\dim_\Gamma \operatorname{im} j_{m,n}$ is decreasing in n, so its limit exists and is increasing in m. Thus, we may define

$$\beta_k(X;\Gamma) := \lim_{m\to\infty} \lim_{n\to\infty} \dim_\Gamma \operatorname{im} j_{m,n}\,.$$

It is easy to see that this does not depend on the exhaustion chosen. Now in the amenable case, if $b_k(X) = 0$, then $Z_k(X_m) \subseteq B_k(X) = \bigcup_{n\ge m} B_k(X_n)$, whence $\lim_{n\to\infty} \dim_\Gamma \operatorname{im} j_{m,n} = 0$, so that $\beta_k(X;\Gamma) = 0$. This is a new proof of a result of Cheeger and Gromov (1986).

Denote the number of k-cells in X/Γ by $f_k = f_k(X/\Gamma)$. Write $\mathfrak{F}$ for a sample from $\mathbf{P}_k$.

PROPOSITION 4.9. *Let Γ be amenable and act freely on a complex X whose k-skeleton is cocompact. If $\tilde{b}_{k-1}(X) = 0$, then the $\mathbf{P}_k$-expected number of k-cells in $\mathfrak{F}$ per vertex of X equals*

$$f_{k-1}/f_0 + \sum_{j=0}^{k-2} (-1)^{k+j-1}\big(f_j - \beta_j(X;\Gamma)\big)/f_0\,.$$

This also equals the average number of k-cells in $\mathfrak{F}$ per vertex of X $\mathbf{P}_k$-a.s.

Proof. The case $k = 0$ is easy, so assume that $k \geq 1$. We use the notation above. Let $\mathfrak{F}$ be a sample from the matroidal measure. Since $X_{\mathfrak{F}}$ has no k-cycles, the Euler-Poincaré formula yields

$$\begin{aligned}\sum_{j=0}^{k-1}(-1)^j|\Xi_j A_n|&+(-1)^k|\Xi_k(X_{\mathfrak{F}} \cap A_n)| \\ &= \sum_{j=0}^{k}(-1)^j|\Xi_j(X_{\mathfrak{F}} \cap A_n)| \\ &= \sum_{j=0}^{k}(-1)^j b_j(X_{\mathfrak{F}} \cap A_n) \\ &= \sum_{j=0}^{k-2}(-1)^j b_j(A_n) + (-1)^{k-1} b_{k-1}(X_{\mathfrak{F}} \cap A_n)\,. \end{aligned} \tag{4.4}$$

Thus, if we divide both sides of (4.4) by $|F_n| f_0$ and use Proposition 4.6 and (4.1), we obtain as a limit the equalities desired. ∎

Write $\mathbb{X}^d$ for the natural d-dimensional CW-complex determined by the tiling of $\mathbb{R}^d$ by a unit cube and all its translates by elements of $\mathbb{Z}^d$. The following result is suggested by duality.

Corollary 4.10. *The $\mathbf{P}_k$-probability that a given k-cell belongs to $\mathfrak{F}$ in $\mathbb{X}^d$ is k/d.*

Proof. In this case, we have $f_j = \binom{d}{j}$ and $\beta_j(\mathbb{X}^d; \mathbb{Z}^d) = 0$, whence the $\mathbf{P}_k$-expected number of k-cells per vertex equals $\binom{d-1}{k-1}$. Since the number of k-cells of $\mathbb{X}^d$ per vertex is $\binom{d}{k}$, the result follows by symmetry, all k-cells having the same probability. ∎

We are interested in the $\mathbf{P}_k$-expected number of k-cells per vertex of X in the non-amenable case as well. In the case of Cayley graphs, the action of Γ is not free when the edges are undirected and there are involutions among the generators. Since the graph case is of special interest, we give the following result first. For simplicity of notation, we write $\deg_{\mathfrak{F}}$ for the degree in the graph spanned by $\mathfrak{F}$.

Proposition 4.11. *Let G be the Cayley graph of a group Γ with respect to a symmetric generating set, S. (The edges are undirected and S does not contain the identity.) Let o be a vertex of G. Let H be a Γ-invariant closed subspace of $C_1^{(2)}(G)$ and $\mathfrak{F} \sim \mathbf{P}^H$. Then*

$$\mathbf{E}^H[\deg_{\mathfrak{F}} o] = 2 \dim_\Gamma H\,.$$

Proof. Let the standard basis elements of $\ell^2(\Gamma \times S)$ be $\{f_{\gamma,s}\,;\ \gamma \in \Gamma, s \in S\}$. Identify $C_1^{(2)}(G)$ with the range in $\ell^2(\Gamma \times S)$ of the map defined by sending the oriented edge $\langle \gamma, \gamma s\rangle$ to the vector $(f_{\gamma,s} - f_{\gamma s, s^{-1}})/\sqrt{2}$. These vectors form an orthonormal basis of the range. Then H becomes identified with a subspace H_S. Write Q for the orthogonal projection of $\ell^2(\Gamma \times S)$ onto H_S. We may choose o to be the identity of Γ. Since $\big(f_{\gamma,s} + f_{\gamma s,s^{-1}}\big) \perp H_S$, we have

$$Q f_{o,s} = -Q f_{s,s^{-1}}\,.$$

Therefore,

$$\begin{aligned}
\mathbf{E}^H[\deg_{\mathfrak{F}} o] &= \sum_{s \in S} \mathbf{P}^H\big[[o,s] \in \mathfrak{F}\big] = \sum_{s\in S} \|Q(f_{o,s} - f_{s,s^{-1}})/\sqrt{2}\|^2 \\
&= \sum_{s\in S} \|\sqrt{2} Q f_{o,s}\|^2 = 2 \sum_{s \in S} (Q f_{o,s},\, f_{o,s}) \\
&= 2 \dim_\Gamma H_S = 2 \dim_\Gamma H\,.
\end{aligned}$$

∎

A complex K is called a $K(\Gamma, 1)$ **CW-model** if K is a CW-complex with fundamental group equal to Γ and vanishing higher homotopy groups. In this case, if X is the universal cover of K, we define $\beta_k(\Gamma) := \beta_k(X;\Gamma)$; it depends only on Γ and not on K. For example, if $k = 1$ and Γ is finitely presented, then $\mathscr{H}_1^{(2)}(X)$ consists of the 1-chains that are orthogonal to both $B^1_{(2)}(X)$ and $B_1(X)$; the latter space is the space generated by the cycles in the Cayley graph, G. Hence, even when Γ is not finitely presented, $\beta_1(\Gamma) = \dim_\Gamma B^1_{(2)}(G)^\perp \cap Z_1(G)^\perp$.

COROLLARY 4.12. *In any Cayley graph of a group Γ, we have*

$$\mathbf{E}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} o] = 2\beta_1(\Gamma) + 2\,.$$

Proof. By Proposition 4.11, we have

$$\mathbf{E}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} o] = 2\,\dim_\Gamma Z_1(G)^\perp = 2\beta_1(\Gamma) + 2\,\dim_\Gamma \bar{B}^1_{(2)}(G) = 2\beta_1(\Gamma) + 2$$

because $\delta : C^0_{(2)}(G) \to C^1_{(2)}(G)$ is injective and $\dim_\Gamma C^0_{(2)}(G) = 1$. ∎

This identity was extended to transitive unimodular graphs by Lyons, Peres, and Schramm (2006) (see the proof of Corollary 3.24), which depends on a definition of Gaboriau (2005).

Now we extend the identity to higher dimensions.

PROPOSITION 4.13. *Suppose that Γ is a countable group and X is a cocompact Γ-CW-complex. Let D be a fundamental domain for the action of Γ on X. Let H be a Γ-invariant closed subspace of $C_k^{(2)}(X)$. Then $\mathbf{E}^H\big[|\mathfrak{F}\cap D|\big] = \dim_\Gamma H$. In particular, $\mathbf{E}_{\mathrm{F}}^k\big[|\mathfrak{F}\cap D|\big] - \mathbf{E}_k^{\mathrm{W}}\big[|\mathfrak{F}\cap D|\big] = \beta_k(X;\Gamma)$.*

Proof. Let the standard basis elements of $C_k^{(2)}(X)$ be $\{f_{\gamma,e}\,;\ \gamma\in\Gamma, e\in\Xi_k D\}$. Write Q for the orthogonal projection onto H. Let o be the identity of Γ. Then

$$\mathbf{E}^H\big[|\mathfrak{F}\cap D|\big] = \sum_{e\in\Xi_k D}\mathbf{P}^H\big[e\in\mathfrak{F}\big] = \sum_{e\in\Xi_k D}(Qf_{o,e},\ f_{o,e}) = \dim_\Gamma H\,.$$ ∎

COROLLARY 4.14. *If K is a $K(\Gamma,1)$ CW-model with finite k-skeleton and X is its universal cover with fundamental domain D, then on X, we have $\mathbf{E}_k^{\mathrm{F}}\big[|\mathfrak{F}\cap D|\big]-\mathbf{E}_k^{\mathrm{W}}\big[|\mathfrak{F}\cap D|\big] = \beta_k(\Gamma)$.*

Proof. Since the higher homotopy groups of X also vanish, so do its homology groups. Thus, $\mathbf{P}_k^{\mathrm{F}} = \mathbf{P}_{\mathrm{F}}^k$. By definition, $\beta_k(\Gamma) = \beta_k(X;\Gamma)$. ∎

We now give an extension of (4.1) to the non-amenable setting. Our proof also gives an alternative proof that in the amenable case, $\beta_k(X;\Gamma) = 0$.

COROLLARY 4.15. *Suppose that Γ is a countable group and X is a Γ-CW-complex whose k-skeleton is cocompact for some fixed $k\geq 1$. Let D be a fundamental domain for the action of Γ on X. If $\tilde{b}_{k-1}(X) = 0$, then*

$$\inf\left\{\frac{|\mathsf{bnd}_{k-1}(F\bar{D})|}{|F|}\,;\ F\subset\Gamma \text{ is finite}\right\}\geq\beta_k(X;\Gamma)\,.$$

Proof. Let $F\subset\Gamma$ be finite and $A := F\bar{D}$. The same reasoning that led to (4.4) shows that

$$\sum_{j=0}^{k-1}(-1)^j|\Xi_j A| + (-1)^k|\Xi_k(X_{\mathfrak{F}}\cap A)| = \sum_{j=0}^{k-2}(-1)^j b_j(A) + (-1)^{k-1}b_{k-1}(X_{\mathfrak{F}}\cap A)$$

when $\mathfrak{F}$ is a sample from any of the four matroidal measures. Apply this to a monotone coupling $(\mathfrak{F},\mathfrak{F}^*)$ of $\mathbf{P}_{\mathrm{F}}^k$ and $\mathbf{P}_k^{\mathrm{W}}$ and subtract the resulting equations to get

$$|\mathfrak{F}\cap A| - |\mathfrak{F}^*\cap A| = b_{k-1}(X_{\mathfrak{F}^*}\cap A) - b_{k-1}(X_{\mathfrak{F}}\cap A)\leq|\mathsf{bnd}_{k-1}(A)|\,,$$

where we have applied Proposition 4.6 in the last step. Therefore,

$$\mathbf{E}_{\mathrm{F}}^k\big[|\mathfrak{F}\cap A|\big] - \mathbf{E}_k^{\mathrm{W}}\big[|\mathfrak{F}\cap A|\big]\leq|\mathsf{bnd}_{k-1}(A)|\,.$$

The left-hand side is equal to

$$|F|\cdot\Big(\mathbf{E}_{\mathrm{F}}^k\big[|\mathfrak{F}\cap D|\big] - \mathbf{E}_k^{\mathrm{W}}\big[|\mathfrak{F}\cap D|\big]\Big) = |F|\beta_k(X;\Gamma)$$

by Proposition 4.13, which gives the desired inequality. ∎

We immediately deduce the following inequality.

COROLLARY 4.16. *Fix $k \geq 1$. For a countable group Γ, every contractible Γ-CW-complex X with fundamental domain D and for which $\Xi_k X/\Gamma$ is finite satisfies*

$$\inf\left\{\frac{|\mathsf{bnd}_{k-1}(F\bar{D})|}{|F|}\,;\ F \subset \Gamma \textit{ is finite}\right\} \geq \beta_k(\Gamma)\,.$$

Corollary 4.15 extends Corollary 7 of Lyons, Pichot, and Vassout (2008) to quasi-transitive graphs acted on by Γ and, of course, to higher dimensions.

Very interesting questions remain for the standard cubical CW-decomposition $\mathbb{X}^d$ of $\mathbb{R}^d$. Recall that all four measures coincide.

• What is the $(k-1)$-dimensional (co)homology of the random k-subcomplex? In the case $k = 1$ of spanning forests, this asks how many trees there are, the question answered by Pemantle (1991).

• If one takes the 1-point compactification of the random subcomplex, what is the k-dimensional (co)homology? In the case of spanning forests, this asks how many ends there are in the tree(s), the question answered partially by Pemantle (1991) and completely by Benjamini, Lyons, Peres, and Schramm (2001).

Note that by translation-invariance of (co)homology and ergodicity of $\mathbf{P}_k$, we have that the values of the (co)homology groups are constants a.s.

It follows trivially from the Alexander duality theorem and the results of Pemantle (1991) and Benjamini, Lyons, Peres, and Schramm (2001) that for $k = d - 1$, we have $H_{k-1}(\mathfrak{F}) = 0$ $\mathbf{P}_k$-a.s., while $\mathbf{P}_k$-a.s., the Čech-Alexander-Spanier cohomology group $\check{H}^k(\mathfrak{F} \cup \infty)$ is 0 for $2 \leq d \leq 4$ and is (naturally isomorphic to) a direct sum of infinitely many copies of $\mathbb{Z}$ for $d \geq 5$. It also follows from the Alexander duality theorem and from equality of free and wired limits that if $d = 2k$, then the a.s. values of $\check{H}^k(\mathfrak{F} \cup \infty)$ and $H_{k-1}(\mathfrak{F})$ are the same (naturally isomorphic), so that the two bulleted questions above are dual in that case.

§5. Analogy to Percolation.

In the 1-dimensional case, there is a suggestive analogy to phase transitions in Bernoulli percolation theory. In that theory, given a connected graph G, one considers for $0 < p < 1$ the random subgraph left after deletion of each edge independently with probability $1-p$. A **cluster** is a connected component of the remaining graph. In the case of transitive graphs, there are two numbers $p_{\mathrm{c}}, p_{\mathrm{u}} \in [0,1]$ such that if $0 < p < p_{\mathrm{c}}$, then there are no infinite clusters a.s.; if $p_{\mathrm{c}} < p < p_{\mathrm{u}}$, then there are infinitely many infinite clusters a.s.; and if $p_{\mathrm{u}} < p < 1$, then there is exactly 1 infinite cluster a.s. See Häggström, Peres, and Schonmann (1999).

PROPOSITION 5.1. *Let G be a Cayley graph of an infinite group Γ and H be a Γ-invariant closed subspace of $C_1^{(2)}(G)$.*

(i) *If $H \subsetneq \bar{B}_c^1(G)$, then $\mathbf{P}^H$-a.s. infinitely many components of $\mathfrak{F}$ are finite.*

(ii) *If $B_c^1(G) \subseteq H \subsetneq Z_1(G)^\perp$, then $\mathbf{P}^H$-a.s. there are infinitely many infinite components of $\mathfrak{F}$ and no finite components.*

Proof. Suppose that $H \subsetneq \bar{B}_c^1(G)$. Since $\mathbf{E}^H[\deg_{\mathfrak{F}} o] = 2 \dim_\Gamma H < 2 \dim_\Gamma \bar{B}_c^1(G) = 2$, where $o \in \Gamma$, it follows from Theorem 6.1 of Benjamini, Lyons, Peres, and Schramm (1999) that some component is finite with positive $\mathbf{P}^H$-probability. However, $\mathbf{P}^H$ has a trivial tail σ-field by Theorem 2.4, which implies ergodicity of the Γ-action, whence this event has probability 1. Now if there were only finitely many finite components, then picking a vertex uniformly at random from their union would give a way to pick a vertex at random in an invariant way, which is clearly impossible. This proves part (i).

Now suppose that $B_c^1(G) \subseteq H \subsetneq Z_1(G)^\perp$. By Theorem 2.3, we have $\mathbf{P}^H \preccurlyeq \mathsf{FSF}$. Since $\mathbf{P}^H \neq \mathsf{FSF}$, it follows that in a monotone coupling $(\mathfrak{F}, \mathfrak{F}^*)$ of the two measures, $A := \mathfrak{F}^* \setminus \mathfrak{F}$ is non-empty with positive probability. Let e_0 be an edge that lies in A with positive probability and let B be the Γ-orbit of e_0, which is necessarily infinite. Because $\mathbf{P}[e \in A] = \mathsf{FSF}[e \in \mathfrak{F}] - \mathbf{P}^H[e \in \mathfrak{F}]$ and both terms on the right-hand side are the same for all $e \in B$, we have that $\mathbf{P}[e \in A]$ also is the same for all $e \in B$. Therefore $E\big[|A|\big] = \infty$. The number of components of $\mathfrak{F}$ is at least the size of A. Since the number of components of $\mathfrak{F}$ is an invariant random variable, it is constant, whence infinite a.s. On the other hand, since $\mathsf{WSF} \preccurlyeq \mathbf{P}^H$, each component is infinite. This proves (ii). ∎

We believe that more is true, namely, that if $H \subsetneq \bar{B}_c^1(G)$, then $\mathbf{P}^H$-a.s. all components are finite. However, there is no part (iii) in general, i.e., it is not true that for every Γ-invariant $H \supsetneq Z_1(G)^\perp$, we have $\mathbf{P}^H$-a.s. there is a unique infinite component, i.e., $\mathbf{P}^H$-a.s. $\mathfrak{F}$ is connected. For a counter-example, let $\Gamma := \mathbb{Z}^2 * \mathbb{Z}^5$, a free product, and let G be its

Cayley graph with respect to its natural generators. We may decompose the edges of G into those, E_2, that come from the generators of $\mathbb{Z}^2$ and those, E_5, that come from $\mathbb{Z}^5$. Let $H := C_1^{(2)}(E_2) + Z_1(E_5)^\perp$. Clearly H is Γ-invariant and strictly contains $Z_1(G)^\perp$. (One way to see the strict containment is to note that $\mathbf{P}^H$-a.s. every edge in E_2 is present, while this is not true for $\mathsf{FSF}(G)$.) However, $\mathbf{P}^H$ is the measure gotten by taking a sample from $\mathsf{FSF}(E_5)$ and adding to it all of E_2. Since $\mathsf{FSF}(\mathbb{Z}^5)$ has infinitely many trees by a result of Pemantle (1991), our claim follows. Nevertheless, if for every $\epsilon > 0$ there were *some* Γ-invariant $H \supset Z_1(G)^\perp$ with the two properties that $\dim_\Gamma H < \dim_\Gamma Z_1(G)^\perp + \epsilon$ and that $\mathbf{P}^H$-almost every sample is connected, then it would follow that $\beta_1(\Gamma) + 1$ equals the cost of Γ, which would answer an important question of Gaboriau (2002). An analogous result *is* known for the free minimal spanning forest; see Lyons, Peres, and Schramm (2006). The first property is not hard to ensure, i.e., that for every $\epsilon > 0$ there is some Γ-invariant $H \supset Z_1(G)^\perp$ with $\dim_\Gamma H < \dim_\Gamma Z_1(G)^\perp + \epsilon$. I am indebted to Vaughan Jones for the following proof of this fact. We first prove a lemma.

LEMMA 5.2. *Let $\mathscr{R}$ be a von Neumann algebra such that every non-0 projection in $\mathscr{R}$ has infinite rank (in the ordinary sense) and such that its commutant $\mathscr{R}'$ is a finite von Neumann algebra. Then $\mathscr{R}$ has no minimal projections.*

Proof. Let $p \neq 0$ be a projection in $\mathscr{R}$ on the Hilbert space $\mathscr{H}$. By Corollary 5.5.7 of Kadison and Ringrose (1997a), we have $(p\mathscr{R}p)' = p\mathscr{R}'$. If p is minimal, then $p\mathscr{R}' = \mathscr{B}\big(p(\mathscr{H})\big)$ by Proposition 6.4.3 of Kadison and Ringrose (1997b). Let $p^\perp := I - p$. Since $p^\perp A = Ap^\perp = p^\perp A p^\perp$ for all $A \in \mathscr{R}'$, it follows that $\{A \in \mathscr{R}'\,;\ p^\perp A = 0\} = p\mathscr{R}'$. Now $\{A \in \mathscr{R}'\,;\ p^\perp A = 0\}$ is easily checked to be a two-sided ideal in $\mathscr{R}'$ that is closed in the weak operator topology. Therefore it is equal to $q\mathscr{R}'$ for some central projection $q \in \mathscr{R}'$ by Theorem 6.8.8 of Kadison and Ringrose (1997b). From the above, we conclude that $q\mathscr{R}' = \mathscr{B}\big(p(\mathscr{H})\big)$. Since $\mathscr{R}'$ is finite, it has a center-valued trace, τ. It is easily checked that $A \mapsto q\tau(A)$ is a center-valued trace on $q\mathscr{R}'$, so that $\mathscr{B}\big(p(\mathscr{H})\big)$ is finite. This means that the rank of p is finite, contradicting our assumption on $\mathscr{R}$. ∎

To apply this lemma, let $L(\Gamma)$ denote the left group von Neumann algebra of Γ. By Theorem 6.7.2 of Kadison and Ringrose (1997b), we have $L(\Gamma)' = R(\Gamma)$, the right group von Neumann algebra. Combining this with Lemma 6.6.2 of Kadison and Ringrose (1997b), we obtain $M_n\big(L(\Gamma)\big)' = R(\Gamma) \otimes I_n$. Every projection in $L(\Gamma)$ has infinite rank since Γ is infinite. Since $R(\Gamma)$ is finite, we deduce from Lemma 5.2 that $M_n\big(L(\Gamma)\big)$ has no minimal projections. Thus for every Γ-invariant closed subspace $K \subseteq C_1^{(2)}(G)$, there is a Γ-invariant closed subspace $\{0\} \neq K' \subsetneq K$. Our claim follows easily from this by using $K := Z_1(G)^\perp$ and its subspaces.

§6. Enumeration.

Recall that $t_j(T) := |[H_j(X_T;\mathbb{Z})]|$. The normalizing constant a_k in Proposition 3.1 is the reciprocal of the sum

$$h_{k-1}(X) := \sum t_{k-1}(T)^2\,,$$

where the sum is over all k-bases T of X. Does this have an explicit expression? We answer this here, following the method of Duval, Klivans, and Martin (2008). Although our analogue of spanning tree is simpler than that of theirs, their enumeration is simpler because their definition implies the finiteness of certain homology groups. We may clearly assume that the dimension d of X is equal to k. We assume $d > 1$, since the case $d = 1$ is the standard matrix-tree theorem.

In this section, all coefficients of chain groups are in $\mathbb{Z}$ except where otherwise indicated explicitly. Given a set $S \subseteq \Xi_k X$ of k-cells, let $Q_k(S)$ denote the quotient of $Z_k(X)$ by $\big(Z_k(X) \cap B_k(X;\mathbb{Q})\big) + Z_k(X_{S^c})$ and let $t'_k(S)$ denote its order, where $S^c := \Xi_k X \setminus S$ denotes the set of k-cells that are not in S.

The key lemma is:

Lemma 6.1. *Let X be a finite CW-complex of dimension d, T be a d-base of X, and S be a $(d-1)$-cobase of X. Then*

$$|\det \partial_{S,T}| = t_{d-1}(T)t_{d-2}(S^c)t'_{d-1}(S)/t_{d-2}(X)\,.$$

Proof. Let $\Gamma := (X_T, X_{S^c})$. As in Proposition 4.1 of Duval, Klivans, and Martin (2008), we have $H_d(\Gamma) = 0$ since $\partial_{S,T}$ is nonsingular. As in Proposition 4.2 of Duval, Klivans, and Martin (2008), we also have that $|\det \partial_{S,T}| = |H_{d-1}(\Gamma)|$. The homology sequence of the pair Γ is exact, which, since $H_d(\Gamma) = 0$ and $\Xi_k X_T = \Xi_k X_{S^c}$ for $k \le d-2$, becomes

$$0 \to H_{d-1}(X_{S^c}) \overset{i_{d-1}}{\to} H_{d-1}(X_T) \overset{j_{d-1}}{\to} H_{d-1}(\Gamma) \overset{\partial_{d-1}}{\to} H_{d-2}(X_{S^c}) \overset{i_{d-2}}{\to} H_{d-2}(X_T) \to 0\,. \tag{6.1}$$

We claim that this induces an exact sequence of finite groups,

$$0 \to H_{d-1}(X_T)/\ker j_{d-1} \overset{[j_{d-1}]}{\to} H_{d-1}(\Gamma) \overset{[\partial_{d-1}]}{\to} [H_{d-2}(X_{S^c})] \overset{[i_{d-2}]}{\to} [H_{d-2}(X_T)] \to 0\,, \tag{6.2}$$

and that

$$|H_{d-1}(X_T)/\ker j_{d-1}| = t_{d-1}(T)t'_{d-1}(S)\,. \tag{6.3}$$

Since $H_{d-2}(X_T) = H_{d-2}(X)$, the result follows.

Any homomorphism of abelian groups restricts to an homomorphism of their torsion subgroups; this is how we define the last two maps $[\partial_{d-1}]$ and $[i_{d-2}]$ above. Since

$\ker i_{d-2} = \operatorname{im} \partial_{d-1}$ is finite, it is contained in the torsion subgroup, whence (6.2) is exact at $[H_{d-2}(X_{S^c})]$. In addition, since $\ker i_{d-2}$ contains only torsion elements, the inverse image of $[H_{d-2}(X_T)]$ also contains only torsion elements, whence $[i_{d-2}]$ is onto. This gives exactness of (6.2) at $[H_{d-2}(X_T)]$.

Define $[j_{d-1}] : H_{d-1}(X_T)/\ker j_{d-1} \to H_{d-1}(\Gamma)$ as the injective map induced by j_{d-1}. This gives exactness of (6.2) at the remaining places automatically.

It remains to prove (6.3). Now $H_{d-1}(X_{S^c}) = Z_{d-1}(X_{S^c})$ is free since $\dim X_{S^c} = d-1$. We have i_{d-1} is injective by exactness of (6.1) at $H_{d-1}(X_{S^c})$. Therefore $\operatorname{im} i_{d-1} \cap [H_{d-1}(X_T)] = 0$, so that we may identify $[H_{d-1}(X_T)]$ with a subgroup G of $K := H_{d-1}(X_T)/\operatorname{im} i_{d-1} = H_{d-1}(X_T)/\ker j_{d-1}$. Thus, the proof will be completed once we show that K/G is isomorphic to $Q_{d-1}(S)$. Now

$$L := H_{d-1}(X_T)/[H_{d-1}(X_T)] = Z_{d-1}(X_T)/\big(B_{d-1}(X_T;\mathbb{Q}) \cap Z_{d-1}(X_T)\big)\,.$$

Also, $Z_{d-1}(X_T) = Z_{d-1}(X)$ since $C_{d-1}(X_T) = C_{d-1}(X)$ and, since T is a d-base, $B_{d-1}(X_T;\mathbb{Q}) = B_{d-1}(X;\mathbb{Q})$. Therefore,

$$L = Z_{d-1}(X)/\big(B_{d-1}(X;\mathbb{Q}) \cap Z_{d-1}(X)\big)\,. \tag{6.4}$$

Since $\operatorname{im} i_{d-1} \cap [H_{d-1}(X_T)] = 0$, we may identify $\operatorname{im} i_{d-1}$ as a subgroup M of L. We have L/M is isomorphic to K/G and

$$M = Z_{d-1}(X_{S^c})/\big(B_{d-1}(X;\mathbb{Q}) \cap Z_{d-1}(X_{S^c})\big)\,. \tag{6.5}$$

The quotient of (6.4) by (6.5) is isomorphic to $Q_{d-1}(S)$ because of the fact that for any group D and subgroups D_1, D_2, we have an isomorphism between $(D/D_1)/(D_2/(D_1 \cap D_2))$ and $D/(D_1D_2)$, where $D_2/(D_1 \cap D_2)$ is identified with a subgroup of D/D_1. ∎

By straightforward applications of the Cauchy-Binet identity as in Duval, Klivans, and Martin (2008), we obtain the following corollary:

Corollary 6.2. *Let X be a finite CW-complex. Write*

$$h'_k(X) := \sum t_k(S^c)^2 t'_{k+1}(S)^2\,,$$

where the sum is over all $(k+1)$-cobases S of X. Then for any $(d-1)$-cobase S of X, we have

$$h_{d-1}(X) = \frac{t_{d-2}(X)^2}{t_{d-2}(S^c)^2 t'_{d-1}(S)^2} \det \partial_{S,X_d} \partial^*_{S,X_d}$$

and the product of the non-zero eigenvalues of $\partial_d \partial_d^*$ *equals*

$$\frac{h_{d-1}(X) h'_{d-2}(X)}{t_{d-2}(X)^2} .$$

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