

# Induced graphs of uniform spanning forests

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## Abstract

Given a subgraph  $H$  of a graph  $G$ , the induced graph of  $H$  is the largest subgraph of  $G$  whose vertex set is the same as that of  $H$ . Our paper concerns the induced graphs of the components of  $\text{WSF}(G)$ , the wired spanning forest on  $G$ , and, to a lesser extent,  $\text{FSF}(G)$ , the free uniform spanning forest. We show that the induced graph of each component of  $\text{WSF}(\mathbb{Z}^d)$  is almost surely recurrent when  $d \geq 8$ . Moreover, the effective resistance between two points on the ray of the tree to infinity within a component grows linearly when  $d \geq 9$ . For any vertex-transitive graph  $G$ , we establish the following resampling property: Given a vertex  $o$  in  $G$ , let  $\mathcal{T}_o$  be the component of  $\text{WSF}(G)$  containing  $o$  and  $\overline{\mathcal{T}}_o$  be its induced graph. Conditioned on  $\overline{\mathcal{T}}_o$ , the tree  $\mathcal{T}_o$  is distributed as  $\text{WSF}(\overline{\mathcal{T}}_o)$ . For any graph  $G$ , we also show that if  $\mathcal{T}_o$  is the component of  $\text{FSF}(G)$  containing  $o$  and  $\overline{\mathcal{T}}_o$  is its induced graph, then conditioned on  $\overline{\mathcal{T}}_o$ , the tree  $\mathcal{T}_o$  is distributed as  $\text{FSF}(\overline{\mathcal{T}}_o)$ .

## 1 Introduction

Given a finite, connected graph  $G$ , the *uniform spanning tree* (UST) on  $G$ , which we denote by  $\text{UST}(G)$ , is the uniform measure on the set of spanning trees of  $G$ . Given a (locally finite) infinite, connected graph  $G$ , notions of “uniform spanning tree” can be defined via limiting procedures. Suppose  $(G_n)$  is a sequence of finite connected subgraphs of  $G$ . We call  $(G_n)$  an *exhaustion* of  $G$  if  $G_n \subset G_{n+1}$  and  $\bigcup G_n = G$ . According to [Pem91], given an exhaustion  $(G_n)$  of  $G$  and a fixed, finite subgraph  $H$  of  $G$ , the weak limit of  $\text{UST}(G_n) \cap H$  exists. Varying  $H$ , one obtains a probability measure on subgraphs of  $G$ , which is called the *free spanning forest* (FSF) of  $G$  and denoted by  $\text{FSF}(G)$ . On the other hand, for any finite connected subgraph  $H$  of  $G$ , let  $\widehat{H}$  be the graph obtained by identifying all vertices not in  $H$  to a single vertex. We call  $\text{UST}(\widehat{H})$  the *wired spanning forest* (WSF) of  $H$  (relative to  $G$ ), which we denote by  $\text{WSF}(H)$ . Given an exhaustion  $(G_n)$  of  $G$  consisting of induced subgraphs and a fixed, finite subgraph  $H$  of  $G$ , the weak limit of  $\text{WSF}(G_n) \cap H$  exists. Varying  $H$ , one obtains a probability measure on subgraphs of  $G$ , which is called the wired spanning forest of  $G$  and denoted by  $\text{WSF}(G)$ .

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Both WSF and FSF must have no cycles but can have more than one (connected) component. This justifies the notion of spanning forest. UST and its infinite-volume extensions have been an important object in probability and mathematical physics for the last three decades. See [BLPS01, LP16] for a comprehensive reference.

For infinite graphs, WSF is much better understood than FSF. In [Wil96], David Wilson provided an efficient algorithm to sample UST on finite graphs. It was soon extended to sample WSF on infinite graphs [BLPS01] (see also Section 3). This powerful tool allows one to study WSF directly via simple random walk. In particular, it is proved in [BLPS01] that  $\text{WSF}(G)$  is concentrated on the set of forests with a unique component if and only if two simple random walks on  $G$  intersect a.s. In contrast, there is no known simple condition to determine whether an FSF has a unique component. If  $G$  is the  $d$ -dimensional integer lattice  $\mathbb{Z}^d$  for  $d \in \mathbb{N}$ , then  $\text{WSF}(\mathbb{Z}^d)$  and  $\text{FSF}(\mathbb{Z}^d)$  coincide. Moreover, it was shown in [Pem91] that  $\text{WSF}(\mathbb{Z}^d)$  has a unique component when  $1 \leq d \leq 4$  and infinitely many components when  $d \geq 5$ . In the latter case, the collections of components exhibit an intriguing geometry [BKPS04, HP17].

If  $H$  is a subgraph of a graph  $G$ , including possibly the case that  $H$  has no edges, then the *induced subgraph* determined by  $H$  is the largest subgraph of  $G$  whose vertex set is the same as that of  $H$ . If  $H$  is a subgraph of a graph  $G$ , we define the *induced-component graph*  $\overline{H}$  of  $H$  to be the largest subgraph of  $G$  whose vertex set is the same as that of  $H$  and that has the same connected components as  $H$ . Namely, an edge of  $G$  belongs to  $\overline{H}$  if and only if both its endpoints belong to the same component of  $H$ . We also have that  $\overline{H}$  is the union of the induced subgraphs determined by the components of  $H$ .

Before stating our main results, we make the following conventions throughout the paper. We will use WSF, FSF, UST to denote either probability measures or their samples as long as it is clear from the context what we are referring to. When there is a risk of ambiguity, we use  $\text{UST}(G)$ ,  $\text{FSF}(G)$ ,  $\text{WSF}(G)$  to represent probability measures and  $\mathfrak{T}(G)$ ,  $\mathfrak{F}_f(G)$ ,  $\mathfrak{F}_w(G)$  to represent their corresponding samples. Similarly, we write  $\overline{\text{WSF}(G)}$  for either the law of  $\overline{\mathfrak{F}_w(G)}$  or for its sample,  $\overline{\mathfrak{F}_w(G)}$ , and likewise for the free versions.

The main object of interest in this paper is  $\overline{\text{WSF}(\mathbb{Z}^d)}$ , which reflects the geometry of  $\text{WSF}(\mathbb{Z}^d)$  as a subgraph embedded in  $\mathbb{Z}^d$ . Since  $\overline{\text{WSF}(\mathbb{Z}^d)} = \mathbb{Z}^d$  when  $1 \leq d \leq 4$ , the only interesting case is when  $d \geq 5$ . On the one hand, components of  $\overline{\text{WSF}(\mathbb{Z}^d)}$  have stochastic dimension 4 for all  $d \geq 5$  [BKPS04]. On the other hand, Morris [Mor03] proved that for any graph  $G$ , simple random walk on each component of  $\overline{\text{WSF}(G)}$  is a.s. recurrent. This leads to the intriguing question of whether the components of  $\overline{\text{WSF}(\mathbb{Z}^d)}$  are recurrent or transient.

**Theorem 1.1.** *If  $d \geq 8$ , almost surely each connected component of  $\overline{\text{WSF}(\mathbb{Z}^d)}$  is recurrent.*

For a graph  $G = (V, E)$ , let  $f$  be a real function from  $V$  to  $\mathbb{R}$ , and let

$$\mathcal{E}(f) := \frac{1}{2} \sum_{x, y \in G; x \sim y} (f(x) - f(y))^2, \quad (1.1)$$

where  $x \sim y$  means  $x$  and  $y$  are adjacent in  $G$ . Given two disjoint subsets  $A$  and  $B$  of  $V$ , the *effective resistance* between  $A$  and  $B$  is defined by

$$R_{\text{eff}}^G(A, B) := \left( \inf \{ \mathcal{E}(f); f|_A = 1, f|_B = 0 \} \right)^{-1}.$$

Given  $v \in \mathbb{Z}^d$ , let  $\mathcal{T}_v$  be the component of  $\text{WSF}(\mathbb{Z}^d)$  containing  $v$ . By the one-ended property of  $\mathcal{T}_v$  (see the text around Corollary 1.6), there exists a unique infinite path  $\langle \text{Ray}_v(n) \rangle_{n \geq 0}$  on  $\mathcal{T}_v$  starting from  $v$  that does not visit any vertex twice.

**Theorem 1.2.** *Given  $v \in \mathbb{Z}^d$  and a sample of  $\text{WSF}(\mathbb{Z}^d)$ , let  $\mathcal{T}_v$  and  $\text{Ray}_v$  be defined as in the paragraph above. If  $d \geq 9$ , then  $\liminf_{n \rightarrow \infty} n^{-1} R_{\text{eff}}^{\overline{\mathcal{T}_v}}(v, \text{Ray}_v(n)) > 0$  almost surely.*

Since  $R_{\text{eff}}^{\overline{\mathcal{T}_v}}(v, \text{Ray}_v(n)) \leq n$ , Theorem 1.2 means  $R_{\text{eff}}^{\overline{\mathcal{T}_v}}(v, \text{Ray}_v(n))$  grows linearly.

Theorem 1.1 and 1.2 leave open the natural questions whether components of  $\overline{\text{WSF}(\mathbb{Z}^d)}$  are recurrent or transient for  $d = 5, 6, 7$  and what the growth rate of  $R_{\text{eff}}^{\overline{\mathcal{T}_v}}(v, \text{Ray}_v(n))$  is for  $d = 5, 6, 7, 8$ . Although we do not address these problems here, we prove the following resampling property of  $\text{WSF}(\mathbb{Z}^d)$  for all dimensions, which has implications for the behavior of random walks on the components of  $\text{WSF}(\mathbb{Z}^d)$ .

We will use the following notion. Given a graph  $G$ , let  $H$  be a random subgraph of  $G$  whose components are infinite graphs. We write  $\text{WSF}(H)$  as the unconditional law of the random subgraph of  $G$  obtained by first sampling  $H$  and then sampling a  $\text{WSF}$  independently on each component of this instance of  $H$ . We similarly define  $\text{FSF}(H)$ .

**Theorem 1.3.** *For all  $d \in \mathbb{N}$ ,  $\text{WSF}(\overline{\mathfrak{F}_w(\mathbb{Z}^d)}) = \text{WSF}(\mathbb{Z}^d)$ , that is, the two measures agree.*

Theorem 1.3 implies that for each  $v \in \mathbb{Z}^d$ ,  $\text{WSF}(\overline{\mathcal{T}_v})$  a.s. has a single component for  $\mathcal{T}_v$  as in Theorem 1.2. Therefore, two independent simple random walks on  $\overline{\mathcal{T}_v}$  a.s. intersect.

Theorem 1.1 and 1.2 are proved in Section 3 and 4 respectively, using quantitative arguments. A **vertex-transitive graph** is a graph such that given any two vertices, there exists a graph automorphism mapping one vertex to the other (see Section 3.2). Theorem 1.1 can be extended to all vertex-transitive graphs whose volume growth is at least  $r \mapsto r^8$ , while the argument for Theorem 1.2 works for unimodular vertex-transitive graphs whose volume growth is at least  $r \mapsto r^9$  (see Section 3.2 for background on unimodular graphs).

On the other hand, Theorem 1.3 is a corollary of the following set of general results, which will be proved in Section 5 by qualitative arguments.

**Theorem 1.4.** *For any locally finite, infinite, connected graph  $G$ , we have*

$$\text{FSF}(\overline{\mathfrak{F}_w(G)}) = \text{WSF}(G) \quad \text{and} \quad \text{FSF}(\overline{\mathfrak{F}_f(G)}) = \text{FSF}(G). \quad (1.2)$$

*In particular, the FSF on each component of  $\overline{\mathfrak{F}_w(G)}$  and  $\overline{\mathfrak{F}_f(G)}$  has a unique component a.s.*

An immediate corollary of Theorem 1.4 is

**Corollary 1.5.**  $\text{WSF}(\overline{\mathfrak{F}_w(G)}) = \text{WSF}(G)$  if and only if each component of  $\overline{\mathfrak{F}_w(G)}$  has the property that  $\text{FSF} = \text{WSF}$ .  $\square$

Corollary 1.5 implies Theorem 1.3 as follows. Recall  $\mathcal{E}$  defined as in (1.1). A necessary and sufficient condition for  $\text{FSF}(G) = \text{WSF}(G)$  is that the only harmonic functions  $f$  on  $G$  with  $\mathcal{E}(f) < \infty$  are constant functions [BLPS01]. This is known to be the case when  $G$  is transitive and **amenable**, i.e.,  $\inf_K \#\partial K / \#K = 0$ , where the infimum is over all finite vertex sets  $K$  of  $G$ . By [BLS99, Theorem 5.5], every amenable transitive graph has the property

that each component of every random subgraph with automorphism-invariant law also a.s. has no nonconstant harmonic functions  $f$  with  $\mathcal{E}(f) < \infty$ . This gives Theorem 1.3.

An *end* of a tree is an equivalence classes of infinite simple paths in the tree, where two paths are equivalent if their symmetric difference is finite.

**Corollary 1.6.**  $\text{WSF}(\overline{\mathfrak{F}_w(G)}) = \text{WSF}(G)$  (resp.,  $\text{FSF}(\overline{\mathfrak{F}_w(G)}) = \text{FSF}(G)$ ) if each component of  $\mathfrak{F}_w(G)$  (resp.,  $\mathfrak{F}_f(G)$ ) is one-ended, that is, has a single end a.s.

In [LMS08], it is proved that the one-end property of WSF components holds for all transient vertex-transitive graphs (also see [LP16, Theorem 10.49]). Thus  $\text{WSF}(\overline{\mathfrak{F}_w(G)}) = \text{WSF}(G)$  in this case. This in particular gives another proof of Theorem 1.3. For more general results on the one-ended property of FSF and WSF, see [LMS08, Hut15].

Inspired by Morris' aforementioned result that each component of WSF on every graph is a.s. recurrent, we conjecture that  $\text{WSF}(\overline{\mathfrak{F}_w(G)}) = \text{WSF}(G)$  for every locally finite, connected graph  $G$  as in Theorem 1.4.

Neither  $\text{WSF}(\overline{\mathfrak{F}_f(G)}) = \text{FSF}(G)$  nor  $\text{WSF}(\overline{\mathfrak{F}_f(G)}) = \text{WSF}(G)$  holds for all graphs. For counterexamples of the first equality, let  $G$  be a tree with the property that  $\mathfrak{F}_w(G) \neq G$  a.s. (For example,  $G$  could be a regular tree.) Then  $\mathfrak{F}_f(G) = G$  a.s. while  $\text{WSF}(\overline{\mathfrak{F}_f(G)}) = \text{WSF}(G)$ . A counterexample for the second equality will be given in Section 5.2.

## 2 Preliminaries

### 2.1 Basic notations

The set of positive integers is denoted by  $\mathbb{N}$ . Given a finite set  $A$ , we write  $\#A$  for the cardinality of  $A$ . Given two sets  $A, B$ , their symmetric difference  $(A \setminus B) \cup (B \setminus A)$  is denoted by  $A \triangle B$ . We use the asymptotic notation that two nonnegative functions  $f(x)$  and  $g(x)$  satisfy  $f \lesssim g$  if there exists a constant  $C > 0$  independent of  $x$  such that  $f(x) \leq Cg(x)$ . We write  $f \gtrsim g$  if  $g \lesssim f$  and write  $f \asymp g$  if  $f \lesssim g$  and  $f \gtrsim g$ .

Given a graph  $G$ , write  $V(G)$  and  $E(G)$  for the vertex and edge sets of  $G$ , respectively. When  $G = \mathbb{Z}^d$  for some  $d \in \mathbb{N}$ , we write  $o$  for its origin. If  $v, u \in V(G)$  are adjacent, we write  $v \sim u$  and write  $(u, v)$  for the edge between them. For  $v \in V(G)$ , let  $\deg(v)$  be the degree of  $v$ , which is the number of vertices adjacent to  $v$ . In our paper, graphs are assumed to be **locally finite**, that is,  $\deg(v) < \infty$  for every  $v \in V(G)$ . A graph  $H$  is called a **subgraph** of  $G$  if  $V(H) \subset V(G)$  and  $E(H) \subset E(G)$ . If  $H$  and  $H'$  are subgraphs of  $G$ , we write  $E(G) \setminus E(H)$  as  $G \setminus H$  and  $E(H) \triangle E(H')$  as  $H \triangle H'$ .

Given a family of probability measures  $\langle \mu_t \rangle_{t \in T}$  with index set  $T$ , a **coupling** of  $\langle \mu_t \rangle_{t \in T}$  is a family of random variables  $\langle X_t \rangle_{t \in T}$  on one probability space such that  $X_t$  is distributed as  $\mu_t$  for all  $t \in T$ . Suppose  $A$  and  $B$  are two probability measures on the space of subgraphs of a graph  $G$ . If there is a coupling  $(\mathfrak{A}, \mathfrak{B})$  of  $(A, B)$  such that  $\mathfrak{A} \subset \mathfrak{B}$  a.s., we say  $A$  is **stochastically dominated** by  $B$ , written as  $A \preceq B$ .

Let  $I$  be an interval in  $\mathbb{Z}$ . Suppose  $\mathcal{P} = \langle v_i \rangle_{i \in I}$  is a sequence of vertices in  $G$  indexed by  $I$  such that  $v_i \sim v_{i+1}$  whenever  $i$  and  $i+1$  are both in  $I$ . Then we call  $\mathcal{P}$  a **path** in  $G$ . If  $v_i \neq v_j$  as long as  $i \neq j$ , we say  $\mathcal{P}$  is **simple**. If  $I = \{0, \dots, n\}$ , then  $\mathcal{P}$  is called a finite path and  $|\mathcal{P}| := n$  is called the **length** of  $\mathcal{P}$ . We call the path  $\langle v_{n-i} \rangle_{0 \leq i \leq n}$  the **reversal** of

$\mathcal{P}$ . If we further have  $v_0 = v_n$ , then we call  $\mathcal{P}$  a (rooted) **loop**<sup>1</sup> and  $v_0$  the **root** of  $\mathcal{P}$ . If  $I = \mathbb{N} \cup \{0\}$  (resp.,  $I = \mathbb{Z}$ ), we call  $\mathcal{P}$  an **infinite** (resp., **bi-infinite**) path. We call  $t$  a **cut time** of  $\mathcal{P}$  if  $\{v_i\}_{i < t} \cap \{v_i\}_{i > t} = \emptyset$ .

Given  $x, y \in V(G)$ , let  $d_G(x, y)$  be the minimal length of a path starting from  $x$  and ending at  $y$  if  $x, y$  are in the same component of  $G$  and  $\infty$  otherwise. We call  $d_G(\cdot, \cdot)$  the **graph distance** on  $G$ . For  $v \in V(G)$  and  $r > 0$ , let  $B_G(v, r) := \{x \in V(G); d_G(v, x) \leq r\}$ . We identify  $B_G(v, r)$  with its induced subgraph.

A graph is called a **forest** if for any pair of distinct vertices there exists at most one simple path connecting them. A connected forest is called a **tree**. Given a connected graph  $G$ , a **spanning tree** (resp., **forest**) on  $G$  is a subgraph  $T \subset G$  such that  $T$  is a tree (resp., forest) and  $V(T) = V(G)$ .

The simple random walk on  $G$  is the Markov chain  $\langle S(n) \rangle_{n \geq 0}$  on the state space  $V(G)$  such that  $\mathbb{P}[S(n+1) = u \mid S(n) = v] = \deg(v)^{-1}$  for all  $u \sim v$  and  $n \geq 0$ . The **heat kernel**  $p$  of  $G$  is defined by  $p_t(x, y) = \mathbb{P}[S(t) = y]$  for  $x, y \in V(G)$  and  $t \in \mathbb{N} \cup \{0\}$  where  $S$  is a simple random walk on  $G$  starting from  $x$ . When  $G = \mathbb{Z}^d$ , it is well known that  $p_t(o, o) \asymp t^{-d/2}$ .

## 2.2 Wilson’s algorithm

Given a finite path  $\mathcal{P} = \langle v_i \rangle_{0 \leq i \leq n}$  in a graph  $G$  of length  $n \in \mathbb{N}$ , the (**forward**) **loop erasure** of  $\mathcal{P}$  (denoted by  $\mathbf{LE}[\mathcal{P}]$ ) is the path defined by erasing cycles in  $\mathcal{P}$  chronologically. More precisely, we define  $\mathbf{LE}[\mathcal{P}]$  inductively as follows. The first vertex  $u_0$  of  $\mathbf{LE}[\mathcal{P}]$  equals  $v_0$ . Supposing that  $u_j$  has been set, let  $k$  be the last index such that  $v_k = u_j$ . Set  $u_{j+1} := v_{k+1}$  if  $k < n$ ; otherwise, let  $\mathbf{LE}[\mathcal{P}] = \langle u_i \rangle_{0 \leq i \leq j}$ . If  $\mathcal{P}$  is an infinite path that visits no vertex infinitely many times, then we define  $\mathbf{LE}[\mathcal{P}]$  in a similar fashion. In particular, if  $S$  is a sample of simple random walk on a transient graph  $G$ , then  $\mathbf{LE}[S]$  is defined a.s. In such a case, we call the law of  $\mathbf{LE}[S]$  the **loop-erased random walk** (LERW) on  $G$ .

In [Wil96], Wilson discovered an algorithm for sampling uniform spanning trees on finite graphs using loop-erased random walk. In [BLPS01], Wilson’s algorithm was adapted to sample WSF on a transient graph  $G$ . This method is called **Wilson’s algorithm rooted at infinity**, which we now review. The algorithm goes by sampling a growing sequence of subgraphs of  $G$  as follows. Set  $\mathcal{T}_0 := \emptyset$ . Inductively, for each  $n \in \mathbb{N}$ , choose  $v_n \in V(G) \setminus V(\mathcal{T}_{n-1})$  and run a simple random walk starting at  $v_n$ . Stop the walk when it hits  $\mathcal{T}_{n-1}$  if it does; otherwise, let it run indefinitely. Denote the resulting path by  $\mathcal{P}_n$ , and set  $\mathcal{T}_n := \mathcal{T}_{n-1} \cup \mathbf{LE}[\mathcal{P}_n]$ . Write  $\mathfrak{F}_w := \bigcup_n \mathcal{T}_n$ . According to [BLPS01, Theorem 5.1], no matter how  $\langle v_n \rangle_{n \geq 1}$  are chosen, as long as  $V(\mathfrak{F}_w) = V(G)$ , the law of  $\mathfrak{F}_w$  is WSF( $G$ ).

## 2.3 Bounds on effective resistance

Nash-Williams’ inequality (see, e.g., [LP16, Section 2.5]) is a useful lower bound for the effective resistance. Here we record a generalization of Nash-Williams’ inequality.

**Lemma 2.1.** *Given a graph  $G$  with two disjoint subsets  $A$  and  $B$  of  $V(G)$ , a set  $\mathcal{C} \subset E(G)$  is called a **cut set** between  $A$  and  $B$  if  $\forall o \in A$  and  $\forall z \in B$ , every path from  $o$  to  $z$  must use*

<sup>1</sup>This is a topological loop, also called a cycle in graph theory, as opposed to the term “loop” in graph theory.

an edge in  $\mathcal{C}$ . Suppose  $\mathcal{C}_1, \dots, \mathcal{C}_n$  are cut sets between  $A$  and  $B$  for some  $n \in \mathbb{N}$ . For  $e \in E$ , let  $j(e) := \#\{k; e \in \mathcal{C}_k\}$ . Then  $R_{\text{eff}}^G(A, B) \geq \sum_{k=1}^n \left( \sum_{e \in \mathcal{C}_k} j(e)c(e) \right)^{-1}$ .

*Proof.* The proof is the same as the classical case in [LP16, Section 2.5], with a slight modification when applying the Cauchy–Schwarz inequality. We leave the details to the reader.  $\square$

The next lemma says that effective resistance is stable under local modification.

**Lemma 2.2.** *Suppose  $H$  and  $H'$  are two connected subgraphs of a graph  $G$  such that  $\#(H \triangle H') < \infty$ . Then there exists a constant  $c > 0$  depending on  $G, H, H'$  such that  $R_{\text{eff}}^H(u, v) \leq R_{\text{eff}}^{H'}(u, v) + c$  for all  $u, v \in V(H) \cap V(H')$ .*

*Proof.* It suffices to show that if  $H' \subset H$  and  $\#(E(H) \setminus E(H')) = 1$ , then there exists  $c' > 0$  depending only on  $H$  and  $H'$  but not on  $u, v \in V(H')$  such that

$$0 \leq R_{\text{eff}}^{H'}(u, v) - R_{\text{eff}}^H(u, v) \leq c'. \quad (2.1)$$

Once this is proved, a similar statement then follows for  $H' \subset H$  and  $\#(E(H) \setminus E(H')) < \infty$ . Then the general case follows by comparing both  $H$  and  $H'$  to the union graph  $H \cup H'$  of the two.

By Rayleigh’s monotonicity principle (see, e.g., [LP16, Section 2.4]), adding an edge can only decrease the effective resistance, hence  $R_{\text{eff}}^{H'}(u, v) \geq R_{\text{eff}}^H(u, v)$ . To prove the other direction of (2.1), we use Thomson’s principle (see, e.g., [LP16, Section 2.4]) that the effective resistance between two vertices is the minimum energy (i.e., the sum of the squares of all edge flows) among all unit flows between the two vertices. We may start from the minimizing flow for  $H$  from  $u$  to  $v$  and then construct a flow on  $H'$  between the same vertices by replacing the current flow along the removed edge  $e$  with a flow along a path in  $H'$  connecting the two endpoints of  $e$ . This increases the flow energy by an additive constant that depends only on  $H$  and  $H'$ .  $\square$

## 2.4 Indistinguishability of WSF components

In this subsection, we review a basic ergodic-theoretic property of components in WSF on transient vertex-transitive graphs. We call a triple  $(G, \rho, \omega)$  a **subgraph-decorated rooted graph** if  $G$  is a locally finite, connected graph,  $\rho$  is a distinguished vertex in  $G$  called the **root**, and  $\omega$  is a function from  $E(G)$  to  $\{0, 1\}$ . We think of  $\omega$  as a distinguished subgraph spanned by the edges  $\{e \in E(G); \omega(e) = 1\}$ . Given two such triples  $(G, \rho, \omega)$  and  $(G', \rho', \omega')$ , an isomorphism between them is a graph isomorphism between  $G$  and  $G'$  that preserves the root and the subgraph. Let  $\mathcal{G}_{\bullet}^{\{0,1\}}$  be the space of subgraph-decorated rooted graphs modulo isomorphisms. We endow  $\mathcal{G}_{\bullet}^{\{0,1\}}$  with the **local topology** where two elements  $(G, \rho, \omega)$  and  $(G', \rho', \omega')$  in  $\mathcal{G}_{\bullet}^{\{0,1\}}$  are close if and only if  $(B_G(\rho, r), \rho, \omega)$  and  $(B_{G'}(\rho', r), \rho', \omega')$  are isomorphic to each other for some large  $r$ .

Given  $(G, v, \omega) \in \mathcal{G}_{\bullet}^{\{0,1\}}$ , we define  $K_{\omega}(v)$  to be the connected component of  $v$  in  $\omega$ . A Borel-measurable set  $\mathcal{A} \subset \mathcal{G}_{\bullet}^{\{0,1\}}$  is called a **component property** if  $(G, v, \omega) \in \mathcal{A}$  implies  $(G, u, \omega) \in \mathcal{A}$  for all  $u \in K_{\omega}(v)$ . Given a component property  $\mathcal{A}$ , we say that a connected

component  $K$  of  $\omega$  has property  $\mathcal{A}$  if  $(G, u, \omega) \in \mathcal{A}$  for some (and equivalently every)  $u \in V(K)$ . A component property  $\mathcal{A}$  is called a **tail component property** if  $(G, v, \omega) \in \mathcal{A}$  implies  $(G, v, \omega') \in \mathcal{A}$  for all  $\omega' \subseteq E(G)$  such that  $\omega \Delta \omega'$  and  $K_\omega(v) \Delta K_{\omega'}(v)$  are both finite.

As a corollary of [HN17, Theorem 1.20], we have

**Lemma 2.3.** *Suppose  $G$  is a transient vertex-transitive graph. For every tail component property  $\mathcal{A}$ , either almost surely every connected component of  $\text{WSF}(G)$  has property  $\mathcal{A}$ , or almost surely none of the connected components of  $\text{WSF}(G)$  have property  $\mathcal{A}$ .*

By Lemma 2.2, for a vertex-transitive graph, both the properties in Theorems 1.1 and 1.2 are tail component properties. Therefore, we have

**Lemma 2.4.** *Consider  $\text{WSF}(\mathbb{Z}^d)$  for  $d \geq 5$ . Recall the notations  $\mathcal{T}_v$  and  $\text{Ray}_v$  in Theorem 1.2. Let  $E_v$  be the event that  $\overline{\mathcal{T}}_v$  is recurrent and  $F_v$  be the event  $\liminf_{n \rightarrow \infty} n^{-1} \mathbb{R}_{\text{eff}}^{\overline{\mathcal{T}}_v}(v, \text{Ray}_v(n)) > 0$ . Then neither  $\mathbb{P}[E_v]$  nor  $\mathbb{P}[F_v]$  depends on  $v$ . Moreover, both  $\mathbb{P}[E_v]$  and  $\mathbb{P}[F_v]$  belong to  $\{0, 1\}$ . The same holds with  $\mathbb{Z}^d$  replaced by any transient vertex-transitive graph.*

## 2.5 Two-sided random walk and loop-erased random walk

For  $d \in \mathbb{N}$ , let  $S^1$  and  $S^2$  be two independent simple random walks on  $\mathbb{Z}^d$  starting from the origin of  $\mathbb{Z}^d$ . For  $n \in \mathbb{Z}$ , let  $S(n) := S_1(n)$  if  $n \geq 0$  and  $S(n) := S^2(-n)$  if  $n < 0$ . We call the law of the bi-infinite path  $\langle S(n) \rangle_{n \in \mathbb{Z}}$  the **two-sided random walk** on  $\mathbb{Z}^d$ . It is standard that  $\mathbb{P}[S_1([0, \infty)) \cap S_2([1, \infty)) = \emptyset] > 0$  if and only if  $d \geq 5$  (see, e.g., [LP16, Theorem 10.24]). For  $d \geq 5$ , consider the event

$$E = \{\mathbf{LE}[S^1](m) \neq S^2(n) \text{ for all } m \geq 0, n \geq 1\}. \quad (2.2)$$

Since with positive probability 0 is a cut time of  $S$ , we have that  $\mathbb{P}[E] > 0$ . Define  $\tilde{S}(n)$  to be  $\mathbf{LE}[S^2](-n)$  for  $n \leq 0$  and  $\mathbf{LE}[S^1](n)$  for  $n \geq 0$ . The conditional law of  $\langle \tilde{S}(n) \rangle_{n \in \mathbb{Z}}$  conditioned on  $E$  is called the **two-sided loop-erased random walk** on  $\mathbb{Z}^d$ . It is clear that without loop-erasures,  $\langle S(n+1) - S(n) \rangle_{n \in \mathbb{Z}}$  is stationary and ergodic; indeed, it is an IID sequence. In fact, two-sided LERW also has stationary ergodic increments:

**Lemma 2.5.** *Suppose  $X$  is a sample of the two-sided loop-erased random walk  $\mathbb{Z}^d$  for  $d \geq 5$ . Then  $\langle X(n+1) - X(n) \rangle_{n \in \mathbb{Z}}$  is stationary and ergodic.*

Lawler [Law80] introduced the two-sided LERW on  $\mathbb{Z}^d$  ( $d \geq 5$ ) and showed that it is the local limit of the usual LERW viewed from nodes with large index. An essential ingredient to the proof of Lemma 2.5 is the reversibility of the loop-erasing operation for simple random walk, which was also first proved in [Law80]. Given the reversibility, we observe that Lemma 2.5 can be deduced from the ergodicity of the two-sided random walk and the following basic fact from ergodic theory (see, e.g., [Pet83]).

**Lemma 2.6** (Kac's Lemma). *Suppose  $\Omega$  is a measurable space and  $T: \Omega \rightarrow \Omega$  is measurable. Suppose  $\mathbb{P}$  is a probability measure on  $\Omega$  which is preserved by  $T$  and is ergodic. Let  $E \subset \Omega$  be an event such that  $\mathbb{P}[E] > 0$  and let  $\tau(\omega) := \inf\{n \in \mathbb{N}; T^n(\omega) \in E\}$  for all  $\omega \in \Omega$ . Let  $T_E(\omega) := T^{\tau(\omega)}(\omega)$  for all  $\omega \in E$ . Then  $T_E$  is an ergodic measure-preserving map from  $E$  to  $E$  under the conditional probability measure  $\mathbb{P}[\cdot | E]$ . Moreover,  $\mathbb{E}[\tau | E] = \mathbb{P}[E]^{-1}$ .*

To put Lemma 2.5 into the setting of Lemma 2.6, let us consider the two-sided simple random walk  $S$ . Since  $S$  can be almost surely decomposed into finite paths separated by cut times, the *forward* loop-erasure of the path  $\langle S(n) \rangle_{n \leq 0}$  is well defined, which we denote by  $\mathbf{LE}[S(-\infty, 0]]$ . By the reversibility of the loop-erasing operation, the path  $\mathbf{LE}[S(-\infty, 0]]$  has the same law as  $\mathbf{LE}[S[0, \infty)]$ . Now we use the event  $A := \{\mathbf{LE}[S(-\infty, 0]] \cap S[1, \infty) = \emptyset\}$ , which plays the same role as the event  $E$  in (2.2). Let  $T$  be the forward shift operator of  $\langle S(n) \rangle_{n \in \mathbb{Z}}$ . Now applying Lemma 2.6 we get Lemma 2.5.

Using estimates for random walk on  $\mathbb{Z}^d$ , it was shown in [Law80] that the two-sided LERW is weakly mixing, which is a property stronger than ergodicity. For this paper, we need only stationarity (see Section 4 for its use) and our argument can be readily extended to more general unimodular vertex-transitive graphs.

## 2.6 Loop space, loop measure, and cut time

Fix  $d \in \mathbb{N}$ . For  $z \in \mathbb{Z}^d$ , let  $\Omega^z$  be the space of loops in  $\mathbb{Z}^d$  rooted at  $z$  (see Section 2.1 for the definition). Define a measure  $\mu$  on  $\Omega^z$  by requiring  $\mu(\gamma) := (2d)^{-|\gamma|}$  for all  $\gamma \in \Omega^z$ . We call  $\mu$  the **loop measure** and  $\mu(\gamma)$  the **weight** of  $\gamma$ . Here we drop the dependence of  $\mu$  on  $z$  for simplicity of notation. In different places, we will consider loops with additional markings. For example, let  $\Omega_t^z := \{(\gamma, i); \gamma \in \Omega^z, i \in [0, |\gamma|] \cap \mathbb{Z}\}$  be the space of loops rooted at  $z$  with a **marked time**. Assigning each element in  $\Omega_t^z$  the weight of its loop, we define a measure on  $\Omega_t^z$ , which we still denote by  $\mu$  in a slight abuse of notation.

Let  $p$  be the heat kernel of  $\mathbb{Z}^d$ , so that  $p_t(o, o) \asymp t^{-d/2}$ . Let

$$Z_i := \sum_{t=0}^{\infty} (t+1)^i p_t(o, o) \quad \text{for } i \in \mathbb{N}.$$

Then  $Z_1 = \sum_{\omega \in \Omega_t^z} \mu(\omega)$  for  $z \in \mathbb{Z}^d$ . Suppose  $d \geq 5$ , so that  $Z_1 < \infty$ . Then  $\bar{\mu} := Z_1^{-1} \mu$  is a probability measure on  $\Omega^z$ . Note that  $\mathbb{E}^{\bar{\mu}}[|\gamma|] = Z_2/Z_1$ , which is finite if and only if  $d \geq 7$ .

**Proposition 2.7.** *For  $d \geq 7$ , let  $\langle S(t) \rangle_{t \in \mathbb{Z}}$  be a sample of two-sided random walk on  $\mathbb{Z}^d$ . Let  $T_0 := \sup\{t \geq 0; S(t) \in S((-\infty, 0])\}$ . For  $i \in \mathbb{N}$ , let*

$$T_i := \{t > T_{i-1}; t \text{ is a cut time of } \langle S(t) \rangle_{t \in \mathbb{Z}}\}.$$

Then  $\mathbb{E}[T_n] \leq Z_1 n + Z_2$  for all  $n \in \mathbb{N} \cup \{0\}$ .

*Proof.* Let  $T_- := \inf\{t \in \mathbb{Z}; S(t) = S(T_0)\}$ . Since  $\#(S((-\infty, 0]) \cap S([0, \infty))) < \infty$  a.s. (see, e.g., [LP16, Theorem 10.24]), we have  $-\infty < T_- \leq 0 \leq T_0 < \infty$  a.s. Let  $\gamma_0(j) := S(j)$  for  $0 \leq j \leq T_0$  and  $\gamma_0(j) := S(T_- - T_0 + j)$  for  $T_0 \leq j \leq T_0 - T_-$ . Then  $(\gamma_0, T_0) \in \Omega_t^o$ . Let  $(\gamma, \tau)$  be sampled from  $\bar{\mu}$  on  $\Omega_t^o$ . Conditioning on  $(\gamma, \tau)$ , let  $(\tilde{S}^1, \tilde{S}^2)$  be sampled from two independent simple random walks starting from  $\gamma(\tau)$ . Let  $\tilde{S}(t) := \tilde{S}^1(t)$  for  $t \geq 0$  and  $\tilde{S}(t) = \tilde{S}^2(-t)$  for  $t < 0$ . Let

$$A := \{0 \text{ is a cut time for } \tilde{S}\} \quad \text{and} \quad B = \{\tilde{S}^1([1, \infty)) \cap \gamma([\tau, |\gamma|]) = \emptyset\}.$$



Let  $\mathbb{P}$  be the measure corresponding to  $S$  and  $\tilde{\mathbb{P}}$  be the probability measure corresponding to the quadruple  $(\gamma, \tau, \tilde{S}^1, \tilde{S}^2)$ . Then

$$\mathbb{P}[(\gamma_0, T_0) = \omega] = \mu(\omega) \tilde{\mathbb{P}}[A, B \mid (\gamma, \tau) = \omega] \quad \forall \omega \in \Omega_t^o. \quad (2.3)$$

Since  $\bar{\mu} = Z_1^{-1} \mu$ , it follows that

$$\tilde{\mathbb{P}}[A, B] = \sum_{\omega \in \Omega_t^o} \bar{\mu}(\omega) \tilde{\mathbb{P}}[A, B \mid \omega] = Z_1^{-1} \sum_{\omega \in \Omega_t^o} \mathbb{P}[(\gamma_0, T_0) = \omega] = Z_1^{-1}. \quad (2.4)$$

In other words, the law of  $(\gamma_0, T_0)$  equals the conditional law of  $(\gamma, \tau)$  under  $\tilde{\mathbb{P}}[\cdot \mid A, B]$ . Let  $S_0^1(n) := S(T_0 + n)$  and  $S_0^2(n) := S(-n + T_-)$  for all  $n \in \mathbb{N} \cup \{0\}$ . Then  $(\gamma, \tau, \tilde{S}^1, \tilde{S}^2)$  under the conditioning  $\tilde{\mathbb{P}}[\cdot \mid A, B]$  has the same law as  $(\gamma_0, T_0, S_0^1, S_0^2)$ . Therefore  $\mathbb{E}^{\mathbb{P}}[T_0] = \mathbb{E}^{\tilde{\mathbb{P}}}[\tau; A, B] / \tilde{\mathbb{P}}[A, B] \leq \mathbb{E}^{\bar{\mu}}[|\gamma|] \cdot Z_1 = Z_2$ . For  $n \in \mathbb{N}$ , let  $\tilde{T}_n$  be the  $n$ th positive cut time for  $\tilde{S}$ . Then  $\mathbb{E}^{\mathbb{P}}[T_n - T_0] = \mathbb{E}^{\tilde{\mathbb{P}}}[\tilde{T}_n \mid A, B]$ . Applying Lemma 2.6 to  $\tilde{S} - \tilde{S}(0)$  and the event  $A$ , we have  $\mathbb{E}^{\tilde{\mathbb{P}}}[\tilde{T}_n \mid A] = n \mathbb{E}^{\tilde{\mathbb{P}}}[\tilde{T}_1 \mid A] = n \tilde{\mathbb{P}}[A]^{-1}$ , in other words,  $\mathbb{E}^{\tilde{\mathbb{P}}}[\tilde{T}_n; A] = n$ . It follows that  $\mathbb{E}^{\tilde{\mathbb{P}}}[\tilde{T}_n \mid A, B] = \mathbb{E}^{\tilde{\mathbb{P}}}[\tilde{T}_n; A, B] / \tilde{\mathbb{P}}[A, B] \leq Z_1 n$ , thanks to (2.4). This concludes the proof.  $\square$

As a corollary of Proposition 2.7, we have

**Lemma 2.8.** *Given  $d \geq 7$ , let  $S$  be a simple random walk on  $\mathbb{Z}^d$  started from the origin. For all  $n \in \mathbb{N} \cup \{0\}$ , let  $L_n := \#\{k \geq 0; |\mathbf{LE}[S([0, k])]| \leq n\}$ . Then  $\mathbb{E}[L_n] \leq Z_1 n + Z_2 + 1$ .*

*Proof.* We extend  $S$  to a two-sided random walk  $\langle S(n) \rangle_{n \in \mathbb{Z}}$ . Define  $T_i$  as in Proposition 2.7 for  $i \in \mathbb{N} \cup \{0\}$ . Since  $|\mathbf{LE}[S([0, k])]| > n$  for  $k > T_n$ , we have  $\{k \geq 0; |\mathbf{LE}[S([0, k])]| \leq n\} \subset [0, T_n]$ , so that  $L_n \leq T_n + 1$ . Now Lemma 2.8 follows from Proposition 2.7.  $\square$

### 3 Recurrence when $d \geq 8$

In this section, we first prove Theorem 1.1 and then extend the result to vertex-transitive graphs in Section 3.2. Recall  $\mathcal{T}_o$  and  $\text{Ray}_o$  as in Theorem 1.2 for the origin  $o$  of  $\mathbb{Z}^d$ . Given  $n \in \mathbb{N} \cup \{0\}$ , we call the connected component of  $\mathcal{T}_o \setminus (\text{Ray}_o[0, n-1] \cup \text{Ray}_o[n+1, \infty))$  containing  $\text{Ray}_o(n)$  the  $n$ th **bush** of  $\mathcal{T}_o$  and denote it by  $\text{Bush}_n$ . Given an edge  $e$  in  $\mathbb{Z}^d$  and two subgraphs  $H_1$  and  $H_2$  of  $\mathbb{Z}^d$  with  $V(H_1) \cap V(H_2) = \emptyset$ , we say that  $e$  **joins**  $H_1$  and  $H_2$  if one endpoint of  $e$  is in  $H_1$  and the other is in  $H_2$ .

**Lemma 3.1.** *Fix  $n, m \in \mathbb{N}$ . For  $0 \leq j \leq n$  and  $\ell \geq m$ , let  $N_{j, \ell}$  be the number of edges joining  $\text{Bush}_{n-j}$  and  $\text{Bush}_{n+\ell}$ . Suppose  $d \geq 8$ . Then there exists a constant  $C > 0$  such that*

$$\sum_{0 \leq j \leq n} \sum_{\ell \geq m} \mathbb{E}[N_{j, \ell}] \leq \frac{Cn}{m} \quad \text{for all } n < m \quad (3.1)$$

and

$$\min_{\xi \in [n, 2n]} \left( \sum_{0 \leq j \leq \xi} \sum_{\ell \geq m} \mathbb{E}[N_{j, \ell}] \right) \leq C \log \left( \frac{2n + m}{m} \right) \quad \text{for all } m, n \in \mathbb{N}. \quad (3.2)$$

We postpone the proof of Lemma 3.1 to Section 3.1 and proceed to prove Theorem 1.1.

*Proof of Theorem 1.1.* By Lemma 2.4, we see that Theorem 1.1 is equivalent to the statement that  $\overline{\mathcal{T}}_o$  is recurrent a.s.

By (3.2) with  $m = 1$ , there exists a constant  $C > 0$  and a sequence  $n_k \in [k^{2k}, 2k^{2k}]$  such that

$$\sum_{0 \leq j \leq n_k} \sum_{\ell \geq 1} \mathbb{E}[N_{j,\ell}] \leq C \log n_k \quad \text{for all } k \in \mathbb{N}. \quad (3.3)$$

Define  $\mathcal{C}_k$  to be the set consisting of edges joining  $\bigcup_{m \leq n_k} \text{Bush}_m$  and  $\bigcup_{m > n_k} \text{Bush}_m$ . Then removing  $\mathcal{C}_k$  leaves  $o$  in a finite component. Since  $\mathbb{E}[\#\mathcal{C}_k] \lesssim \log n_k$  by (3.3) and

$$\sum_{k=1}^{\tau} \log(n_k) \lesssim \tau^2 \log \tau \quad \text{for all } \tau \geq 1,$$

the argument in [BLPS01, Lemma 13.5 and Remark 13.6] yields  $\sum_1^{\infty} (\#\mathcal{C}_k)^{-1} = \infty$  a.s.

Let  $I_k$  be the event that there exists an edge joining  $\bigcup_{m \leq n_k} \text{Bush}_m$  and  $\bigcup_{m \geq n_{k+1}} \text{Bush}_m$ . Since  $\sum_1^{\infty} \frac{n_k}{n_{k+1} - n_k} < \infty$ , by (3.1) and the Borel–Cantelli lemma, we know that almost surely only finitely many events  $I_k$  occur. Therefore there exists a (random)  $K \in \mathbb{N}$  such that the elements in  $\{\mathcal{C}_k; k \geq K\}$  are all disjoint. By the Nash-Williams criterion (see, e.g. [LP16, Sec. 2.5]), it follows that  $\overline{\mathcal{T}}_o$  is recurrent a.s.  $\square$

### 3.1 Proof of Lemma 3.1

By linearity of expectation, we estimate  $\mathbb{E}[N_{j,\ell}]$  by estimating the probability of joining  $\text{Bush}_{n-j}$  and  $\text{Bush}_{n+\ell}$  for each edge of  $\mathbb{Z}^d$ . Let  $x$  and  $y$  be two adjacent vertices in  $\mathbb{Z}^d$  and  $S, S^1$ , and  $S^2$  be three independent simple random walks on  $\mathbb{Z}^d$  starting from  $o, x$ , and  $y$  respectively. Suppose that  $\text{WSF}(\mathbb{Z}^d)$  is sampled via Wilson’s algorithm rooted at infinity by first sampling  $S, S^1$ , and  $S^2$  and then other random walks. Fix  $0 \leq j \leq n$  and  $\ell \geq m$ . Given  $s, s', t' \in \mathbb{N} \cup \{0\}$  and  $z, w \in \mathbb{Z}^d$ , let  $E^{x,y}(s, s', t', z, w)$  be the event that

1.  $S(s) = z$  and  $|\mathbf{LE}[S([0, s])]| = n - j$ ;
2.  $S^1(s') = z$  and  $S^2(t') = w$ ;
3.  $\lambda := \sup\{k; S(k) = w\} \in [s, \infty)$  and  $|\mathbf{LE}[S([s, \lambda])]| = j + \ell$ ; and
4.  $t' = \inf\{k; S^2(k) \in \mathbf{LE}[S([s, \lambda])]\}$ .

Then  $\{x \in V(\text{Bush}_{n-j}) \text{ and } y \in V(\text{Bush}_{n+\ell})\} \subset \bigcup E^{x,y}(s, s', t', z, w)$ , where the union ranges over all possible tuples  $(s, s', t', z, w)$ .

Recall the loop space  $(\Omega^z, \mu)$  in Section 2.6, where  $z \in \mathbb{Z}^d$ . Now we consider another variant of  $\Omega^z$ , defined by  $\Omega_s^z := \{(\gamma, i); \gamma \in \Omega^z, i \in [0, |\gamma| - 1] \cap \mathbb{Z}\}$ . Each element in  $\Omega_s^z$  is a loop  $\gamma$  rooted at  $z$  with a **marked step** being the ordered pair  $\langle \gamma(i), \gamma(i+1) \rangle$ . By assigning each element in  $\Omega_s^z$  the weight of its loop, we define a measure on  $\Omega_s^z$ , which we still denote by  $\mu$ . Let  $\Omega_s^z(x, y, w, t', s') \subset \Omega_s^z$  be the set of  $(\gamma, i)$  that satisfy

1.  $\gamma(i) = y$  and  $\gamma(i+1) = x$ ;

2.  $s' = |\gamma| - i - 1$ ;
3.  $\mathbf{LE}[\gamma([0, i])](j + \ell) = w$ ; and
4.  $\max\{k \leq i; \gamma(k) = w\} = i - t'$ .

On  $E^{x,y}(s, s', t', z, w)$ , by concatenating  $S([s, \lambda])$ , the reversal of  $S^2([0, t'])$ , the edge from  $y$  to  $x$ , and  $S^1([0, s'])$ , we obtain an element in  $\Omega_s^z$  whose marked step is  $\langle y, x \rangle$ . Therefore  $\mathbb{P}[E^{x,y}(s, s', t', z, w)]$  equals

$$\mathbb{P}[S(s) = z \text{ and } |\mathbf{LE}[S([0, s])]| = n - j] \cdot (2d) \cdot \mu[\Omega_s^z(x, y, w, t', s')], \quad (3.4)$$

where the factor  $2d$  comes from the fact that the step  $\langle y, x \rangle$  need not be traversed by  $S$ ,  $S^1$ , or  $S^2$ . Note that  $\Omega_s^z(x, y, w, s', t') \subset \{(\gamma, i) \in \Omega_s^z; |\gamma| \geq j + \ell\}$ . Now let  $x$  and  $y$  vary. For different tuples  $(x, y, w, s', t')$ , the corresponding sets  $\Omega_s^z(x, y, w, s', t')$  are disjoint (because the definition (1)–(4) determines  $x, y, w, s'$  and  $t'$  from  $(\gamma, i)$ ). Therefore

$$\sum_{x,y,w,s',t'} \mu[\Omega_s^z(x, y, w, s', t')] \leq \mu[\{(\gamma, i) \in \Omega_s^z; |\gamma| \geq j + \ell\}]. \quad (3.5)$$

Let  $p$  be the heat kernel of  $\mathbb{Z}^d$ . By the definition of  $\Omega_s^z$  and  $\mu$ , for all  $t \in \mathbb{N}$ , we have

$$\mu[\{(\gamma, i) \in \Omega_s^z; |\gamma| = t\}] = tp_t(z, z) = tp_t(o, o).$$

Since  $d \geq 8$  and  $tp_t(o, o) \asymp t^{-d/2}$ , we see from (3.5) that

$$\sum_{x,y,w,s',t'} \mu[\Omega_s^z(x, y, w, s', t')] \leq \sum_{t \geq j + \ell} tp_t(o, o) \lesssim (j + \ell)^{-2}.$$

Set  $K_i := \#\{s; |\mathbf{LE}[S([0, s])]| = i\}$  for all  $0 \leq i \leq n$ . By (3.4), we have

$$\begin{aligned} \sum_{x,y,s,s',t',z,w} \mathbb{P}[E^{x,y}(s, s', t', z, w)] &\lesssim (j + \ell)^{-2} \sum_{s,z} \mathbb{P}[S(s) = z, |\mathbf{LE}[S([0, s])]| = n - j] \\ &= (j + \ell)^{-2} \sum_s \mathbb{P}[|\mathbf{LE}[S([0, s])]| = n - j] = \mathbb{E}[K_{n-j}](j + \ell)^{-2}. \end{aligned}$$

Since  $\mathbb{E}[N_{j,\ell}] \leq \sum_{x,y,s,s',t',z,w} \mathbb{P}[E^{x,y}(s, s', t', z, w)]$ , we see that

$$\sum_{0 \leq j \leq n} \sum_{\ell \geq m} \mathbb{E}[N_{j,\ell}] \leq \sum_{0 \leq j \leq n} \sum_{\ell \geq m} \mathbb{E}[K_{n-j}](j + \ell)^{-2} \lesssim \sum_{0 \leq j \leq n} \mathbb{E}[K_{n-j}](j + m)^{-1}. \quad (3.6)$$

Recall  $L_k$  in Lemma 2.8. We have  $L_k = \sum_{i=0}^k K_i$  for all  $k \in \mathbb{N} \cup \{0\}$ .

For  $n < m$ , by (3.6) we have  $\sum_{0 \leq j \leq n} \sum_{\ell \geq m} \mathbb{E}[N_{j,\ell}] \leq \frac{L_n}{m}$  and (3.1) follows from Lemma 2.8.

To prove (3.2), let  $Q_n := \sum_{0 \leq j \leq n} \mathbb{E}[K_{n-j}](j + m)^{-1}$ . Then

$$\begin{aligned} \sum_{\xi=n}^{2n} Q_\xi &= \sum_{\xi=n}^{2n} \sum_{i=0}^{\xi} \mathbb{E}[K_i](\xi - i + m)^{-1} \leq \sum_{i=0}^{2n} \mathbb{E}[K_i] \sum_{\xi=i}^{2n} (\xi - i + m)^{-1} \\ &\lesssim L_{2n} \log \left( \frac{2n + m}{m} \right) \lesssim n \log \left( \frac{2n + m}{m} \right), \end{aligned}$$

where the last inequality uses Lemma 2.8. This gives (3.2).  $\square$

## 3.2 Extensions to vertex-transitive graphs

Suppose  $G$  is a vertex-transitive graph and  $o \in V(G)$ . We call  $V(r) := \#B_G(o, r)$  the **volume growth function** of  $G$ . In this subsection, we explain the following extension of Theorem 1.1.

**Theorem 3.2.** *If  $G$  is a vertex-transitive graph with  $V(r) \gtrsim r^8$ , then almost surely each connected component of  $\overline{\text{WSF}(G)}$  is recurrent.*

Let  $p$  be the heat kernel of  $G$ . It is standard that  $p_t(o, o) \lesssim t^{-4}$  when  $V(r) \gtrsim r^8$  (see, e.g., [LP16, Corollary 6.32]). Given this, Lemma 2.8 would imply Lemma 3.1, hence Theorem 1.1 in the same way with  $\mathbb{Z}^d$  replaced by  $G$  in Theorem 1.1. Let  $S$  be the two-sided random walk on  $G$  defined as in Section 2.5 with  $\mathbb{Z}^d$  and its origin replaced by  $G$  and  $o \in V(G)$ . The proof of Proposition 2.7 still works with  $\mathbb{Z}^d$  replaced by a vertex-transitive graph  $G$  with  $V(r) \gtrsim r^7$  as long as the following additional condition holds:

$$\langle S(n) \rangle_{n \in \mathbb{Z}} \text{ is stationary and ergodic viewed as path-decorated rooted graphs.} \quad (3.7)$$

Therefore Theorem 3.2 holds under this additional condition. A vertex-transitive graph  $G$  satisfies (3.7) if it is **unimodular**, that is, the automorphism group  $\text{Aut}(G)$  of  $G$  is unimodular.<sup>2</sup> In other words,  $\text{Aut}(G)$  admits a nontrivial Borel measure that is invariant under both left and right multiplication by group elements. We will not elaborate on the notion of unimodularity but refer to [BLPS99] or [LP16, Chapter 8] for more background.

If  $G$  is nonunimodular, then Theorem 3.2 is essentially already known. In fact, by [LP16, Proposition 8.14],  $G$  is **nonamenable** in this case, that is,  $\inf_K \#\partial K / \#K > 0$ , where the infimum is over all finite vertex sets  $K$  of  $G$ . Therefore, by [BLPS01, Theorem 13.1], we have  $\mathbb{E}[\#(\overline{\mathcal{T}}_o \cap B_G(0, n))] \asymp n^2$ , where  $\mathcal{T}_o$  is the component of  $\text{WSF}(G)$  containing  $o$ . Now [BLPS01, Lemma 13.5] yields that  $\overline{\mathcal{T}}_o$  is a.s. recurrent. Since  $G$  is transient, Lemma 2.4 concludes Theorem 3.2 in the nonunimodular case.

## 4 Linear growth of resistance when $d \geq 9$

Recall the two-sided LERW defined in Section 2.5. Now we define the two-sided WSF.

**Definition 4.1.** *Given  $d \geq 5$ , sample a random spanning forest  $\mathfrak{F}_w^2(\mathbb{Z}^d)$  on  $\mathbb{Z}^d$  as follows.*

1. *Sample a two-sided loop-erased random walk  $\tilde{S}$ .*
2. *Conditioning on  $\tilde{S}$ , sample a WSF (denoted by  $\mathfrak{F}_w$ ) on the graph obtained from  $\mathbb{Z}^d$  by identifying the trace of  $\tilde{S}$  as a single vertex.*
3. *Set  $\mathfrak{F}_w^2(\mathbb{Z}^d)$  to be the union of  $\mathfrak{F}_w$  and the trace of  $\tilde{S}$ , where  $\mathfrak{F}_w$  is viewed as a random subgraph of  $\mathbb{Z}^d$ .*

We call the law of  $\mathfrak{F}_w^2(\mathbb{Z}^d)$  the **two-sided wired spanning forest** on  $\mathbb{Z}^d$  and denote it by  $\text{WSF}^2(\mathbb{Z}^d)$ .

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<sup>2</sup>It can be shown that unimodularity is, in fact, equivalent to (3.7).

It is clear that  $\text{WSF}^2(\mathbb{Z}^d)$  can be sampled from a modified version of Wilson's algorithm rooted at infinity: first sample a two-sided LERW and treat it as the first walk in Wilson's algorithm; then proceed as in the original Wilson's algorithm to form a spanning forest on  $\mathbb{Z}^d$ . The stationary two-sided LERW on  $\mathbb{Z}^d$  was extended to  $d = 4$  in [LSW16] and to  $d = 2, 3$  in [Law18] by a limiting procedure. Therefore  $\text{WSF}^2(\mathbb{Z}^d)$  can be defined for all  $d \in \mathbb{N}$ . However, we will not need the lower-dimensional cases.

By Lemma 2.5, as a subgraph-decorated rooted graph,  $(\mathbb{Z}^d, o, \mathfrak{F}_w^2(\mathbb{Z}^d))$  is stationary under shifting along the trace of  $\tilde{S}$ . We will use this stationarity and the ergodic theorem to prove Theorem 1.2. The following lemma will be needed.

**Lemma 4.2.** *In the setting of Theorem 1.2, for  $v \in \mathbb{Z}^d$  such that  $v \neq o$ , let  $\mathcal{N}_v$  be the number of edges joining<sup>3</sup>  $\mathcal{T}_o$  and  $\mathcal{T}_v$  if  $\mathcal{T}_o \neq \mathcal{T}_v$  and be 0 otherwise. Then  $\mathbb{E}[\mathcal{N}_v] < \infty$  for  $d \geq 9$ .*

*Proof.* We follow a similar argument as in Lemma 3.1. Given two neighboring vertices  $x$  and  $y$ , let  $\mathcal{I}_{x,y}$  be the event that  $x \in \mathcal{T}_o$  and  $y \in \mathcal{T}_v$ . Suppose that  $S^o, S^v, S^x$ , and  $S^y$  are independent simple random walks on  $\mathbb{Z}^d$  starting from  $o, v, x$ , and  $y$ , respectively. By Wilson's algorithm,

$$\mathbb{P}[\mathcal{I}_{x,y}] \leq \mathbb{P}[S^x([0, \infty)) \cap S^o([0, \infty)) \neq \emptyset \text{ and } S^y([0, \infty)) \cap S^v([0, \infty)) \neq \emptyset].$$

Therefore

$$\mathbb{E}[\mathcal{N}_v] \leq \sum_{x \sim y} \sum_{k,l,m,n \geq 0} \mathbb{P}[S^o(k) = S^x(l) \text{ and } S^y(m) = S^v(n)]. \quad (4.1)$$

Now let  $\Omega^{o,v}$  be the space of quadruples  $(\gamma, \sigma, \tau, i)$  where  $\gamma$  is a path in  $\mathbb{Z}^d$  from  $o$  to  $v$ , and  $\sigma, \tau \in [0, |\gamma|] \cap \mathbb{Z}$ , and  $i \in [0, |\gamma| - 1] \cap \mathbb{Z}$ . Here,  $\sigma$  and  $\tau$  are considered as two marked times of  $\gamma$  and  $\langle \gamma(i), \gamma(i+1) \rangle$  is considered as a marked step. Define the measure  $\mu$  on  $\Omega^{o,v}$  by assigning weight  $(2d)^{-|\gamma|}$  to each  $(\gamma, \sigma, \tau, i) \in \Omega^{o,v}$ . Then

$$\mu[\Omega^{o,v}] = \sum_{t=0}^{\infty} \mu[\{(\gamma, \sigma, \tau, i) \in \Omega^{o,v}; |\gamma| = t\}] = \sum_{t=0}^{\infty} t(t+1)^2 p_t(o, v).$$

Since  $p_t(o, v) \lesssim t^{-d/2}$  (see [LP16, Corollary 6.32(ii)]), we see that  $\mu[\Omega^{o,v}] < \infty$  if  $d \geq 9$ .

Let  $\Omega^{o,v}(k, l, m, n) := \{(\gamma, k, k + \ell + 1 + m, k + \ell) \in \Omega^{o,v}; |\gamma| = k + \ell + 1 + m + n\}$ . By concatenating  $S^o([0, k])$ , the reversal of  $S^x([0, \ell])$ , the edge from  $x$  to  $y$ , the path  $S^y([0, m])$ , and the reversal of  $S^v([0, n])$ , we see that  $\sum_{x \sim y} \mathbb{P}[S^o(k) = S^x(l) \text{ and } S^y(m) = S^v(n)]$  is no larger than  $\mu[\Omega^{o,v}(k, l, m, n)]$ . On the other hand,  $\Omega^{o,v}(k, l, m, n) \cap \Omega^{o,v}(k', l', m', n') = \emptyset$  if  $(k, l, m, n) \neq (k', l', m', n')$ . Now interchanging the summation in (4.1), we get  $\mathbb{E}[\mathcal{N}_v] \leq \mu[\Omega^{o,v}]$ , which is finite if  $d \geq 9$ .  $\square$

*Proof of Theorem 1.2.* By Lemma 2.4, it suffices to prove that

$$\mathbb{P}[F] > 0 \quad \text{where} \quad F := \left\{ \liminf_{n \rightarrow \infty} n^{-1} \text{R}_{\text{eff}}^{\overline{\mathcal{T}}_o}(o, \text{Ray}_o(n)) > 0 \right\}. \quad (4.2)$$

Let us perform a particular Wilson's algorithm rooted at infinity to sample  $\text{WSF}(G)$ .

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<sup>3</sup>Recall the notion from Section 3.

1. Sample a simple random walk  $S^1$  from  $o$  as the first walk in Wilson's algorithm.
2. Run a simple random walk  $W$  from  $o$  until the time  $\tau := \{t \geq 0; W(t) \notin \mathbf{LE}[S^1]\}$ . Set  $v := W(\tau)$ .
3. Assign  $v$  to be the starting point of the second simple random walk in Wilson's algorithm and denote this walk by  $S^v$ .
4. Sample the rest of  $\text{WSF}(G)$  according to Wilson's algorithm in an arbitrary way.

Let  $\mathbb{P}$  be the probability measure from the above sampling and let  $\tilde{\mathbb{P}}$  be  $\mathbb{P}$  conditioned on the event  $B := \{\tau = 1 \text{ and } \mathcal{T}_v \neq \mathcal{T}_o\}$ . Set  $S^2(n) := W(n)$  for  $0 \leq n \leq \tau$  and  $S^2(n) := S^v(n - \tau)$  for  $n \geq \tau$ . Then conditional on  $v$ ,  $(S^1, S^2)$  is a pair of independent random walks on  $\mathbb{Z}^d$  and  $B$  is exactly the event  $E$  in (2.2). We define  $\tilde{S}$  in terms of  $(S^1, S^2)$  as in Lemma 2.5, so that under  $\tilde{\mathbb{P}}$  it is a two-sided LERW. On the event  $B$ , let  $\mathfrak{F}_w^2$  consist of the edges of  $\text{WSF}(G)$  and the edge  $(o, v)$ , and let  $\tilde{\mathcal{T}}_o$  consist of the edges of  $\mathcal{T}_o$ ,  $\mathcal{T}_v$  and  $(o, v)$ . By Lemma 2.5 and Definition 4.1, under  $\tilde{\mathbb{P}}$ , we see that  $\mathfrak{F}_w^2$  is distributed as  $\text{WSF}^2(G)$  and  $\tilde{\mathcal{T}}_o$  is the component of  $\mathfrak{F}_w^2$  containing  $o$ . We claim that

$$\tilde{\mathbb{P}}[F] > 0, \text{ where } F \text{ is as in (4.2).} \quad (4.3)$$

To prove (4.3), recall the notion of  $\text{Bush}_n$  in Section 3. For  $k \in \mathbb{N}$ , let  $\mathcal{C}_k$  be the set of edges joining  $\bigcup_{m \leq k} \text{Bush}_m$  and  $\bigcup_{m \geq k+1} \text{Bush}_m$ . For any edge  $e$  of  $\mathbb{Z}^d$ , let  $j(e) := \#\{k; e \in \mathcal{C}_k\}$ . Let  $J_k := \sum_{e \in \mathcal{C}_k} j(e)$ .

Under  $\tilde{\mathbb{P}}$ , for  $n \in \mathbb{Z}$ , let  $\widetilde{\text{Bush}}_n$  be the connected component of  $\tilde{\mathcal{T}}_o \setminus \tilde{S}(\mathbb{Z} \setminus \{n\})$  containing  $\tilde{S}(n)$ . Let  $\tilde{\mathcal{C}}_k$  be the set of edges joining  $\bigcup_{m \leq k} \widetilde{\text{Bush}}_m$  and  $\bigcup_{m \geq k+1} \widetilde{\text{Bush}}_m$ . Let  $\tilde{j}(e) := \#\{k; e \in \tilde{\mathcal{C}}_k\}$  and  $\tilde{J}_k := \sum_{e \in \tilde{\mathcal{C}}_k} \tilde{j}(e)$ . By Lemma 4.2,  $\#\tilde{\mathcal{C}}_{-1} < \infty$   $\tilde{\mathbb{P}}$ -a.s. By the stationarity of  $\text{WSF}^2(\mathbb{Z}^d)$ , both  $\langle \tilde{\mathcal{C}}_k \rangle_{k \in \mathbb{Z}}$  and  $\langle \tilde{J}_k \rangle_{k \in \mathbb{Z}}$  are stationary under  $\tilde{\mathbb{P}}$ . On the other hand, if  $e \in \mathcal{C}_k$  joins  $\text{Bush}_m$  and  $\text{Bush}_n$  for some  $n > m$ , we must have  $e \in \tilde{\mathcal{C}}_k$  and  $j(e) = \tilde{j}(e) = n - m$ . Therefore  $\mathcal{C}_k \subset \tilde{\mathcal{C}}_k$  and  $J_k \leq \tilde{J}_k < \infty$   $\tilde{\mathbb{P}}$ -a.s. for all  $k \in \mathbb{N} \cup \{0\}$ . By the stationarity of  $\langle \tilde{J}_k \rangle_{k \in \mathbb{Z}}$  under  $\tilde{\mathbb{P}}$  and Birkhoff's ergodic theorem, there exists a random variable  $Y$  such that  $\mathbb{E}^{\tilde{\mathbb{P}}}[Y] = \mathbb{E}^{\tilde{\mathbb{P}}}[\tilde{J}_0^{-1}] > 0$  and  $\lim_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} \tilde{J}_k^{-1} = Y$   $\tilde{\mathbb{P}}$ -a.s. Since  $J_k \leq \tilde{J}_k$ , with positive probability under  $\tilde{\mathbb{P}}$  (hence under  $\mathbb{P}$ ), we have

$$\liminf_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} J_k^{-1} > 0. \quad (4.4)$$

By the one-end property of  $\mathcal{T}_o$ , we see that (4.4) defines a tail component property for  $\text{WSF}(\mathbb{Z}^d)$ . By Lemma 2.3, the event (4.4) holds almost surely under both  $\mathbb{P}$  and  $\tilde{\mathbb{P}}$ .

According to Lemma 4.2, there exists  $m_0 \in (0, \infty)$  that  $\tilde{\mathbb{P}}[\#\tilde{\mathcal{C}}_0 = m_0] > 0$ . For  $k \in \mathbb{Z}$ , let  $I_k$  be the indicator of the event that  $\#\tilde{\mathcal{C}}_k = m_0$ . By Birkhoff's ergodic theorem, there exists a random variable  $I$  such that  $\mathbb{E}^{\tilde{\mathbb{P}}}[I] > 0$  and  $\lim_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} I_k = I$   $\tilde{\mathbb{P}}$ -a.s. Now we work on the event  $E_\delta := \{I > \delta \text{ and } \liminf_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} J_k^{-1} > \delta\}$ , where  $\delta > 0$  is chosen so small that  $\tilde{\mathbb{P}}[E_\delta] > 0$ . By Lemma 2.1,  $R_{\text{eff}}^{\tilde{\mathcal{T}}_o}(o, \text{Ray}_o(n)) \geq \sum_{k=0}^{n-1} J_k^{-1}$ . By the definition of  $E_\delta$ , we see that  $\tilde{\mathbb{P}}[E_\delta \setminus F] = 0$ , which gives (4.3), hence (4.2).  $\square$

We conclude this section by the following straightforward extension of Theorem 1.2.

**Theorem 4.3.** *Theorem 1.2 still holds if  $\mathbb{Z}^d$  is replaced by a unimodular vertex-transitive graph  $G$  such that  $V(r) \gtrsim r^9$ . (Recall the notions in Section 3.2.)*

*Proof.* Note that  $V(r) \gtrsim r^9$  implies that the heat kernel satisfies  $p_t(o, o) \lesssim t^{-9/2}$ . By inspection, the proof of Theorem 1.2 still works given this heat-kernel estimate and the fact that  $\mathcal{T}_o$  can be coupled with the (stationary) two-sided WSF as in the proof of Theorem 1.2 via Wilson's algorithm. This holds as long as the two-sided LERW can be sampled from the two-sided simple random walk as in Section 2.5. By (3.7) and Lemma 2.6, this is true if  $G$  is unimodular.  $\square$

We expect that the unimodularity assumption in Theorem 4.3 can be removed. However, this would require a different approach, because for nonunimodular vertex transitive graphs, although the two-sided LERW can still be defined by a limiting procedure, it is not related to the two-sided simple random walk that we defined earlier.

## 5 Resampling property

In this section, we first prove Theorem 1.4 and Corollary 1.6 in Section 5.1. Then we provide a counterexample to  $\text{WSF}(\overline{\mathfrak{F}_f(G)}) = \text{WSF}(G)$  in Section 5.2.

### 5.1 Proof of Theorem 1.4 and Corollary 1.6

We introduce the following notation. Given a graph  $G$ , suppose  $H$  is a random finite subgraph of  $G$ . Let us sample a random forest on  $G$  as follows. First sample  $H$ . Conditioning on  $H$ , uniformly sample a spanning tree on each component of  $H$ . The unconditional law of the resulting random forest is denoted by  $\text{USF}(H)$ .

*Proof of Theorem 1.4.* We prove only  $\text{FSF}(\overline{\mathfrak{F}_w(G)}) = \text{WSF}(G)$  since  $\text{FSF}(\overline{\mathfrak{F}_f(G)}) = \text{FSF}(G)$  can be proved in exactly the same way.

Fix  $o \in V(G)$ . For a positive integer  $n$ , let  $\mathfrak{F}_w^n$  be a sample of  $\text{WSF}(B_G(o, n))$ . For  $0 < m < n$ , thinking of  $\mathfrak{F}_w^n \cap B_G(o, m)$  and  $\mathfrak{F}_w \cap B_G(o, m)$  as subgraphs of  $B_G(o, m)$ , let  $K_{m,n}$  and  $K_m$  be their induced-component graphs, respectively. For a fixed  $m$ , as  $n$  tends to  $\infty$ , the laws of  $\mathfrak{F}_w^n \cap B_G(o, m)$  and  $\mathfrak{F}_w \cap B_G(o, m)$  can be coupled so that they are identical with probability  $1 - o_n(1)$ . Hence the same is true for  $K_{m,n}$  and  $K_m$ . Conditioning on  $K_{m,n} = K$ , the conditional law of  $\mathfrak{F}_w^n \cap B_G(o, m)$  is  $\text{USF}(K)$  because every spanning forest of  $K$  that is connected in each component of  $K$  extends to a spanning tree of  $\widehat{B_G(o, n)}$  in the same number of ways. Letting  $n$  tend to  $\infty$ , we see that the law of  $\mathfrak{F}_w \cap B_G(o, m)$  is  $\text{USF}(K_m)$ . Note that  $(K_m)_{m \geq 1}$  is an exhaustion of  $\overline{\mathfrak{F}_w(G)}$ . (More precisely, each component of  $\overline{\mathfrak{F}_w(G)}$  is exhausted by the corresponding sequence of components of  $K_m$ .) Therefore by the definition of FSF, the measures  $\text{USF}(K_m)$  converge to  $\text{FSF}(\overline{\mathfrak{F}_w(G)})$  as  $m \rightarrow \infty$  (restricted to any finite subgraph of  $G$ ). Since the law of  $\mathfrak{F}_w \cap B_G(o, m)$  is  $\text{USF}(K_m)$ , by letting  $m$  tend to  $\infty$ , we obtain  $\text{WSF}(G) = \text{FSF}(\overline{\mathfrak{F}_w(G)})$ .  $\square$

*Proof of Corollary 1.6.* Recall that  $\text{WSF}(G) \preceq \text{FSF}(G)$  for any locally finite connected graph  $G$  (see, e.g., [LP16, Section 10.2]). Together with Theorem 1.4, we obtain

$$\text{WSF}(\overline{\mathfrak{F}_w(G)}) \preceq \text{FSF}(\overline{\mathfrak{F}_w(G)}) = \text{WSF}(G).$$

Let  $(\mathfrak{F}'_w, \mathfrak{F}_w)$  be a coupling of  $\text{WSF}(\overline{\mathfrak{F}_w(G)})$  and  $\text{WSF}(G)$  such that  $\mathfrak{F}'_w \subset \mathfrak{F}_w$ . Since each connected component of  $\mathfrak{F}'_w$  is an infinite graph a.s., while each component of  $\mathfrak{F}_w$  has a single end, we must have  $\mathfrak{F}_w = \mathfrak{F}'_w$  a.s. This proves the first assertion; the second is proved similarly.  $\square$

## 5.2 A counterexample for $\text{WSF}(\overline{\text{FSF}(G)}) = \text{WSF}(G)$

Recall that for any graph  $G$  and neighbors  $x, y$  in  $G$ , Kirchhoff's formula extended to the wired spanning forest gives that  $\mathbb{P}[(x, y) \in \mathfrak{F}_w(G)] = R_{w\text{-eff}}(x, y)$ , see [LP16, Equation (10.3)]. Here, the **wired effective resistance** between  $x$  and  $y$  is defined by

$$R_{w\text{-eff}}^G(x, y) = \left( \inf \{ \mathcal{E}(f); f|_A = 1, f|_B = 0, \#(f^{-1}[\mathbb{R} \setminus \{0\}]) < \infty \} \right)^{-1}.$$

If  $H$  is a subgraph of  $G$  that includes  $(x, y)$  but does not include at least one edge  $(u, w)$  for which the wired current  $i_w^{(x,y)}(u, w) \neq 0$ , then Rayleigh's monotonicity principle yields  $\mathbb{P}[(x, y) \in \mathfrak{F}_w(G)] < \mathbb{P}[(x, y) \in \mathfrak{F}_w(H)]$ ; see [LP16, Section 9.1] for the definition of wired current.

Now let  $G$  be the graph consisting of two copies of  $\mathbb{Z}^5$ , which we denote by  $\mathbb{Z}^5 \times \{0\}$  and  $\mathbb{Z}^5 \times \{1\}$ , and an edge  $e$  connecting  $o_1 := (o, 0)$  and  $o_2 := (o, 1)$ . As before,  $o$  represents the origin of  $\mathbb{Z}^5$ . Since  $\mathbb{Z}^5$  is transient, the wired current  $i_w^e$  is nonzero on infinitely many edges of  $\mathbb{Z}^5 \times \{i\}$  for each  $i \in \{0, 1\}$ . (In fact, it can be proved that all edges have nonzero current.) Recall that  $\text{FSF}(\mathbb{Z}^5) = \text{WSF}(\mathbb{Z}^5)$ . Since  $\mathfrak{F}_f(\mathbb{Z}^5)$  contains infinitely many trees a.s., its induced components are not all of  $\mathbb{Z}^5$ . Furthermore,  $e$  is not contained in any cycle, whence  $e \in \mathfrak{F}_f(G)$  a.s., and  $\text{FSF}(G)$  may be coupled with  $\text{FSF}(\mathbb{Z}^5 \times \{0\})$  and  $\text{FSF}(\mathbb{Z}^5 \times \{1\})$  so that  $\mathfrak{F}_f(G) = \{e\} \cup \mathfrak{F}_f(\mathbb{Z}^5 \times \{0\}) \cup \mathfrak{F}_f(\mathbb{Z}^5 \times \{1\})$ . Let  $\mathcal{T}_e$  be the component of  $\mathfrak{F}_f(G)$  containing  $e$ . Then  $\mathcal{T}_e$  consists of  $e$  and the component of  $\mathfrak{F}_f(\mathbb{Z}^5 \times \{i\})$  containing  $o_i$ , where  $i = 0, 1$ . Each edge of  $\mathbb{Z}^5$  has the same probability of being in  $\mathfrak{F}_f(\mathbb{Z}^5)$ , whence infinitely many edges  $(u, w)$  with  $i_w^e(u, w) \neq 0$  are not in  $\overline{\mathcal{T}_e}$  a.s. It follows from the preceding paragraph that  $\mathbb{P}[(x, y) \in \mathfrak{F}_w(G)] < \mathbb{P}[(x, y) \in \mathfrak{F}_w(\overline{\mathcal{T}_e}) | \overline{\mathcal{T}_e}]$ . Taking the expectation gives the result.

This same method answers negatively a long-standing question of whether  $\text{WSF}(\overline{\mathfrak{F}_f(G)}) = \text{WSF}(G)$  for Cayley graphs,  $G$ . We may take  $G$  to be the natural Cayley graph of the free product of  $\mathbb{Z}^3$  with  $\mathbb{Z}_2$  to obtain a counterexample. The analysis is similar to the preceding.

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