Random Walks on Groups and the Kaimanovich–Vershik Conjecture

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Microsoft

Simple Random Walk

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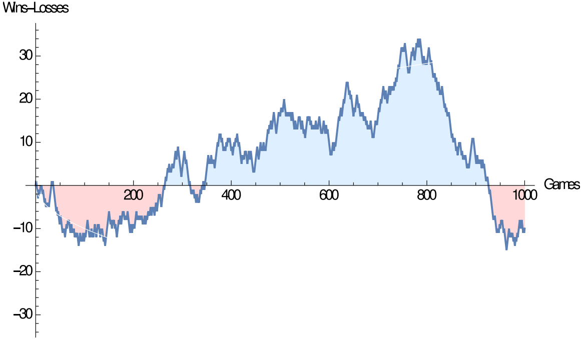

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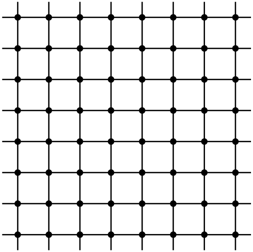

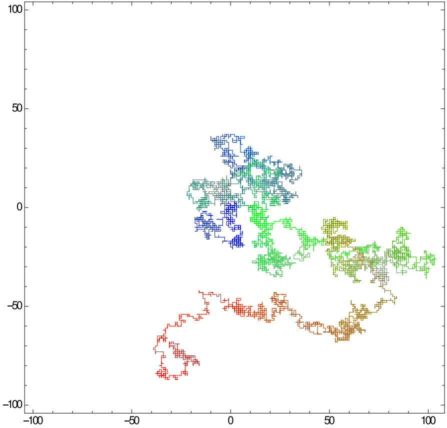

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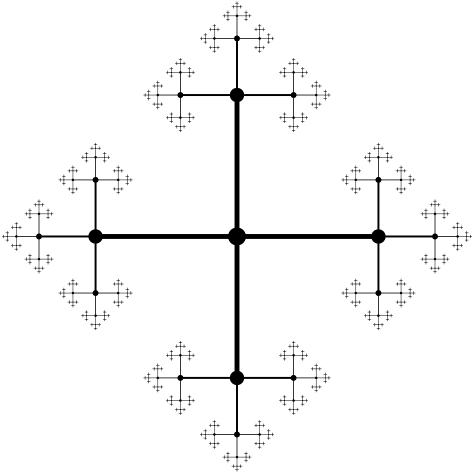

Simple Random Walk on Cayley Graphs

Let $X_0, X_1, X_2, \ldots$ be simple random walk on a Cayley graph of a group Γ with respect to a symmetric generating set S. Thus, $X_{n+1} = X_n \cdot s$ for a random generator $s \in S$. We start with X_0 equal to the identity of Γ. (Assume laziness if necessary to avoid periodicity: make S contain the identity.)

Simple Random Walk on Cayley Graphs

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Simple random walk on $\mathbb{Z}^d$ is recurrent for $d = 1, 2$ and transient for $d \geq 3$.

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The Lamplighter Groups

$\mathbb{Z}^{d\odot} := \left(\sum_{x\in\mathbb{Z}^d} \mathbb{Z}_2\right) \rtimes \mathbb{Z}^d$. For $(\psi, m) \in \mathbb{Z}^{d\odot}$, the lamplighter is at $m \in \mathbb{Z}^d$ and the light is on at $x \in \mathbb{Z}^d$ iff $\psi(x) = 1$; only finitely many lamps are on.

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Asymptotic Behavior

Transience and positive limiting speed are examples of limiting (asymptotic) behavior that have probability 0 or 1.

We can describe asymptotic behavior either via the invariant σ-field, consisting of the events that are invariant under the left shift on the space of trajectories $(x_0, x_1, \ldots) \mapsto (x_1, x_2, \ldots)$,

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Question

When is $\mathcal{T}$ trivial (i.e., consist only of sets of probability 0 and 1)?

Equivalences

Subadditivity shows that the speed $\lim_{n\to\infty} \frac{1}{n}\mathbf{E}\left[|X_n|\right]$ exists.

---

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Bounded functions measurable with respect to the tail $\mathcal{T}$[1] can be identified with the space **BH** of bounded harmonic functions, where $u \colon \Gamma \to \mathbb{R}$ is called harmonic if for all $x \in \Gamma$,

$$u(x) = \frac{1}{|S|}\sum_{s\in S} u(xs)\,.$$

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Theorem of Blackwell (1955), Varopoulos (1985)

The following are equivalent:

- $\mathcal{T}$ is trivial;
- **BH** contains only constants;
- the speed is 0.

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Tail Triviality: Examples

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- In the case of regular trees of degree at least 3, there are asymptotic "directions" (limiting rays), so $\mathcal{T}$ is not trivial.
- What happens on the lamplighter groups $\mathbb{Z}^{d^{\odot}}$? [demo]

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This is easy to see for $d \geq 3$: if $X_n = (\Phi_n, Y_n)$, where $\Phi_n \in \sum_{x \in \mathbb{Z}^d} \mathbb{Z}_2$ and $Y_n \in \mathbb{Z}^d$, then there is a final lamp configuration $\Phi_\infty = \lim_{n \to \infty} \Phi_n$ because the random walk is transient. [demo] Thus, $\sigma(\Phi_\infty) \subseteq \mathcal{T}$.

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Question

Does $\sigma(\Phi_\infty) = \mathcal{T}$ up to sets of measure 0 (for $d \ge 3$)?
Kaimanovich–Vershik conjectured yes.

Tail on Lamplighters

Why is $\mathcal{T}$ is trivial on $\mathbb{Z}^{\odot}$? After n steps, the lamplighter is very likely never to have gone distance more than $O(\sqrt{n})$ and so the set of lamps that are on is very likely to be within distance $O(\sqrt{n})$. This means that the walk is very likely to be within a lamplighter ball of radius $O(\sqrt{n})$, so speed $= 0$.

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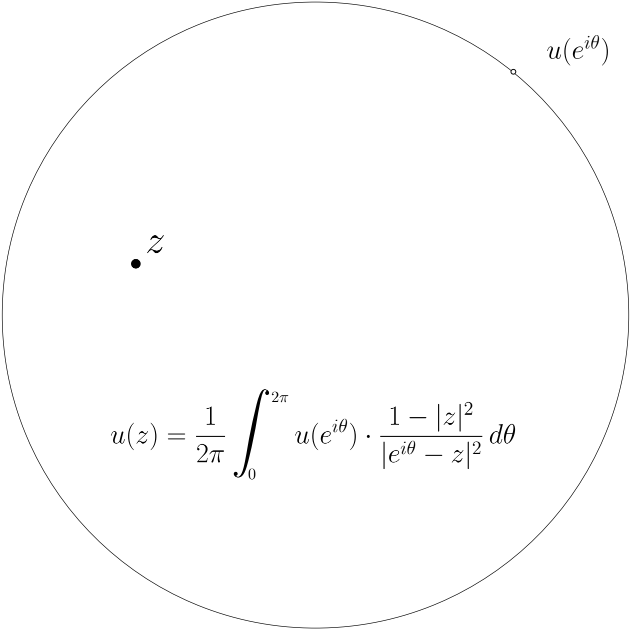
The same can be shown on ${\mathbb{Z}^2}^{\odot}$, with the radius $\sqrt{n}$ replaced by $n/\log n$ at the end.

Poisson–Furstenberg Boundaries

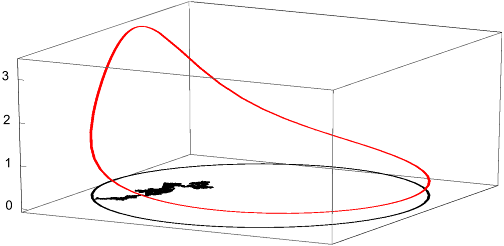
When the tail σ-field is not trivial, how can we describe all bounded harmonic functions?

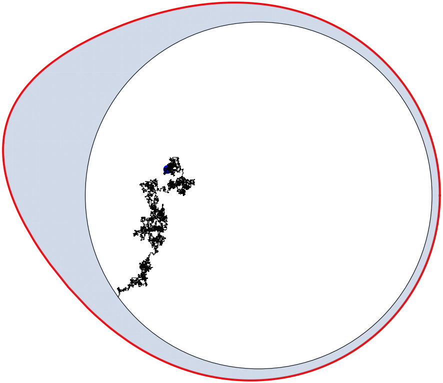
Poisson Formula for Harmonic Functions in the Unit Disk



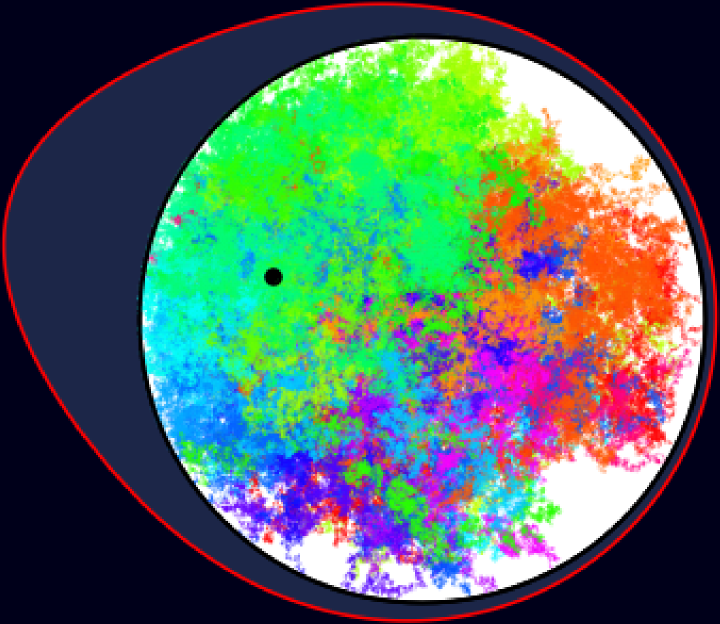
$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \cdot \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \, d\theta$$

Poisson Formula: Interpretation by Kakutani (1944) of Harmonic Measures



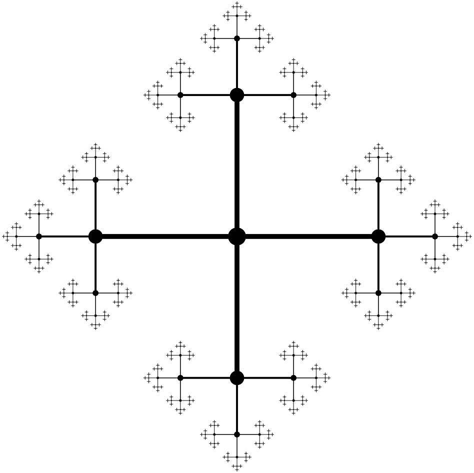


Poisson Formula: Interpretation by Kakutani (1944), 100 Samples



Poisson–Furstenberg Boundary of $\mathbb{F}_2$

Free group on two letters:



Kaimanovich–Vershik Conjecture (1983)

For $d \geq 3$, the Poisson–Furstenberg boundary of $\mathbb{Z}^{d\odot}$ is $\mathbb{Z}_2^{\mathbb{Z}^d}$ with the laws of Φ_∞.

Entropy

Recall that

$$H\big((p_1, p_2, \ldots, p_k)\big) := \sum_{i=1}^{k} p_i \log \frac{1}{p_i} \le \log k\,.$$

We have

$$H(\mu \times \nu) = H(\mu) + H(\nu)\,,$$

so for the increments ξ_k of random walk (i.e., $X_{k+1} = X_k \xi_{k+1}$), we have

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whence $\lim_{n\to\infty} \frac{1}{n} H(X_n) =: \boldsymbol{h}$ exists, the Avez asymptotic entropy.

Entropy and Tail Triviality

Theorem of Avez (1974), Kaimanovich–Vershik (1979), Derriennic (1980)

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$\boldsymbol{h} = 0$ iff $\mathcal{T}$ is trivial.

Thus, $\mathcal{T}$ is trivial when Γ has subexponential growth.

Criterion to Identify the Tail

Suppose that $\mathcal{T}' \subseteq \mathcal{T}$ is a Γ-closed σ-field. Write $H(X_n \mid \mathcal{T}')$ for the expected entropy of the conditional distribution of X_n given $\mathcal{T}'$.

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Criterion of Kaimanovich (1985, 1994, 2000)

$\boldsymbol{h}' := \lim_{n\to\infty} \frac{1}{n} H(X_n \mid \mathcal{T}')$ exists with $\boldsymbol{h}' = 0$ iff $\mathcal{T}' = \mathcal{T}$ (mod 0).

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Corollary

If $\mathbf{Q}_n$ are $\mathcal{T}'$-measurable random finite subsets of Γ with

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathbf{Q}_n| = 0 \quad \text{a.s.}$$

and

$$\lim_{n\to\infty} \mathbf{P}[X_n \in \mathbf{Q}_n] = 1\,,$$

then $\mathcal{T}' = \mathcal{T}$ (mod 0).

Enhanced Criterion to Identify the Tail

Enhanced Corollary, L.-Peres (2015+)

If $\mathbf{Q}_n$ are $\mathcal{T}'$-measurable random finite subsets of Γ with

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$$\lim_{n\to\infty} \mathbf{P}[\exists m \geq n \;\; X_m \in \mathbf{Q}_n] > 0\,,$$

then $\mathcal{T}' = \mathcal{T}$ (mod 0).

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Resolution of the Kaimanovich–Vershik Conjecture

Erschler (2008, pub. 2011)

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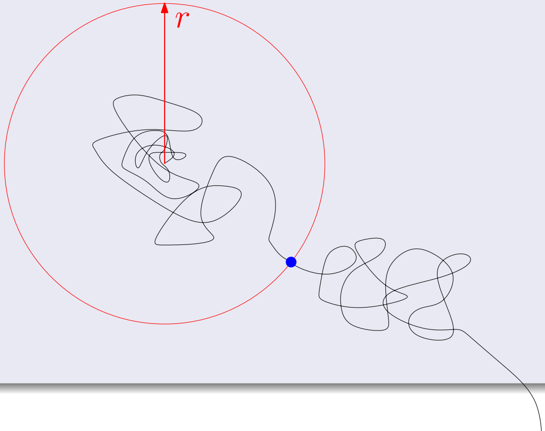
We can "guess" (enumerate the possibilities of) the location of the lamplighter at time *n* easily—there are only polynomially many choices. But it is harder to guess which lamps were on at time *n*, seeing only Φ_∞. [demo]

Proof of the Kaimanovich–Vershik Conjecture

Proof.

For $r > 1$, consider the events

$$\mathtt{cut}_r := \left[\exists m \geq r \ \left(\forall k < m \ |Y_k| < r \text{ and } \forall j > m \ |Y_j| > r\right)\right].$$



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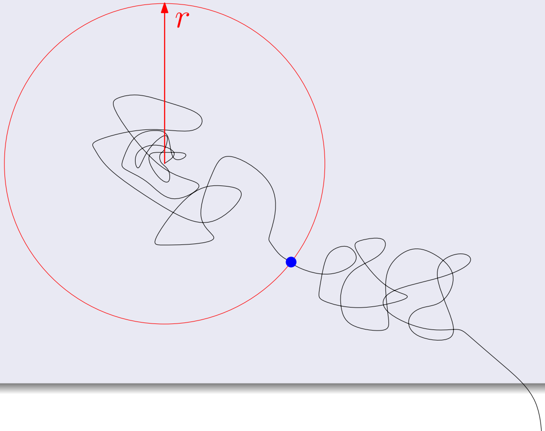

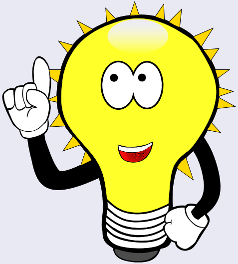

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$$\phi(z) = \begin{cases} \Phi_\infty(z) & \text{if } |z| < |y|, \\ 0 & \text{if } |z| \geq |y|. \end{cases}$$

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Since $\limsup_{n\to\infty} \mathbf{P}_o\left[\exists m \geq n \ X_m \in \mathbf{Q}_n\right] \geq c_3 > 0$

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For $r > 1$, consider the events

$$\mathtt{cut}_r := \Big[\exists m \geq r \ \ \big(\forall k < m \ \ |Y_k| < r \text{ and } \forall j > m \ \ |Y_j| > r\big)\Big] .$$

James and Peres (1996) showed that $\mathbf{P}_o(\mathtt{cut}_r) \geq c_1/r$ and $\mathbf{P}_o(\mathtt{cut}_r \cap \mathtt{cut}_{r+j}) \leq c_2/(r \cdot j)$. The second moment method applied to $\sum_{r=n}^{n^2} \mathbf{1}_{\mathtt{cut}_r}$ then yields that $\mathbf{P}_o\big(\bigcup_{r=n}^{n^2} \mathtt{cut}_r\big) \geq c_3 \frac{(\log n)^2}{(\log n)^2} = c_3 > 0$. Define $\mathbf{Q}_n := \mathbf{Q}_n(\Phi_\infty)$ to be the set of (ϕ, y) such that $n \leq |y| \leq n^2$ and

$$\phi(z) = \begin{cases} \Phi_\infty(z) & \text{if } |z| < |y|, \\ 0 & \text{if } |z| \geq |y|. \end{cases}$$

Since $\limsup_{n \to \infty} \mathbf{P}_o\big[\exists m \geq n \ \ X_m \in \mathbf{Q}_n\big] \geq c_3 > 0$ and $|\mathbf{Q}_n| \leq c \cdot n^{2d}$,

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Supplementary Material

Proof of Blackwell's Theorem

To say that u is harmonic is equivalent to

$$u(X_n) = \mathbf{E}[u(X_{n+1}) \mid X_n] = \mathbf{E}[u(X_{n+1}) \mid X_0, X_1, \ldots, X_n]\,,$$

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In particular, $\mathcal{T}$ is trivial iff **BH** contains only constants.

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We have

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so if $\boldsymbol{h} = 0$, then $X_{[1,k]}$ is independent of $\mathcal{T}$, which implies that $\mathcal{T}$ is independent of $\mathcal{T}$, i.e., $\mathcal{T}$ is trivial. ☐

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Write $p_n^{\mathcal{T}'}(x, y) := \mathbf{P}_x[X_n = y \mid \mathcal{T}']$ for the transition probabilities of the Markov chain conditioned on $\mathcal{T}'$. We use the following result of Kaimanovich (2000):

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contradicting the hypothesis. ☐

Our proof for $d \geq 5$ not using the enhanced corollary.

Guess (the change in) the location of the lamplighter every $\log n$ steps.
That amounts to $\left(O(\log n)^d\right)^{\frac{n}{\log n}} = e^{O(d\frac{n}{\log n}\log\log n)}$ choices altogether.

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To take care of the balls that might be visited in the future, enumerate the possible $Y_{n-\sqrt{n}}, \ldots, Y_n$ and their lamps at time n; there are only $O(n^d)(2d)^{\sqrt{n}} 2^{\sqrt{n}} = e^{O(\sqrt{n})}$ such possibilities. $\square$

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$$\binom{n/\log n}{n^{3/4}} 2^{O\left((\log n)^d n^{3/4}\right)} .$$

□

General Results

Theorem (Rosenblatt (1981), Kaimanovich–Vershik (1983), conjecture of Furstenberg (1973))

A group Γ is amenable iff there is a symmetric μ whose support generates Γ such that (Γ, μ) is Liouville. Moreover, if Γ is nonamenable, then every μ is non-Liouville.

Theorem (Dynkin–Maljutov (1961) $+$ Gromov (1981))

If a group Γ has polynomial growth, then every μ is Liouville.

Theorem (Frisch, Hartman, Tamuz, and Vahidi Ferdowsi (2018))

If a group Γ does not have polynomial growth, then there exists a symmetric μ with finite entropy that is non-Liouville.

Open Question (Kaimanovich–Vershik (1983))

Is the Liouville property for simple random walk on a Cayley graph of a finitely generated group stable under change of generators?