Poisson Boundaries of Lamplighter Groups: Proof of the Kaimanovich–Vershik Conjecture

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Abstract. We answer positively a question of Kaimanovich and Vershik from 1979, showing that the final configuration of lamps for simple random walk on the lamplighter group over $\mathbb{Z}^d$ ($d \ge 3$) is the Poisson boundary. For $d \ge 5$, this had been shown earlier by Erschler (2011). We extend this to walks of more general types on more general groups.

§1. Introduction.

Suppose that Γ is a countable infinite group and μ is a probability measure on Γ whose support generates Γ (as a group). A function $f\colon \Gamma \to \mathbb{R}$ is called ***harmonic*** if $f(x) = \sum_z \mu(z) f(xz)$ for all $x \in \Gamma$. If all bounded harmonic functions are constant, then (Γ, μ) is said to have the ***Liouville property***. A general theory for the non-Liouville case was initiated by Furstenberg (1963, 1971a, 1971b), who defined the notion of Poisson boundary to describe the set of bounded harmonic functions. Such harmonic functions are closely linked to the μ***-walk***, which is the Markov chain with transition probabilities $p(x, y) := \mu(x^{-1}y)$. Earlier work on boundaries for general Markov chains is due to Blackwell (1955), Feller (1956), Doob (1959), Hunt (1960), and Feldman (1962); a special case for groups was established by Dynkin and Maljutov (1961).

Rosenblatt (1981) and Kaimanovich and Vershik (1983) proved a conjecture of Furstenberg (1973) that Γ is amenable iff there is a symmetric μ whose support generates Γ such that (Γ, μ) is Liouville. Another open question had been whether there exists an amenable group with a symmetric non-Liouville measure. To answer this, Vershik and Kaimanovich (1979, 1983) utilized certain restricted wreath products $\mathbb{Z}_2 \wr \mathbb{Z}^d$, now commonly called lamplighter groups, where $\mathbb{Z}_2 := \mathbb{Z}/(2\mathbb{Z})$ is referred to as the lamp group and $\mathbb{Z}^d$ as the

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base group. These are solvable (hence amenable) groups of exponential growth. To define them more generally, let $\mathfrak{L}$ and Λ be two groups. Then $\mathfrak{L} \wr \Lambda$ is the semidirect product $\left(\sum_{z\in\Lambda} \mathfrak{L}\right) \rtimes \Lambda$, where Λ acts on $\sum_{z\in\Lambda} \mathfrak{L}$ by

$$(x\Phi)(z) := \Phi(x^{-1}z)\,.$$

Thus, if $(\Phi, x), (\Psi, y) \in \sum_{x\in\Lambda} \mathfrak{L} \times \Lambda$, then

$$(\Phi, x)(\Psi, y) = \big(\Phi \cdot (x\Psi), xy\big)\,.$$

The interpretation of an element (Φ, x) is that a lamplighter is at x, there is one lamp at each element of Λ, each lamp has a state in $\mathfrak{L}$, and Φ gives the states of all the lamps. If $\mathfrak{L}$ and Λ are both finitely generated, then so is their restricted wreath product. To see this, write o for the identity in Λ and id for the identity in $\mathfrak{L}$. Write $\mathbf{ID}$ for the function that is equal to id identically on Λ. Also, write δ^s for the element of $\sum_{z\in\Lambda} \mathfrak{L}$ that equals s at o and equals id elsewhere; thus, $\mathbf{ID} = \delta^{\mathsf{id}}$. If S_1 and S_2 are generating sets for $\mathfrak{L}$ and Λ, respectively, then an often-used generating set for $\mathfrak{L} \wr \Lambda$ is $\big\{(\delta^{s_1}, o)\,;\ s_1 \in S_1\big\} \cup \big\{(\mathbf{ID}, s_2)\,;\ s_2 \in S_2\big\}$. Multiplying (Φ, x) on the right by a generator (δ^{s_1}, o) changes the state of the lamp at x by s_1, while multiplying (Φ, x) on the right by a generator $(\mathbf{ID}, s_2)$ moves the lamplighter to xs_2. Since every element of $\sum_{z\in\Lambda} \mathfrak{L}$ is the identity of $\mathfrak{L}$ at all but finitely many $z \in \Lambda$, the above set does indeed generate $\mathfrak{L} \wr \Lambda$.

Let μ be a finitely supported symmetric probability measure whose support generates $\mathbb{Z}_2 \wr \mathbb{Z}^d$. Kaimanovich and Vershik (1983), Proposition 6.4, showed that $(\mathbb{Z}_2 \wr \mathbb{Z}^d, \mu)$ is Liouville iff $d \le 2$. Vershik and Kaimanovich (1979, 1983) also asked for a description of the Poisson boundary for finitely supported μ on the lamplighter groups $\mathbb{Z}_2 \wr \mathbb{Z}^d$ when it is nontrivial, i.e., $d \ge 3$. Moreover, they suggested a natural candidate, namely, $(\mathbb{Z}_2)^{\mathbb{Z}^d}$ with the probability measure given by the final configuration of lamps under the μ-walk. On $\mathbb{Z}_2 \wr \mathbb{Z}^d$, the final configuration of lamps, which we will denote by Φ_∞, exists because the projection of the walk to the base group $\mathbb{Z}^d$ is transient.

In 2008, a breakthrough was achieved by Erschler (2011, 2010), who proved that the conjecture of Vershik and Kaimanovich (1979) is correct when $d \ge 5$.

We show here that the conjecture of Vershik and Kaimanovich (1979) is correct for all $d \ge 3$. In fact, we prove the following result that, for a commonly considered type of random walk on lamplighter groups, assumes only finite entropy on the base walk.

Theorem 1.1. *Let $\mathfrak{L}$ be a nontrivial finite group. Let Λ be a finitely generated group of at least cubic growth. Let μ be a probability measure of finite entropy on $\mathfrak{L} \wr \Lambda$ whose support generates $\mathfrak{L} \wr \Lambda$. Suppose that μ is a convex combination of a probability measure* *on $\mathfrak{L}$ and a probability measure on Λ in the sense that μ is concentrated on $\{(\delta^s, o)\,;\ s \in \mathfrak{L}\} \cup \{(\mathbf{ID}, x)\,;\ x \in \Lambda\}$. Then the Poisson boundary of $(\mathfrak{L} \wr \Lambda, \mu)$ is $\mathfrak{L}^\Lambda$ endowed with the law of Φ_∞.*

This theorem is proved in Section 4. When $d \le 2$, the Poisson boundary is trivial, just like when $\mathfrak{L} = \mathbb{Z}_2$.

Entropy is a key quantity in the study of Poisson boundaries. Few results exist that identify a Poisson boundary without an entropy assumption.

We also introduce an enhanced version of the celebrated entropy criterion of Kaimanovich (2000), which has been the key tool for identification of Poisson boundaries. This is presented in Corollary 2.3 and used in Section 3. We also discuss it informally below in the context of the history of the subject.

Poisson boundaries are related to other important aspects of random walks. One fundamental aspect is to determine, given a random walk on a group Γ, its set of possible asymptotic behaviors, by which we mean the σ-field $\mathcal{I}$ on the path space $\Gamma^{\mathbb{N}}$ invariant under time shifts. There is a well known correspondence between $\mathcal{I}$ and the space $\mathbf{BH}$ of bounded harmonic functions on Γ. In particular, the invariant σ-field is trivial (i.e., consists only of sets of probability 0 or 1) iff all bounded harmonic functions are constant.

Combining the introduction of asymptotic entropy by Avez (1972, 1974, 1976a, 1976b) and an important technical result by Derriennic (1976), together with their own new ideas and examples, a foundational paper by Kaimanovich and Vershik (1983), announced in Vershik and Kaimanovich (1979), developed a general theory to analyze Poisson boundaries. In particular, Avez, Derriennic (1980), and Kaimanovich–Vershik proved that if μ has finite entropy, then the Avez (asymptotic) entropy of the μ-walk is 0 iff the walk is Liouville. Varopoulos (1985) showed that for finitely supported symmetric μ, the rate of escape of the μ-walk is sublinear iff (Γ, μ) is Liouville. This was extended by Karlsson and Ledrappier (2007) to symmetric μ with finite first moment with respect to the word metric for a finite generating set.

Erschler (2004b) showed that every finitely generated solvable group of exponential growth admits a symmetric non-Liouville measure, and that every non-degenerate measure on $\mathfrak{L} \wr \Lambda$ whose projection to Λ is transient has nonzero Avez entropy. She also proved a result similar to the latter for the free metabelian groups $\mathbb{F}_d/\mathbb{F}_d''$ with $d \ge 3$. Furthermore, Erschler (2004a) showed that there are groups of intermediate growth with finite-entropy symmetric non-Liouville measures. Frisch, Hartman, Tamuz, and Vahidi Ferdowsi (2018) extended this to show that every finitely generated group that is not of polynomial growth admits a finite-entropy symmetric non-Liouville measure.

Furstenberg (1971b) and Kaimanovich and Vershik (1983) gave entropy criteria for identifying the Poisson boundary. Two notable papers by Ledrappier (1983, 1985) used this criterion to determine the Poisson boundary for discrete matrix groups. Ballmann and Ledrappier (1994) developed further the entropy method in the context of rank-one manifolds. Kaimanovich (1985, 1994, 2000) refined the entropy method more generally and introduced a powerful general criterion for equality of a given boundary and the Poisson boundary.

Informally, Kaimanovich's criterion says that in order that a candidate boundary be the Poisson boundary, it suffices to find a sequence of random finite sets $Q_n \subset \Gamma$, that depend on points of the candidate boundary, such that $|Q_n|^{1/n} \to 1$ as $n \to \infty$ and $\mathbf{P}[\Xi_n \in Q_n]$ is bounded below, where Ξ_n is the location of the random walk at time n. One of Kaimanovich's important observations was that the sets Q_n can often be defined geometrically. This led to his well-known strip and ray criteria. We enhance Kaimanovich's more general criterion so that it suffices that $\Xi_m \in Q_n$ for *some* $m \geq n$.

Kaimanovich's criteria led to much progress in identifying Poisson boundaries, such as the works by Kaimanovich and Masur (1996, 1998), Karlsson (2003), Malyutin (2003), Karlsson and Woess (2007), Sava (2010b), Brofferio and Schapira (2011), Gautero and Mathéus (2012), Malyutin, Nagnibeda, and Serbin (2017), Nevo and Sageev (2013), Maher and Tiozzo (2014), and Malyutin and Svetlov (2014).

Using these methods, Kaimanovich (2001) made some progress on the lamplighter question by showing that for μ whose projection on the base group, $\mathbb{Z}^d$, has nonzero mean, the final lamps do indeed give the Poisson boundary. This problem of identifying the Poisson boundary has been raised repeatedly (e.g., Kaimanovich (1991), Vershik (2000), Karlsson and Woess (2007), Sava (2010b), Erschler (2011, 2010), Georgakopoulos (2016)) and has been considered a major open problem in the field.

Beyond Erschler's result on the Kaimanovich–Vershik conjecture, similar results have been established for random walks μ of finite first moment whose support generates one of the following groups $\mathfrak{L} \wr \Lambda$:

- Λ has subexponential growth and there is a homomorphism $\psi \colon \Lambda \to \mathbb{Z}$ such that if $\pi \colon \mathfrak{L} \wr \Lambda \to \Lambda$ is the canonical projection, then $(\psi\pi)_*\mu$ has nonzero mean (Kaimanovich (2001));
- $\mathfrak{L}$ is finite and Λ is a group with a Cayley graph being a tree of degree at least 3 (Karlsson and Woess (2007));
- $\mathfrak{L} = \mathbb{Z}_2$ and Λ has infinitely many ends or is non-elementary hyperbolic (Sava (2010b)).

In all these cases, the projection of the random walk to Λ has linear rate of escape, and this makes the analysis considerably simpler.

Erschler (2011) also extended her result on $\mathfrak{L} \wr \mathbb{Z}^d$ ($\mathfrak{L}$ finitely generated and $d \geq 5$) beyond finitely supported μ to those with finite third moment, and noted that similar techniques work for free metabelian groups $\mathbb{F}_d/\mathbb{F}_d''$ when $d \geq 5$.

Prior to the work of Erschler (2011), Kaimanovich's entropy criterion was used in a mostly geometric fashion that did not require detailed knowledge of the probabilistic behavior of the random walks. Erschler succeeded in her results by discovering how to leverage such knowledge of random walks in $\mathbb{Z}^d$ for $d \geq 5$. In particular, she relied heavily on the existence of a positive density of cutpoints (for simple random walk—and analogous behavior in general). That is, for the lamplighter random walk $\langle \widehat{X}_n \rangle$ on $\mathbb{Z}_2 \wr \mathbb{Z}^d$, its projection X_n at time n to the base $\mathbb{Z}^d$ is a cutpoint with probability bounded below over all n. This allowed Erschler to define the required random finite sets Q_n that capture X_n with probability bounded below.

Our enhanced criterion allows the use of cut-spheres, which do not occur with positive density, but they do occur infinitely often for $d = 3, 4$. Use of cut-spheres also simplifies considerably the definition of the random sets Q_n. This is a general feature of our enhanced criterion, which we illustrate with a simple proof of a conjecture of Sava (2010a). However, we do not use our enhanced criterion to handle general base groups, where other innovations are used. The innovation that is most closely related to cutpoints is to use upper bounds on the Green function in order to bound the number of times at which the future of the walk can get close to the locations of the past of the walk. In order to handle more general base groups beyond $\mathbb{Z}^d$, other innovations convert small entropy growth found in various places into enumeration with small exponential growth of the required sets Q_n.

We begin with the definition of the Poisson boundary and Kaimanovich's criterion in Section 2. In order to present the proof of the original conjecture of Vershik and Kaimanovich (1979) in the briefest manner, we prove Theorem 1.1 in Section 3 in the special case where the μ-walk is simple random walk on $\mathfrak{L} \wr \mathbb{Z}^d$. We then prove the full Theorem 1.1 in Section 4. As did Erschler, we consider other step distributions μ on $\mathfrak{L} \wr \mathbb{Z}^d$; in Section 5, we extend her result to $d \geq 3$ and to μ having finite second moment. In this broader setting where generators can change lamps arbitrarily far from the location of the lamplighter, some technical condition is needed to ensure existence of the limiting lamp configuration, as discussed at the end of Section 5. In Section 6, we give some details about metabelian groups and similar groups and discuss our extensions to them.

§2. Preliminaries.

For a discrete probability distribution π on a set S, write $H(\pi) := \sum_{s\in S} \pi(s) \log \pi(s)^{-1}$ for the ***entropy*** of π. For a σ-field $\mathscr{F}$ and a discrete random variable X, write $H(X)$ for the entropy of the distribution of X and $H(X \mid \mathscr{F})$ for the conditional entropy of X given $\mathscr{F}$:

$$H(X \mid \mathscr{F}) := -\,\mathbf{E}\Big[\sum_x \mathbf{P}[X = x \mid \mathscr{F}] \cdot \log \mathbf{P}[X = x \mid \mathscr{F}]\Big]\,.$$

Our Markov chains will begin at a fixed point; when that point is x, we use $\mathbf{P}_x$ for the corresponding probability measure. Usually x will be the identity element, $\mathbf{o}$, of a group, Γ. When a transition matrix is given, we often regard $\mathbf{P}_x$ as the law on $\Gamma^{\mathbb{N}}$ of the trajectory of the corresponding Markov chain $\langle \Xi_n\,;\ n \ge 0\rangle$. The σ-field of shift-invariant events is denoted by $\mathcal{I}$. We say that two σ-fields are equal ***mod 0*** if their completions are equal, generally with respect to $\mathbf{P}_{\mathbf{o}}$. The diagonal action of Γ by multiplication on $\Gamma^{\mathbb{N}}$ induces an action of Γ on $\mathcal{I}$; a subset $\mathcal{J} \subseteq \mathcal{I}$ is said to be Γ***-closed*** if $\gamma(A) \in \mathcal{J}$ for all $\gamma \in \Gamma$ and all $A \in \mathcal{J}$.

The following criteria of Kaimanovich (2000) (see Theorem 4.6 and Corollary 4.6 there, or see Theorem 14.35 and Corollary 14.36 of Lyons and Peres (2016)) are essential in identifying Poisson boundaries:

THEOREM 2.1. *Let $\langle \Xi_n\,;\ n \ge 0\rangle$ be a random walk on Γ with $H(\Xi_1) < \infty$. Let $\mathcal{I}$ be the associated invariant σ-field and $\mathcal{J} \subseteq \mathcal{I}$ be a Γ-closed sub-σ-field. Then $\boldsymbol{h}^{\mathcal{J}} := \lim_{n\to\infty} n^{-1} H^{\mathcal{J}}(\Xi_n)$ converges a.s. and in L^1 to the constant $H(\Xi_1 \mid \mathcal{J}) - H(\Xi_1 \mid \mathcal{I})$. Furthermore, $\boldsymbol{h}^{\mathcal{J}} = 0$ iff $\mathcal{J} = \mathcal{I}$ mod 0.*

COROLLARY 2.2. *Let $\langle \Xi_n\,;\ n \ge 0\rangle$ be a random walk on Γ with $H(\Xi_1) < \infty$. Let $\mathcal{I}$ be the associated invariant σ-field and $\mathcal{J} \subseteq \mathcal{I}$ be a Γ-closed sub-σ-field. Suppose that for each $\epsilon > 0$, there is a random sequence $\langle Q_{n,\epsilon}\,;\ n \ge 0\rangle$ of finite subsets of Γ such that*

(i) *$Q_{n,\epsilon}$ is $\mathcal{J}$-measurable;*
(ii) *$\limsup_{n\to\infty} \frac{1}{n} \log |Q_{n,\epsilon}| < \epsilon$ a.s.;*
(iii) *$\limsup_{n\to\infty} \mathbf{P}_{\mathbf{o}}[\Xi_n \in Q_{n,\epsilon}] > 0$.*

Then $\mathcal{J} = \mathcal{I}$ mod 0.

When Γ is replaced by the lamplighter group $\mathfrak{L} \wr \Lambda$, we will apply this to the $\mathfrak{L} \wr \Lambda$-closed σ-field $\mathcal{J} := \sigma(\Phi_\infty) \subseteq \mathcal{I}$ defined by the limiting configuration of lamps. Thus, $Q_{n,\epsilon}$ will be a measurable function of configurations $\phi_\infty \in \mathfrak{L}^\Lambda$.

In Section 3, we will illustrate the use of a more flexible version of the preceding corollary, to wit:

Corollary 2.3. *Let $\langle \Xi_n\,;\ n \ge 0\rangle$ be a random walk on Γ with $H(\Xi_1) < \infty$. Let $\mathcal{I}$ be the associated invariant σ-field and $\mathcal{J} \subseteq \mathcal{I}$ be a Γ-closed sub-σ-field. Suppose that for each $\epsilon > 0$, there is a random sequence $\langle \mathbf{Q}_{n,\epsilon}\,;\ n \ge 0\rangle$ of finite subsets of Γ such that*

(i) *$\mathbf{Q}_{n,\epsilon}$ is $\mathcal{J}$-measurable;*

(ii) *$\limsup_{n\to\infty} \frac{1}{n}\log|\mathbf{Q}_{n,\epsilon}| < \epsilon$ a.s.;*

(iii) *$\limsup_{n\to\infty} \mathbf{P_o}[\exists m \ge n\ \ \Xi_m \in \mathbf{Q}_{n,\epsilon}] > 0$.*

Then $\mathcal{J} = \mathcal{I}$ mod 0.

Proof. Write $p_n^{\mathcal{J}}(x,y) := \mathbf{P}_x[\Xi_n = y \mid \mathcal{J}]$ for the transition probabilities of the Markov chain conditioned on $\mathcal{J}$. We will use the following result of Kaimanovich (2000):

$$\lim_n \frac{1}{n}\log p_n^{\mathcal{J}}(\mathbf{o},\Xi_n) = -\boldsymbol{h}^{\mathcal{J}} \quad \text{a.s.} \tag{2.1}$$

It suffices to show that $\boldsymbol{h}^{\mathcal{J}} = 0$. Suppose that $\boldsymbol{h}^{\mathcal{J}} > 0$ and define the random $\mathcal{J}$-measurable sets

$$S_m := \left\{x \in \Gamma\,;\ p_m^{\mathcal{J}}(\mathbf{o},x) \le \exp(-m\boldsymbol{h}^{\mathcal{J}}/2)\right\}.$$

For $\epsilon > 0$,

$$\mathbf{P_o}\big[\Xi_m \in \mathbf{Q}_{n,\epsilon} \cap S_m \bigm| \mathcal{J}\big] \le |\mathbf{Q}_{n,\epsilon}| \cdot \exp(-m\boldsymbol{h}^{\mathcal{J}}/2).$$

Summing over $m \ge n$, we deduce that for $0 < \epsilon < \boldsymbol{h}^{\mathcal{J}}/2$,

$$\mathbf{P_o}\big[\exists m \ge n\ \ \Xi_m \in \mathbf{Q}_{n,\epsilon} \cap S_m \bigm| \mathcal{J}\big] \le |\mathbf{Q}_{n,\epsilon}| \cdot c\exp(-n\boldsymbol{h}^{\mathcal{J}}/2) \to 0 \text{ a.s.} \tag{2.2}$$

as $n \to \infty$, where $c = c(\boldsymbol{h}^{\mathcal{J}})$ is a constant. Therefore,

$$\mathbf{P_o}\big[\exists m \ge n\ \ \Xi_m \in \mathbf{Q}_{n,\epsilon} \cap S_m\big] \to 0 \quad \text{as } n \to \infty. \tag{2.3}$$

By (2.1), $\mathbf{P_o}\big[\exists m \ge n\ \ \Xi_m \notin S_m\big] \to 0$ as $n \to \infty$. In conjunction with (2.3), this implies that

$$\mathbf{P_o}\big[\exists m \ge n\ \ \Xi_m \in \mathbf{Q}_{n,\epsilon}\big] \to 0 \quad \text{as } n \to \infty,$$

contradicting the hypothesis (iii). ∎

A ***Poisson boundary*** for a random walk on Γ is a quadruple $(\Theta, \mathscr{F}, \nu, \mathbf{b})$, where $(\Theta, \mathscr{F}, \nu)$ is a probability space with $\mathscr{F}$ being countably generated and separating points, and where $\mathbf{b}\colon (\Gamma^{\mathbb{N}}, \mathcal{I}) \to (\Theta, \mathscr{F})$ is a Γ-equivariant measurable map that pushes forward $\mathbf{P_o}$ to ν and such that $\mathbf{b}^{-1}\mathscr{F} = \mathcal{I}$ mod $\mathbf{P_o}$. It is unique up to isomorphism. For more details and background, see Kaimanovich (2000) or Definition 14.28 and Theorem 14.29 of Lyons and Peres (2016).

When we consider random walks on $\mathfrak{L} \wr \Lambda$, we will write $\widehat{X}_n := \Xi_n$ and $\widehat{X}_n =: (\Phi_n, X_n)$. Similarly, write $\langle \widehat{Y}_n ; n \ge 1 \rangle$ for the increments of $\langle \Xi_n ; n \ge 0 \rangle$ on $\mathfrak{L} \wr \Lambda$, i.e., $\widehat{Y}_n := \widehat{X}_{n-1}^{-1} \widehat{X}_n$. Write $\widehat{Y}_n =: (\Psi_n, Y_n)$. Thus, $\langle Y_n ; n \ge 1 \rangle$ are IID elements of Λ, used as increments of the random walk $\langle X_n ; n \ge 0 \rangle$, i.e., $Y_n := X_{n-1}^{-1} X_n$. Note that while $\langle \Psi_n \rangle$ are IID, Ψ_n and Y_n are in general dependent for each n. Also, for $x \in \Lambda$,

$$\Phi_n(x) = \Phi_{n-1}(x) \Psi_n(X_{n-1}^{-1} x) .$$

We generally assume that the support of (the law of) Y_1 generates Λ and, likewise, the support of $\widehat{Y}_1$ generates $\mathfrak{L} \wr \Lambda$.

Let $\operatorname{lit} \phi$ denote the "lit set" of lamps, $\{x \in \Lambda ; \phi(x) \neq \mathsf{id}\}$, of $\phi \in \mathfrak{L}^\Lambda$.

Suppose that $\Phi_\infty := \lim_{n \to \infty} \Phi_n$ exists a.s. For example, this occurs if $\mathbf{E}\big[|\operatorname{lit} \Psi_1|\big] < \infty$ and $\langle X_n \rangle$ is transient (Kaimanovich (1991), Theorem 3.3 or Erschler (2011), proof of Lemma 1.1). In various cases, we will show that $\mathbf{b} \colon \big((\mathfrak{L} \wr \Lambda)^{\mathbb{N}}, \mathcal{I}, \mathbf{P}_{\mathbf{ID}}\big) \to (\mathfrak{L}^\Lambda, \mathscr{F}, \nu)$ is a Poisson boundary, where $\mathscr{F}$ is the product σ-field, ν is the $\mathbf{P}_{\mathbf{ID}}$-law of Φ_∞, and $\mathbf{b}$ takes a sequence to its limiting configuration of lamps; on the set of measure 0 where the limiting configuration does not exist, we define $\mathbf{b}$ to take the value $\mathbf{ID}$ for convenience.

We will use c to stand for a positive constant, whose value can vary from one use to another.

When a group is finitely generated, we use the word metric to define $|x|$ as the distance between x and the identity element.

§3. Proof for the Classical Case.

Here we give a very short proof of the basic conjecture of Vershik and Kaimanovich (1979) concerning random walks on $\mathfrak{L} \wr \mathbb{Z}^d$ for $d \ge 3$ and $\mathfrak{L}$ any nontrivial denumerable group.

Theorem 3.1. *Let $\mathfrak{L}$ be a nontrivial denumerable group. Let $d \ge 3$. Let μ be a probability measure of finite entropy on $\mathfrak{L} \wr \mathbb{Z}^d$ whose support generates $\mathfrak{L} \wr \mathbb{Z}^d$. Suppose that μ is a convex combination of a probability measure on $\mathfrak{L}$ and a probability measure on $\mathbb{Z}^d$ in the sense that μ is concentrated on $\big\{(\delta^s, o) ; s \in \mathfrak{L}\big\} \cup \big\{(\mathbf{ID}, x) ; x \in \mathbb{Z}^d\big\}$. If the projection of μ on $\mathbb{Z}^d$ has mean $\mathbf{0}$ and is finitely supported, then the Poisson boundary of $(\mathfrak{L} \wr \mathbb{Z}^d, \mu)$ is $\mathfrak{L}^{\mathbb{Z}^d}$ endowed with the law of Φ_∞.*

Proof. For $r > 1$, consider the events

$$\mathsf{cut}_r := \big[\exists m \ge 1 \ \ (\forall k < m \ \ |X_k| < r \text{ and } \forall j > m \ \ |X_j| > r)\big] .$$

In the proof of their Proposition 2.1, James and Peres (1996) showed that when the projection of μ is symmetric, $\mathbf{P}_o(\mathsf{cut}_r) \ge c/r$ and $\mathbf{P}_o(\mathsf{cut}_r \cap \mathsf{cut}_{r+j}) \le c/(rj)$. In fact, their proof depends only on estimates of the Green function, and those hold as long as the projection of μ has mean $\mathbf{0}$: see, e.g., Lawler and Limic (2010), Theorem 4.3.1. Thus, the preceding inequalities of James and Peres (1996) hold not only for symmetric μ, but also for those μ whose projection has mean $\mathbf{0}$. The second moment method applied to $\sum_{r=n}^{n^2} \mathbf{1}_{\mathsf{cut}_r}$, just as in the proof of Proposition 2.1 of James and Peres (1996), then yields that $\mathbf{P}_o\big(\bigcup_{r=n}^{n^2} \mathsf{cut}_r\big) \ge c(\log n)^2/(\log n)^2 = c > 0$. Define $\mathbf{Q}_n := \mathbf{Q}_{n,\epsilon}(\Phi_\infty)$ to be the set of (ϕ, x) such that $|x| \le n^2$ and

$$\phi(z) = \begin{cases} \Phi_\infty(z) & \text{if } |z| < |x|, \\ \mathsf{id} & \text{if } |z| \ge |x|. \end{cases}$$

If $r \le n^2$ and the time m witnesses the event cut_r, then $\widehat{X}_m \in \mathbf{Q}_n$. Therefore, $\mathbf{P}_{\mathbf{ID}}\big[\exists m \ge n \;\; \widehat{X}_m \in \mathbf{Q}_n\big] \ge \mathbf{P}_o\big(\bigcup_{r=n}^{n^2} \mathsf{cut}_r\big) \ge c > 0$; since $|\mathbf{Q}_n| \le cn^{2d}$, Corollary 2.3 implies that $\sigma(\Phi_\infty)$ coincides with $\mathcal{I}$ mod 0. ∎

As a further illustration of the usefulness of Corollary 2.3, we prove a conjecture of Sava (2010a). First we remark that the notion of Poisson boundary extends to all Markov chains and criteria such as Corollary 2.3 extend to the setting of transitive Markov chains: see Kaimanovich and Woess (2002) for the required analogues of Theorem 2.1 and Equation (2.1), or see Lyons and Peres (2016), Proposition 14.34 and Theorem 14.35.

Now consider the d-regular tree, $\mathbb{T}_d$, and fix an end ξ of $\mathbb{T}_d$. The group of graph automorphisms that preserve ξ is known as the ***affine group of*** $\mathbb{T}_d$; it acts transitively on the vertex set, $V(\mathbb{T}_d)$. Fix some vertex $o \in V(\mathbb{T}_d)$. There is a horodistance function $d_\xi \colon V(\mathbb{T}_d) \to \mathbb{Z}$ defined by $d_\xi(o) = 0$ and $d_\xi(x) = d_\xi(y) + 1$ when y is the parent of x (the unique neighbor of x in the direction of ξ). The affine group preserves differences of values of the horodistance function.

Let $\mathfrak{L}$ be a finite group. We consider Markov chains $\langle \widehat{X}_n \,;\, n \ge 1\rangle = \langle (\Phi_n, X_n) \,;\, n \ge 1\rangle$ on the state space

$$\mathfrak{L} \wr \mathbb{T}_d := \big\{(\phi, x) \,;\, \phi \in \mathfrak{L}^{V(\mathbb{T}_d)},\ |\mathrm{lit}\,\phi| < \infty,\ x \in V(\mathbb{T}_d)\big\}$$

that only change lamps in a bounded neighborhood of the current location, make only bounded jumps in the base $\mathbb{T}_d$, and whose transition probabilities are invariant under the diagonal action of the affine group. Write R for the maximum distance in $\mathbb{T}_d$ from the current location that one step of the Markov chain can move or at which one step of the Markov chain can change the lamps.

Sava (2010a) conjectured the following Theorem 3.2. She proved that it holds when $\mathbf{E}\big[d_\xi(X_1)\big] \neq 0$ (indeed, with $R < \infty$ replaced by a first moment condition) or when $\langle X_n \rangle$ is a nearest-neighbor random walk that can change lamps only at the location of the lamplighter.

THEOREM 3.2. *Let $\langle \widehat{X}_n \rangle$ be a Markov chain that is invariant under the affine group of $\mathbb{T}_d$ such that $R < \infty$ and the random walk projected to the base $\mathbb{T}_d$ is not constant. Then the Poisson boundary of $\langle \widehat{X}_n \rangle$ is $\mathfrak{L}^{V(\mathbb{T}_d)}$ endowed with the law of Φ_∞.*

Proof. We may assume that $\mathbf{E}\big[d_\xi(X_1)\big] = 0$. Cartwright, Kaimanovich, and Woess (1994) proved that $\langle X_n \rangle$ converges to ξ a.s. Let ξ_n be the ξ-ancestor of o with $d_\xi(\xi_n) = -n$. Define the cone $C_n := \{x\,;\ \xi_n \text{ is an ancestor of } x\}$.

The case when $\langle X_n \rangle$ is a nearest-neighbor random walk is somewhat simpler for our method: To see how it follows from Corollary 2.3, let $\mathbf{Q}_{n,\epsilon}(\phi_\infty)$ be the singleton $\big\{(\phi_n, \xi_n)\big\}$, where $\phi_n(y) = \phi_\infty(y)$ for $y \in C_n$ and $\phi_n(y) = \mathsf{id}$ otherwise. Let $\alpha := \mathbf{P}_x[\forall j \geq 1\ \ X_j \neq x]$; this does not depend on x by transitivity and is positive by transience. With $\mathbf{P_{ID}}$-probability 1, there will be some random smallest time $m \geq n$ such that $X_m = \xi_n$. For this time m, the chance that $X_j \notin C_n$ for all $j > m$ is equal to α by the strong Markov property. Therefore, $\mathbf{P_{ID}}\big[\exists m \geq n\ \ \widehat{X}_m \in \mathbf{Q}_{n,\epsilon}(\Phi_\infty)\big] \geq \alpha > 0$.

For the general case, let τ_n be the first exit time of C_n $(n \geq 0)$. Let K_n be the ball of radius R about ξ_n. By transience, for each $x \in K_0$, there is some time $t_x \geq 0$ such that $\mathbf{P}_x[\forall s \geq t_x\ \ X_s \notin K_0] > 1/2$. Choosing $t_{\max} := \max_{x \in K_0} t_x$ gives a time such that $\mathbf{P}_x[\forall s \geq t_{\max}\ \ X_s \notin K_0] > 1/2$ for all $x \in K_0$. Before time τ_n, a lamp can be changed only in $C_n \cup K_n$. Let A_n be the ball of radius $R(t_{\max} + 1)$ about ξ_n. Then at times in $[\tau_n, \tau_n + t_{\max}]$, the lamplighter must stay in A_n and the changes of lamps must be entirely within A_n. We may define $\mathbf{Q}_{n,\epsilon}(\phi_\infty)$ to consist of those (ϕ_n, x_n) such that $x_n \in A_{Rn}$ and such that

$$\phi_n(y) = \begin{cases} \phi_\infty(y) & \text{if } y \in C_{Rn} \setminus A_{Rn}, \\ \mathsf{id} & \text{if } y \notin C_{Rn} \cup K_{Rn} \cup A_{Rn}. \end{cases}$$

Then $\mathbf{Q}_{n,\epsilon}(\phi_\infty)$ is of bounded size and $\mathbf{P_{ID}}\big[\exists m \geq n\ \ \widehat{X}_m \in \mathbf{Q}_{n,\epsilon}(\Phi_\infty)\big] \geq 1/2$. ∎

§4. Proof for Standard Generators: Theorem 1.1.

In this section, we prove Theorem 1.1. This comes in two parts; one handles base groups Λ that are Liouville for the projected walk (Theorem 4.5), and the other handles the rest (Theorem 4.6). We will write "with high probability" to mean "with probability tending to 1 as $n \to \infty$".

It will be easy to see that the same proofs—indeed, with simplifications—extend to all μ whose support is finite and generates $\mathfrak{L} \wr \Lambda$.

Lemma 4.1. *If $k \le n/3$, then $\sum_{j=0}^{k} \binom{n}{j} \le 2(ne/k)^k$.*

Proof. Since $k! \ge (k/e)^k$ by Stirling's inequality (p. 54 of Feller (1968)), we have $\binom{n}{k} \le (ne/k)^k$. Since $\binom{n}{j+1} \ge 2\binom{n}{j}$ for $j < n/3$, the result follows by comparison with a geometric series. ∎

The following theorem of Shannon is well known and easy to prove via the weak law of large numbers (e.g., Cover and Thomas (2006), Theorem 3.1.2).

Lemma 4.2. *If π is a discrete distribution on a set S with entropy $H(\pi)$ and $Y_n \sim \pi$ are independent, then there are sets $\Lambda_n \subseteq S^n$ $(n \ge 1)$ such that $\lim_{n\to\infty} n^{-1} \log |\Lambda_n| = H(\pi)$ and $\lim_{n\to\infty} \mathbf{P}\big[(Y_1, Y_2, \ldots, Y_n) \in \Lambda_n\big] = 1$.* ∎

Write $\operatorname{dist}(x, y)$ for the distance between x and y in some Cayley graph of Γ and $V_\Gamma(r)$ for the number of points within distance r of the identity, $\mathbf{o}$. Let $B(x, r)$ denote the ball of radius r about x.

Lemma 4.3. *Let $\langle \Xi_n \rangle$ be a μ-walk on a group Γ that satisfies $V_\Gamma(r) \ge cr^d$ for all $r \in \mathbb{N}$. Assume that the support of μ generates Γ and that $\mu(\mathbf{o}) \ge 1/2$. Then $p_t(\mathbf{o}, x) \le ct^{-d/2}$ for all $t \ge 1$ and all $x \in \Gamma$.*

Proof. Let P be the transition matrix for the μ-walk and $\widehat{P}$ denote its transpose, which is the transition matrix for another random walk. Since the support of μ generates Γ and $\mu(\mathbf{o}) > 0$, some power P^j has the property that $(P^j + \widehat{P}^j)/2$ is irreducible. Thus, for such j, we have $\alpha := \min\{p_j(x, y) + \widehat{p}_j(x, y)\,;\ x \sim y\} > 0$. It is well known that $\sum_{x \in S,\ y \notin S} p_j(x, y) = \sum_{x \in S,\ y \notin S} \widehat{p}_j(x, y)$ for all finite $S \subset \Gamma$ (e.g., see Morris and Peres (2005)). Since the sum of these two quantities is at least $\alpha |\{(x, y)\,;\ x \in S,\ y \notin S\}|$, it follows that $\sum_{x \in S,\ y \notin S} p_j(x, y) \ge \alpha |\{(x, y)\,;\ x \in S,\ y \notin S\}|/2$. Now the result follows from the isoperimetric inequality of Coulhon and Saloff-Coste (1993) and Corollary 6.32(i) of Lyons and Peres (2016). ∎

Lemma 4.4. *Let $\langle \Xi_n \rangle$ be a random walk on Γ. Let $0 \le k < m$. Suppose that $\mathcal{M}$ is a random subset of Γ that is measurable with respect to $\langle \Xi_1, \ldots, \Xi_k \rangle$. Then*

$$\mathbf{P}[\Xi_m \in \mathcal{M}] \le \mathbf{E}\big[|\mathcal{M}|\big] \sup_{x \in \Gamma} p_{m-k}(\mathbf{o}, x) \,.$$

Moreover, for every m_0,

$$\mathbf{P}\big[|\{t \ge 1 \,;\; \Xi_t \in \mathcal{M}\}| > k + m_0\big] \le \mathbf{E}\big[|\mathcal{M}|\big] \sum_{j \ge m_0} \sup_{x \in \Gamma} p_j(\mathbf{o}, x) \,.$$

Proof. For each $y \in \Gamma$, we have

$$\mathbf{P}[\Xi_m = y \mid \Xi_1, \ldots, \Xi_k] = p_{m-k}(\mathbf{o}, \Xi_k^{-1} y) \le \sup_{x \in \Gamma} p_{m-k}(\mathbf{o}, x) \,.$$

Summing over $y \in \mathcal{M}$ and then taking expectation gives the first result. This inequality implies that

$$\begin{aligned}
\mathbf{P}\big[|\{t \ge 1 \,;\; \Xi_t \in \mathcal{M}\}| > k + m_0\big] &\le \mathbf{P}\big[|\{t \ge k + m_0 \,;\; \Xi_t \in \mathcal{M}\}| > 0\big] \\
&\le \mathbf{E}\big[|\{t \ge k + m_0 \,;\; \Xi_t \in \mathcal{M}\}|\big] \\
&\le \mathbf{E}\big[|\mathcal{M}|\big] \sum_{j \ge m_0} \sup_{x \in \Gamma} p_j(\mathbf{o}, x) \,. \qquad \blacksquare
\end{aligned}$$

Theorem 4.5. *Let $\mathfrak{L}$ be a nontrivial finite group. Let Λ have at least cubic growth. Let μ be a probability measure of finite entropy on $\mathfrak{L} \wr \Lambda$ whose support generates $\mathfrak{L} \wr \Lambda$ and that is concentrated on $\big\{(\delta^s, o) \,;\; s \in \mathfrak{L}\big\} \cup \big\{(\mathbf{ID}, x) \,;\; x \in \Lambda\big\}$. If the projection of μ on Λ is Liouville, then the Poisson boundary of $(\mathfrak{L} \wr \Lambda, \mu)$ is $\mathfrak{L}^\Lambda$ endowed with the law of Φ_∞.*

A rough sketch of the proof follows. Since $\langle X_n \rangle$ is Liouville, its asymptotic entropy is 0, whence there is some t_0 such that $H(X_{t_0}) < \epsilon t_0$. Lemma 4.2 converts this to a likely set of fewer than $e^{\epsilon n}$ possibilities for $S := \langle X_{j t_0} \,;\; j \le n/t_0 \rangle$. For a large ρ, partially obscure the increments $\langle Y_k \,;\; 1 \le k \le n \rangle$ by replacing those Y_k that satisfy $|Y_k| \le \rho$ by $*$, to mean "unknown"; the resulting sequence $U \in \big(\Lambda \cup \{*\}\big)^n$ has small entropy, so we again have a collection of size $< e^{\epsilon n}$ containing likely values U of the partially obscured increments. In this way, we guess the large jumps and bound the others. Knowing S and U, we define the set $M_i(S, U)$ of possible values for $\{X_j \,;\; (i-1)t_0 < j \le i t_0\}$, and $|M_i(S, U)| \le t_0 V_\Lambda(\rho)^{t_0}$. In most locations $y \in M_i(S, U)$, we have $\Phi_n(y) = \Phi_\infty(y)$, and we can bound the number of possibilities for $\Phi_n(y)$ for the other y.

Proof. Since $H(X_1) < \infty$ and the walk on Λ is Liouville, we have that $H(X_n) = o(n)$.

Let $\epsilon > 0$. Choose t_0 so that $H(X_{t_0}) < \epsilon t_0$. For $n \in t_0 \cdot \mathbb{Z}^+$, set $s_n := n/t_0$. Write $\mathcal{S} := \langle X_{jt_0}\,;\ 1 \le j \le s_n \rangle$. Applying Lemma 4.2 to the t_0-step increments $X_{jt_0}^{-1} X_{(j+1)t_0}$ yields a set $\mathbf{S} = \mathbf{S}_n \subseteq \Lambda^{s_n}$ with $\log |\mathbf{S}_n| < \epsilon s_n \le \epsilon n$ and $\mathbf{P}[\mathcal{S} \in \mathbf{S}_n] \to 1$.

Write

$$u_\rho(x) := \begin{cases} x & \text{if } \operatorname{dist}(o,x) > \rho, \\ * & \text{otherwise.} \end{cases} \tag{4.1}$$

Recall that $\langle Y_k \rangle$ are the increments of the random walk on Λ. Choose ρ so that $H\big(u_\rho(Y_1)\big) < \epsilon$.

Write $\mathcal{U} := \langle u_\rho(Y_k)\,;\ 1 \le k \le n \rangle$. By Lemma 4.2, there is a set $\mathbf{U} = \mathbf{U}_n \subseteq \big(\Lambda \cup \{*\}\big)^n$ with $\log |\mathbf{U}_n| < \epsilon n$ and $\mathbf{P}[\mathcal{U} \in \mathbf{U}_n] \to 1$. For each $U \in \mathbf{U}_n$ and $0 \le j_1 < j_2 \le n$, define the set $L(U, j_1, j_2) \subset \Lambda$ to be the set of possible values of $X_{j_1}^{-1} X_{j_2}$ that are consistent with $\mathcal{U} = U$. That is, let

$$Z_k := \begin{cases} Y_k & \text{if } |Y_k| > \rho, \\ B(o,\rho) & \text{otherwise,} \end{cases}$$

and define

$$L(U, j_1, j_2) := \prod_{j_1 < k \le j_2} Z_k := Z_{j_1+1} Z_{j_1+2} \cdots Z_{j_2}\,. \tag{4.2}$$

When Λ is abelian, this is a ball and $|L(U, j_1, j_2)| \le V_\Lambda\big(\rho(j_2 - j_1)\big)$. More generally,

$$|L(U, j_1, j_2)| \le V_\Lambda(\rho)^{j_2 - j_1} \le V_\Lambda(\rho)^{t_0}\,.$$

Given $S = \langle x_1, x_2, \ldots, x_{s_n} \rangle$ and $1 \le i \le s_n$, write

$$M_i(S, U) := \bigcup_{j=(i-1)t_0+1}^{it_0} x_{i-1} L\big(U, (i-1)t_0, j\big)\,,$$

where $x_0 := o$, for the set of possible values of $\{X_j\,;\ (i-1)t_0 < j \le it_0\}$ that are consistent with $\mathcal{S} = S$ and $\mathcal{U} = U$. Thus, $\bigcup_{i \in [1, s_n]} M_i(S, U)$ is the set of possible values of $\{X_j\,;\ 0 < j \le n\}$ that are consistent with $\mathcal{S} = S$ and $\mathcal{U} = U$; outside this set, every lamp must be the identity at time n. Inside this set, the lamp at time n takes the same value as at time ∞ except possibly at those locations that are visited after time n.

Thus, let $\mathcal{W} := \big\{ i \in [1, s_n]\,;\ \exists m > n\ X_m \in M_i(\mathcal{S}, \mathcal{U}) \big\}$. At the end of the last paragraph, we observed that

$$\Phi_n(z) = \begin{cases} \Phi_\infty(z) & \text{for } z \in \bigcup_{i \in [1,s_n] \setminus \mathcal{W}} M_i(\mathcal{S}, \mathcal{U}), \\ \mathsf{id} & \text{for } z \notin \bigcup_{i \in [1,s_n]} M_i(\mathcal{S}, \mathcal{U}). \end{cases}$$

Write $A_n := \big[|\mathcal{W}| \le n^{3/4}\big]$. It follows from Lemmas 4.4 and 4.3 that

$$\mathbf{E}\big[|\mathcal{W}|\big] \le ct_0 V_\Lambda(\rho)^{t_0} n^{1/2} = o\big(n^{3/4}\big)\,,$$

whence $\mathbf{P}(A_n) \to 1$ as $n \to \infty$ by Markov's inequality.

Let $\phi_\infty \in \mathfrak{L}^\Lambda$ be a possible limiting lamp configuration. For $n \in t_0 \cdot \mathbb{Z}^+$, define $Q_{n,\epsilon}(\phi_\infty)$ to be the set of all (ϕ_n, x) such that there are U, S, and W satisfying

(i) $U = \langle y_1, \ldots, y_n \rangle \in \mathbf{U}$,

(ii) $S = \langle x_1, x_2, \ldots, x_{s_n} \rangle \in \mathbf{S}$ with $x_{s_n} = x$,

(iii) $W \subseteq [1, s_n]$ with $|W| \le n^{3/4}$,

and

(iv) $\phi_n(z) = \begin{cases} \phi_\infty(z) & \text{for } z \in \bigcup_{i \in [1,s_n] \setminus W} M_i(S,U), \\ \mathsf{id} & \text{for } z \notin \bigcup_{i \in [1,s_n]} M_i(S,U). \end{cases}$

We have established that $\mathcal{U}$, $\mathcal{S}$, and $\mathcal{W}$ satisfy (i)–(iv) with high probability as choices for U, S, and W, respectively, when $\phi_\infty = \Phi_\infty$, $\phi_n = \Phi_n$, and $x = X_n$, and thus $\lim_{t_0 \cdot \mathbb{Z}^+ \ni n \to \infty} \mathbf{P}\big[\widehat{X}_n \in Q_{n,\epsilon}(\Phi_\infty)\big] = 1$. To establish the theorem, in light of Corollary 2.2, it suffices to show that $|Q_{n,\epsilon}(\phi_\infty)| < e^{2\epsilon n + o(n)}$ since ϵ was arbitrary.

By definition, the number of choices of $U \in \mathbf{U}$ is at most $e^{\epsilon n}$ and the number of choices of $S \in \mathbf{S}$ is at most $e^{\epsilon n}$. For $n \ge 81$, the number of choices of W is at most $2(n^{1/4}e)^{n^{3/4}} = e^{o(n)}$ by Lemma 4.1. Note that $|M_i(S,U)| \le t_0 V_\Lambda(\rho)^{t_0}$. Thus, given S and W, the number of choices of ϕ_n is at most $|\mathfrak{L}|^{|W| t_0 V_\Lambda(\rho)^{t_0}} \le |\mathfrak{L}|^{t_0 V_\Lambda(\rho)^{t_0} n^{3/4}} = e^{o(n)}$. Therefore, $|Q_{n,\epsilon}(\phi_\infty)| < e^{2\epsilon n + o(n)}$, as desired. ∎

THEOREM 4.6. *Let $\mathfrak{L}$ be a nontrivial denumerable group and Λ be a countable group. Let μ be a probability measure of finite entropy on $\mathfrak{L} \wr \Lambda$ whose support generates $\mathfrak{L} \wr \Lambda$ and that is concentrated on $\big\{(\delta^s, o)\,;\ s \in \mathfrak{L}\big\} \cup \big\{(\mathbf{ID}, x)\,;\ x \in \Lambda\big\}$. If the projection of μ on Λ is non-Liouville, then the Poisson boundary of $(\mathfrak{L} \wr \Lambda, \mu)$ is $\mathfrak{L}^\Lambda$ endowed with the law of Φ_∞.*

A rough sketch of the proof follows. Let $\tau_x := \inf\{n\,;\ X_n = x\}$. Let $\rho_\Lambda(x) := -\log \mathbf{P}_o[\tau_x < \infty]$, the negative log of the probability that the projection of the μ-walk to Λ, started at o, ever visits $x \in \Lambda$. Because the walk on Λ is non-Liouville, its Avez entropy is $\boldsymbol{h}' > 0$. It is known that $n^{-1}\rho_\Lambda(X_n) \to \boldsymbol{h}'$ a.s. Consider the sets $W(r) := \{x \in \Lambda\,;\ \rho_\Lambda(x) \le r\}$, the sizes of which will not concern us. Given $\epsilon > 0$, it is likely that for large n, we have $X_k \in W := W\big(n\boldsymbol{h}'(1+\epsilon)\big)$ for all $k \le n$ and also that $X_m \notin W$ for all $m > n(1+3\epsilon)$. At the same time, there is a reasonable chance that $\Phi_\infty(X_n) \ne \mathsf{id}$. Thus, there is a reasonable chance that Φ_n agrees with Φ_∞ on $W \setminus \{X_{n+1}, \ldots, X_{n(1+\epsilon)}\}$, and it is likely that $\Phi_n(z) = \mathsf{id}$ for all $z \notin W$. Furthermore, there are likely fewer than $n(1+3\epsilon)$ locations $z \in W$ where $\Phi_\infty(z) \ne \mathsf{id}$, whence it is likely, seeing Φ_∞, that there are not many possibilities for where X_n is. Finally, it is likely that $\widehat{Y}_m$ for $n < m \le n(1+3\epsilon)$ belongs to a set of size $e^{c\epsilon n}$ (that does not depend on Φ_∞). From Φ_∞ and these possibilities, we can thus likely deduce Φ_n.

Proof. Let $\boldsymbol{h}' > 0$ be the Avez entropy of the projection of the μ-walk to Λ. By Proposition 6.2 of Benjamini and Peres (1994) in the symmetric case or Blachère, Haïssinsky, and Mathieu (2008) in general, $\lim_{n\to\infty} n^{-1}\rho_\Lambda(X_n) = \boldsymbol{h}'$ a.s.; this result is also proved as Theorem 14.50 of Lyons and Peres (2016). Write $W(r) := \{x \in \Lambda\,;\ \rho_\Lambda(x) \le r\}$. Let $\epsilon \in (0, 1/3)$. Let $W := W\big(n\boldsymbol{h}'(1+\epsilon)\big)$ and $W' := W\big(n(1+3\epsilon)\boldsymbol{h}'(1-\epsilon)\big)$. Since $(1+3\epsilon)(1-\epsilon) - (1+\epsilon) = \epsilon(1-3\epsilon) > 0$, it follows that $W' \supset W$.

Write $\mathcal{U} := \langle \widehat{Y}_m\,;\ n < m \le n + 3\epsilon n\rangle$. By Lemma 4.2, there is a set $\mathbf{U} \subseteq \big(\mathfrak{L} \wr \Lambda\big)^{\lfloor 3\epsilon n\rfloor}$ with $\log|\mathbf{U}| < 6\epsilon n H(\widehat{X}_1)$ and $\mathbf{P}[\mathcal{U} \in \mathbf{U}] \to 1$ as $n \to \infty$.

Let A_n be the event that $X_k \in W$ for all $k \le n$. Since every sequence $\langle r_n\rangle$ with $\lim_{n\to\infty} r_n/n = \boldsymbol{h}'$ has the property that for all sufficiently large n, $\forall k \le n\ \ r_k < n\boldsymbol{h}'(1+\epsilon)$, we have $\lim_{n\to\infty} \mathbf{P}(A_n) = 1$. In addition, at any time, the walk may leave its current location with the lamp not equal to id, after 1 or 2 steps, and never return. Therefore, $\inf_n \mathbf{P}[\Phi_\infty(X_n) \ne \mathsf{id}] > 0$.

Recall that $\operatorname{lit}\phi$ denotes the lit set of lamps, $\{x \in \Lambda\,;\ \phi(x) \ne \mathsf{id}\}$, of $\phi \in \mathfrak{L}^\Lambda$. Let D_n be the event that $X_m \notin W$ for all $m > n + 3\epsilon n$. Since $\mathbf{P}\big[\forall m > n\ \ X_m \notin W\big(n(1+3\epsilon)\boldsymbol{h}'(1-\epsilon)\big)\big] \to 1$ as $n \to \infty$ and $W\big(n(1+3\epsilon)\boldsymbol{h}'(1-\epsilon)\big) = W' \supset W$, it follows that $\mathbf{P}(D_n) \to 1$. On the event D_n, we have that $|W \cap \operatorname{lit}\Phi_\infty| \le n + 3\epsilon n$. Now, the lamp at any $z \in W$ is changed at time $m \in (n, n+3\epsilon n]$ by multiplying by $\Psi_m(X_{m-1}^{-1}z)$, whence the total change from what it was at time n due to the changes in $\mathcal{U}$ is $\prod_{m=n+1}^{n+\lfloor 3\epsilon n\rfloor} \Psi_m(X_{m-1}^{-1}z)$. Therefore, on the event D_n, we have that for every $z \in W$,

$$\Phi_\infty(z) = \Phi_n(z) \prod_{m=n+1}^{n+\lfloor 3\epsilon n\rfloor} \Psi_m(X_{m-1}^{-1}z)\,.$$

Let $\phi_\infty \in \mathfrak{L}^\Lambda$. Define $Q_{n,\epsilon}(\phi_\infty)$ to be the set of all (ϕ_n, x) such that there is U satisfying

(i) $U = \langle(\psi_{n+1}, y_{n+1}), \ldots, (\psi_{n+\lfloor 3\epsilon n\rfloor}, y_{n+\lfloor 3\epsilon n\rfloor})\rangle \in \mathbf{U}$,

(ii) $|W \cap \operatorname{lit}\phi_\infty| \le n + 3\epsilon n$,

(iii) $x \in W \cap \operatorname{lit}\phi_\infty$,

and

(iv) writing $z_m := x\prod_{j=n+1}^m y_j$ for $n \le m \le n + \lfloor 3\epsilon n\rfloor$ and

$$\psi(z) := \prod_{m=n+1}^{n+\lfloor 3\epsilon n\rfloor} \psi_m(z_{m-1}^{-1}z)\,,$$

we have

$$\phi_n(z) = \begin{cases} \phi_\infty(z)\psi(z)^{-1} & \text{for } z \in W, \\ \mathsf{id} & \text{for } z \notin W. \end{cases}$$

Clearly using $U := \mathcal{U}$ satisfies (i) with high probability. We have proved that for $\phi_\infty = \Phi_\infty$ and $x = X_n$, the probability of (iii) is bounded away from 0. In addition, (ii) and (iv) hold on the event D_n, which is likely. Therefore, $\limsup_{n\to\infty} \mathbf{P}\big[\widehat{X}_n \in Q_{n,\epsilon}(\Phi_\infty)\big] > 0$.

By assumption, the number of choices of $U \in \mathbf{U}$ is at most $e^{6\epsilon n H(\widehat{X}_1)}$. The number of choices of x is at most $2n$. Therefore, $|Q_{n,\epsilon}(\phi_\infty)| < e^{6\epsilon n H(\widehat{H}_1)+o(n)}$. This completes the proof. ∎

Define $\zeta_n(x) := -\log \mathbf{P}_o[\tau_x \le n]$. We remark that one may use in the proof the more elementary fact that

$$\lim_{n\to\infty} -n^{-1}\zeta_n(X_n) = \boldsymbol{h}'$$

(Benjamini and Peres (1994), proof of Proposition 6.2) in place of $\lim_{n\to\infty} n^{-1}\rho_\Lambda(X_n) = \boldsymbol{h}'$.

§5. General Generators.

Here we extend the result of Erschler (2011) from finite third moments to finite second moments on $\mathbb{Z}^d$, and from $d \ge 5$ to $d \ge 3$. We also allow infinite lamp groups.

THEOREM 5.1. *Let $\mathfrak{L}$ be a nontrivial finitely generated group and $d \ge 3$. Let μ be a probability measure on $\mathfrak{L} \wr \mathbb{Z}^d$ whose support generates $\mathfrak{L} \wr \mathbb{Z}^d$ with $\sum_x |x|^2 \mu(x) < \infty$. Then the Poisson boundary of $(\mathfrak{L} \wr \mathbb{Z}^d, \mu)$ is $\mathfrak{L}^\Lambda$ endowed with the law of Φ_∞.*

Note that for $x \in \mathbb{Z}^d$, its graph distance $|x|$ to $\mathbf{0} \in \mathbb{Z}^d$ is comparable to the ℓ^2-norm $\|x\| := \langle \mathrm{Cov}(X_1)x, x\rangle^{1/2}$, which we define for $x \in \mathbb{R}^d$. Write $B(r) := \{z\,;\ \|z\| \le r\}$.

We preface the proof of Theorem 5.1 with a sketch. The case when $\mathbf{E}[X_1] \ne \mathbf{0}$ was established by Kaimanovich (2001), so assume that $\mathbf{E}[X_1] = \mathbf{0}$. The main new difficulty compared to our previous proofs is that lamps may be changed at distances arbitrarily far from the lamplighter. Control over this distance is given by the moment assumption. When s is a large constant, for each n there is a high chance that the first n steps of the walk on the base $\mathbb{Z}^d$ do not exit the ball $B(s\sqrt{n})$, nor change any lamps outside the ball $B(2s\sqrt{n})$. In particular, there are only $cn^{d/2}$ possibilities for X_n in this case. There is a tiny, but bounded below, chance that the walk on $\mathbb{Z}^d$ also has the property that it never visits the ball $B(4s\sqrt{n})$ after time $n(1+\epsilon)$; conditional on this event, the chance is very small that any lamp in $B(2s\sqrt{n})$ is changed after time $n(1+\epsilon)$. There is a set of size $e^{c\epsilon n}$ that is likely to contain $\widehat{Y}_{n+1}, \ldots, \widehat{Y}_{n(1+\epsilon)}$. Having guessed $X_n \in B(s\sqrt{n})$, seeing

$\Phi_\infty \restriction B(s\sqrt{n})$, and having changed the lamps therein according to $\widehat{Y}_{n+1}, \ldots, \widehat{Y}_{n(1+\epsilon)}$, we arrive at our guess of $\widehat{X}_n$.

LEMMA 5.2. *Let $d \ge 3$. Consider a random walk $\langle X_n \rangle$ on $\mathbb{Z}^d$ with $\mathbf{E}\big[|X_1|^2\big] < \infty$ and $\mathbf{E}[X_1] = \mathbf{0}$.*

(i) *We have $\lim_{s\to\infty} \inf_n \mathbf{P_0}\big[\forall k \le n \ \ \|X_k\| \le s\sqrt{n}\big] = 1$.*

(ii) *For every $s > 0$,*

$$\lim_{n\to\infty} \inf_{\|x\| \ge 2s\sqrt{n}} \mathbf{P}_x\big[\forall m \ge 0 \ \ \|X_m\| > s\sqrt{n}\big] = 1 - \frac{1}{2^{d-2}}\,.$$

Proof. Part (i) is immediate from Kolmogorov's maximal inequality (Theorem 2.5.2 in Durrett (2010)).

To prove part (ii), we let $A := \mathrm{Cov}(X_1)^{-1/2}$ and define $Y_m := s^{-1} A X_m$. Let $|\cdot|_2$ be the standard Euclidean norm and $B_2(r)$ be the associated closed ball of radius r about the origin. Then (ii) can be rewritten in the form

$$\lim_{n\to\infty} \sup_{|y|_2 \ge 2\sqrt{n}} \mathbf{P}_y\big[\exists m \ge 0 \ \ |Y_m|_2 \le \sqrt{n}\big] = 2^{d-2}\,.$$

First recall that if standard Brownian in $\mathbb{R}^d$ starts at z with $|z|_2 = 2$, then the probability that it ever visits the ball $B_2(\alpha)$ of radius $\alpha < 2$ is $(\alpha/2)^{d-2}$; see, e.g., Mörters and Peres (2010), Corollary 3.19. Given $\epsilon > 0$, we can select $T = T(\epsilon)$ so that the probability this visit occurs before time T is at least $(\alpha/2)^{d-2} - \epsilon$; taking $\alpha := 1 - \epsilon$, we deduce from the d-dimensional Donsker invariance principle (see, e.g., Whitt (2002), Theorem 4.3.5),

$$\lim_{n\to\infty} \sup_{|y|_2 \ge 2\sqrt{n}} \mathbf{P}_y\big[\exists m \in [0, nT] \ \ |Y_m|_2 \le \sqrt{n}\big] \ge \big((1-\epsilon)/2\big)^{d-2} - \epsilon\,,$$

and this gives the lower bound in (ii).

In dimension three, a matching upper bound follows from the asymptotic relation

$$g(0, y) = \big(c_3 + o(1)\big)|y|_2^{-1}$$

for the Green function of $\langle Y_j \rangle$ (see Spitzer (1976), Proposition P26.1), where c_3 is a positive constant. Indeed, if τ is the hitting time of the ball $B_2(\sqrt{n})$ by $\langle Y_j \rangle$ (which may be infinite), then the optional stopping theorem (e.g., Durrett (2010), Theorem 5.7.4) for the bounded martingale $\langle g(0, Y_{\tau\wedge j})\,;\ j \ge 0 \rangle$ yields

$$g(0, y) \ge \mathbf{P}_y[\tau < \infty] \cdot \min_{y_1 \in B_2(\sqrt{n}\,)} g(0, y_1)\,.$$

It follows that

$$\big(c_3+o(1)\big)|y|_2^{-1} \ge \mathbf{P}_y[\tau<\infty]\big(c_3+o(1)\big)n^{-1/2}\,;$$

the two occurrences of $o(1)$ do not necessarily denote the same function. Since $|y|_2 \ge 2\sqrt{n}$, we conclude that $\mathbf{P}_y[\tau<\infty] \le 1/2 + o(1)$.

It remains to prove the upper bound in (ii) for dimensions $d>3$. Given $\epsilon>0$, let $T := \lceil d\epsilon^{-4}\rceil$. Another application of Donsker's theorem yields

$$\lim_{n\to\infty} \sup_{|y|_2\ge 2\sqrt{n}} \mathbf{P}_y\big[\exists m\in[0,nT]\ \ |Y_m|_2 \le \sqrt{n}\,\big] \le (1/2+\epsilon)^{d-2}\,.$$

By the central limit theorem, for every y in $\mathbb{R}^d$ and sufficiently large $n>1$,

$$\mathbf{P}_y\big[|Y_{nT}|_2 \le \epsilon\sqrt{nT}\,\big] \le C_d\epsilon^d\,.$$

If $z\in\mathbb{R}^d$ satisfies $|z|_2 > \epsilon\sqrt{nT} \ge \sqrt{nd}/\epsilon$, then one of the coordinate projections of z must have absolute value greater than $\sqrt{n}/\epsilon$; projecting to a three-dimensional space containing that coordinate, we infer (from optional stopping in three dimensions) that as $n\to\infty$,

$$\mathbf{P}_z\big[\exists m\in[0,\infty)\ \ |Y_m|_2 \le \sqrt{n}\,\big] \le \big(1+o(1)\big)\epsilon\,.$$

By considering whether $|Y_{nT}|_2 \le \sqrt{nd}/\epsilon$, we conclude that for some constant C_d,

$$\sup_{|y|_2\ge 2\sqrt{n}} \mathbf{P}_y\big[\exists m\ge 0\ \ |Y_m|_2 \le \sqrt{n}\,\big] \le (1/2+\epsilon)^{d-2} + C_d\epsilon^d + \big(1+o(1)\big)\epsilon\,. \qquad \blacksquare$$

Let rad denote radius of a subset of Λ, meaning the maximum distance in the word metric of any of its elements from the identity $o\in\Lambda$. Although we will apply the following lemma only for $\Lambda=\mathbb{Z}^d$, we state it in general as it may find other uses. This lemma controls the changes of lamps far from the projection of the walk on the base group.

LEMMA 5.3. *Let $\mathfrak{L}$ and Λ be nontrivial finitely generated groups. Let $\langle(\Psi_k,Y_k)\,;\ k\ge 1\rangle$ be the increments of a μ-walk $\langle\widehat{X}_n\,;\ n\ge 0\rangle$ on $\mathfrak{L}\wr\Lambda$ such that $\mathbf{P}[Y_1=o]\ge 1/2$. Suppose that $V_\Lambda(r)/r^d$ is bounded above and below by positive finite constants for some $d\ge 3$ and that $\mathbf{E}\big[(\operatorname{rad}\operatorname{lit}\Psi_1)^2\big]<\infty$. Then for some constant c_μ, we have for every $a>0$ that*

$$\sum_{k\ge 1}\mathbf{P}\big[\operatorname{rad}\operatorname{lit}\Psi_k > a\cdot|X_{k-1}|\big] \le c_\mu a^{-2}\,\mathbf{E}\big[(\operatorname{rad}\operatorname{lit}\Psi_1)^2\big]\,.$$

Proof. Let R be a random variable independent of $\langle X_k\rangle$ that has the same distribution as $a^{-1}\operatorname{rad}\operatorname{lit}\Psi_1$. Since Ψ_k has the same law as Ψ_1 and is independent of X_{k-1}, we have

$$\sum_{k\ge 1}\mathbf{P}\big[\operatorname{rad}\operatorname{lit}\Psi_k > a\cdot|X_{k-1}|\big] = \sum_{k\ge 1}\mathbf{P}\big[R>|X_{k-1}|\big] = \mathbf{E}\Big[\sum_{k\ge 1}\mathbf{1}_{[R>|X_{k-1}|]}\Big]\,.$$

The idea now is that for $k > R^2$, we control the chance that $|X_{k-1}| < R$ by using Lemma 4.3, summing over the relevant possible values of X_{k-1}. Thus,

$$\begin{aligned}
\mathbf{E}\Big[\sum_{k\geq 1} \mathbf{1}_{[R>|X_{k-1}|]}\Big] &\leq \mathbf{E}\Big[R^2 + \mathbf{E}\big[\sum_{k>R^2} \mathbf{1}_{[R>|X_{k-1}|]} \mid R\big]\Big] \\
&\leq \mathbf{E}\Big[R^2 + \sum_{k>R^2} V_\Lambda(R) c k^{-d/2}\Big] \\
&\leq \mathbf{E}\Big[R^2 + cR^d (R^2)^{1-d/2}\Big] = c\,\mathbf{E}[R^2]\,. \qquad \blacksquare
\end{aligned}$$

Proof of Theorem 5.1. Our assumption is that $\mathbf{E}\big[|\widehat{X}_1|^2\big] < \infty$. However, all we will use of this moment condition is weaker, namely, that $H(\widehat{X}_1) < \infty$, that $\mathbf{E}\big[|X_1|^2\big] < \infty$, and that $\mathbf{E}\big[(\operatorname{rad}\operatorname{lit}\Phi_1)^2\big] < \infty$. The first is a well-known consequence of the weaker assumption $\mathbf{E}\big[|\widehat{X}_1|\big] < \infty$; the latter two follow from $|(\phi, x)| \geq \max\big\{|x|, \operatorname{rad}\operatorname{lit}\phi\big\} + |\operatorname{lit}\phi|$.

The case $\mathbf{E}[X_1] \neq \mathbf{0}$ was done by Kaimanovich (2001), so assume that $\mathbf{E}[X_1] = \mathbf{0}$.

Let $\epsilon \in (0,1)$. Choose s so large that

$$\inf_n \mathbf{P_0}\big[\forall k \leq n \ \ \|X_k\| \leq s\sqrt{n}\,\big] > \frac{1}{2} + \frac{\mathbf{E}\big[(\operatorname{rad}\operatorname{lit}\Phi_1)^2\big]}{s^2}\,;$$

such an s exists by Lemma 5.2. We will define random sets $Q_{n,\epsilon}$ that are Φ_∞-measurable in order to apply Corollary 2.2.

Abbreviate $\Lambda := \mathbb{Z}^d$.

Write $\mathcal{U} := \langle \widehat{Y}_m\,;\ n < m \leq n + \epsilon n\rangle$. By Lemma 4.2, there is a set $\mathbf{U} \subseteq \big(\mathfrak{L} \wr \Lambda\big)^{\lfloor \epsilon n \rfloor}$ with $\log|\mathbf{U}| < 2\epsilon n H(\widehat{X}_1)$ and $\mathbf{P}[\mathcal{U} \in \mathbf{U}] \to 1$.

We wish to define a set $Q_{n,\epsilon}(\Phi_\infty)$ that will contain $\widehat{X}_n$ with reasonable probability and that will have small exponential growth. We will consider the possible increments $\mathcal{U}$ and the possible values of $X_n \in B(s\sqrt{n})$. Given such possible values, we guess values for Φ_n from the ones we see, Φ_∞, by correcting by the changes caused by $\mathcal{U}$. Namely, the lamp at some $z \in B(2s\sqrt{n})$ is changed at time $m \in (n, n+\epsilon n]$ by multiplying by $\Psi_m(z - X_{m-1})$, whence the total change from what it was at time n due to the changes in $\mathcal{U}$ is $\prod_{m=n+1}^{n+\lfloor \epsilon n \rfloor} \Psi_m(z - X_{m-1})$. Provided that the lamps in $B(2s\sqrt{n})$ are not changed after time $n + \epsilon n$, we may multiply $\Phi_\infty(z)$ by the inverse of this product to guess $\Phi_n(z)$.

Thus, we proceed as follows. Let $\phi_\infty \in \mathfrak{L}^\Lambda$. Define $Q_{n,\epsilon}(\phi_\infty)$ to be the set of all (ϕ_n, x) such that there is some $U = \langle(\psi_{n+1}, y_{n+1}), \ldots, (\psi_{n+\lfloor\epsilon n\rfloor}, y_{n+\lfloor\epsilon n\rfloor})\rangle \in \mathbf{U}$ and some $x \in B(s\sqrt{n})$, such that, writing $z_m := x + \sum_{j=n+1}^m y_j$ for $n \leq m \leq n + \epsilon n$ and

$$\psi(z) := \prod_{m=n+1}^{n+\lfloor\epsilon n\rfloor} \psi_m(z - z_{m-1})\,,$$

we have

$$\phi_n(z) = \begin{cases} \phi_\infty(z)\psi(z)^{-1} & \text{for } z \in B(2s\sqrt{n}), \\ \mathsf{id} & \text{for } z \notin B(2s\sqrt{n}). \end{cases}$$

By assumption, the number of choices of $U \in \mathbf{U}$ is at most $e^{2\epsilon n H(\widehat{X}_1)}$. The number of choices of x is at most $cn^{d/2}$. Therefore, $|Q_{n,\epsilon}(\phi_\infty)| < e^{2\epsilon n H(\widehat{X}_1)+o(n)}$.

We will prove that $\limsup_{n\to\infty} \mathbf{P}\big[\widehat{X}_n \in Q_{n,\epsilon}(\Phi_\infty)\big] > 0$.

Let A_n be the event that $\|X_k\| \le s\sqrt{n}$ for all $k \le n$. Let C_n be the event that $\operatorname{rad\,lit} \Psi_k > s\sqrt{n}$ for some $k \le n$. Then

$$\mathbf{P}[\operatorname{rad\,lit} \Psi_k > s\sqrt{n}] \le \frac{\mathbf{E}\big[(\operatorname{rad\,lit} \Psi_k)^2\big]}{s^2 n} = \frac{\mathbf{E}\big[(\operatorname{rad\,lit} \Phi_1)^2\big]}{s^2 n}$$

by Chebyshev's inequality, whence $\mathbf{P}(C_n) \le \mathbf{E}\big[(\operatorname{rad\,lit} \Phi_1)^2\big]/s^2$ by a union bound. Let D_n be the event that $\Phi_n(y) = \mathsf{id}$ for all $y \notin B(2s\sqrt{n})$. Then $A_n \setminus D_n \subseteq C_n$, whence $\mathbf{P}(A_n \cap D_n) = \mathbf{P}(A_n) - \mathbf{P}(A_n \setminus D_n) \ge \mathbf{P}(A_n) - \mathbf{P}(C_n) > 1/2$ by choice of s.

Let E_n be the event that $\|X_{n+\lfloor \epsilon n \rfloor} - X_n\| > 5s\sqrt{n}$. Then $\liminf_{n\to\infty} \mathbf{P}(E_n) > 0$ and E_n is independent of $A_n \cap D_n$; on the event $A_n \cap D_n \cap E_n$, we have $\|X_{n+\lfloor \epsilon n \rfloor}\| > 4s\sqrt{n}$. Let F_n be the event that for all $m > n + \epsilon n$, we have $\|X_m\| > 2s\sqrt{n}$. By Lemma 5.2, $\lim_{n\to\infty} \mathbf{P}(F_n \mid A_n D_n E_n) = 1 - 1/2^{d-2}$.

Let G_n be the event that at no time after $n + \lfloor \epsilon n \rfloor$ does the walk change a lamp in $B(2s\sqrt{n})$. Then $A_n D_n E_n F_n \setminus G_n$ is contained in the event that for some $m > n + \epsilon n$, we have $\operatorname{rad\,lit} \Psi_m > \|X_{m-1}\|/2$, which by the Borel-Cantelli lemma and Lemma 5.3, has probability tending to 0 as $n \to \infty$. Therefore, $\liminf_{n\to\infty} \mathbf{P}(A_n D_n E_n F_n G_n) > 0$.

On the event $A_n D_n E_n F_n G_n$, we have that for every $z \in B(2s\sqrt{n})$,

$$\Phi_\infty(z) = \Phi_n(z) \prod_{m=n+1}^{n+\lfloor \epsilon n \rfloor} \Psi_m(z - X_{m-1}),$$

as desired. ∎

Recall that our proof of Theorem 5.1 did not use the full strength of the hypothesis $\mathbf{E}\big[|\widehat{X}_1|^2\big] < \infty$, but only the weaker hypotheses that $H(\widehat{X}_1) < \infty$, that $\mathbf{E}\big[|X_1|^2\big] < \infty$, and that $\mathbf{E}\big[(\operatorname{rad\,lit} \Phi_1)^2\big] < \infty$. This last assumption cannot be weakened to finiteness of a smaller moment, even if $\langle X_n \rangle$ is simple random walk on $\mathbb{Z}^3$ and $\mathfrak{L} = \mathbb{Z}_2$. To see this, we adapt Kaimanovich (1983), Proposition 1.1, which gave an example of a μ-walk on $\mathbb{Z}_2 \wr \mathbb{Z}$ that yielded a nontrivial Poisson boundary but with no limiting configuration of lamps a.s. Indeed, suppose that $\widehat{X}_1$ has the following distribution: With probability 1/2, $\Psi_1 = \mathbf{0}$ and X_1 is a step of simple random walk on $\mathbb{Z}^3$, while for each $n \ge 1$, with probability c_0/n^3,

$\Psi_1 = \mathbf{1}_{B(n)}$ and $X_1 = \mathbf{0}$, where $c_0 := 1/(2\zeta(3))$ is a normalizing constant. We still have $H(\widehat{X}_1) < \infty$, while $\mathbf{E}\big[(\operatorname{rad}\operatorname{lit}\Phi_1)^a\big] < \infty$ iff $a < 2$. We claim that while Φ_∞ does not exist a.s. for this walk, the Poisson boundary is nontrivial. To see this, condition on the walk in the base, $\langle X_n \rangle$. If $X_n = X_{n+1}$, then the chance that at time $n+1$ the lamp changes at the origin is of order $1/\big(1 + \|X_n\|^2\big)$, independently of all other steps of the walk. Now $\sum_n \big(1+\|X_n\|^2\big)^{-1} = \infty$ a.s. by the law of the iterated logarithm, whence the Borel-Cantelli lemma yields infinitely many changes of the lamp at the origin a.s. On the other hand, the difference between the lamp at the origin and the lamp at $(1, 0, 0)$ changes only finitely many times a.s., again by the Borel-Cantelli lemma, since if $X_n = X_{n+1}$, then the chance that at time $n+1$ this difference changes is of order $1/\big(1 + \|X_n\|^3\big)$, independently of all other steps of the walk, and $\sum_n \big(1 + \|X_n\|^3\big)^{-1} < \infty$ a.s. by Dvoretzky and Erdös (1951). Therefore, the Poisson boundary is nontrivial. On the other hand, if μ has a finite first moment and projects to a transient random walk on $\mathbb{Z}^d$, then a limiting lamp configuration exists; see Lemma 1.1 of Erschler (2011). This general case is still open: is the harmonic measure on the limiting lamp configuration equal to the Poisson boundary? We remark that Erschler (2011) shows that the Poisson boundary can be nontrivial even for some random walks where no combination of lamps stabilizes.

§6. Metabelian Groups.

As Erschler (2011) noted following Vershik (2000), free metabelian groups are sufficiently similar to lamplighter groups on $\mathbb{Z}^d$ that similar results on their Poisson boundaries carry over. A group F is ***metabelian*** if F'' is trivial, where prime indicates commutator subgroup. Those of the form $\mathbb{F}_d/\mathbb{F}_d''$ are called ***free metabelian groups***, where $\mathbb{F}_d$ is the free group on d generators. More generally, consider groups of the form $\mathbb{F}_d/H'$, where H is a normal subgroup of $\mathbb{F}_d$. As explained by Erschler (2004b), with more details given by Vershik and Dobrynin (2005), the groups $\mathbb{F}_d/H'$ are isomorphic to groups of finite configurations on $\Lambda := \mathbb{F}_d/H$ as follows.

Let G be the right Cayley graph of $\mathbb{F}_d/H$ corresponding to the free generators of $\mathbb{F}_d$. Orient each edge of G so as to form the group $C_1(G) = C_1(G, \mathbb{Z})$ of 1-chains. For each $x \in \Lambda$, fix a finite path $\langle e_1, \ldots, e_k \rangle$ of edges from $o \in \Lambda$ to x. To this path associate the 1-chain $\theta_x := \sum_{j=1}^k \pm e_j$, where we choose the plus sign iff e_j is oriented in the direction from o to x along the path. For simplicity, we choose $\theta_o := 0$. Let $Z_1(G)$ denote the space of cycles in $C_1(G)$. (As there are no 2-cells, this is the same as $H_1(G, \mathbb{Z})$.) Note that H is the fundamental group of G, and its abelianization, H/H', is canonically isomorphic to $Z_1(G)$, meaning that the homomorphism $\varphi \colon \mathbb{F}_d \to C_1(G)$ defined by $\varphi(a) := \theta_{aH}$ for generators a of $\mathbb{F}_d$ has kernel H' and $\varphi(H) = Z_1(G)$. Now Λ acts on G by translation from the left, and so also acts on $C_1(G)$, which we denote by $(x, f) \mapsto T_x f$. Define $\widetilde{\Lambda}$ to be the subset $\{\theta_x + f\,;\ x \in \Lambda,\ f \in Z_1(G)\} \subset C_1(G)$; this set is clearly independent of the choices of the chains θ_x. In addition, the map $\theta_x + f \mapsto x$ from $\widetilde{\Lambda} \to \Lambda$ is well defined. Define a multiplication on $\widetilde{\Lambda}$ by

$$(\theta_x + f)(\theta_y + g) := \theta_x + T_x\theta_y + f + T_x g\,.$$

Then $\widetilde{\Lambda}$ is closed under this multiplication because $\theta_x + T_x\theta_y$ corresponds to a path from o to xy. It is easy to check that $\widetilde{\Lambda}$ is a group with identity element 0. Indeed, $\widetilde{\Lambda}$ is canonically isomorphic to $\mathbb{F}_d/H'$ via the homomorphism φ defined above.

A random walk $\langle \theta_{X_n} + \Phi_n \rangle$ on $\widetilde{\Lambda}$ yields a.s. an edgewise limiting configuration in the space of cochains, $C^1(G)$, under weak conditions: As Erschler (2011) proved, it suffices that the walk on $\widetilde{\Lambda}$ has finite first moment and projects to a transient random walk on Λ. Under similar conditions as our previous theorems and with similar proofs, the subset of possible limits, together with harmonic measures, is the Poisson boundary. For example, if $\Lambda = \mathbb{F}_d/H$ has at least cubic growth, then this holds for every finitely supported walk on $\widetilde{\Lambda}$. In the case of free metabelian groups with $d \geq 3$, it holds for every walk having finite second moment. Erschler (2011) had proved this for free metabelian groups with $d \geq 5$ and μ having finite third moment.

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