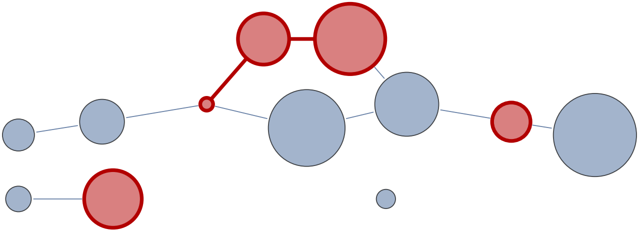
Random Orderings and Unique Ergodicity of Automorphism Groups

Russell Lyons

Indiana University, Bloomington

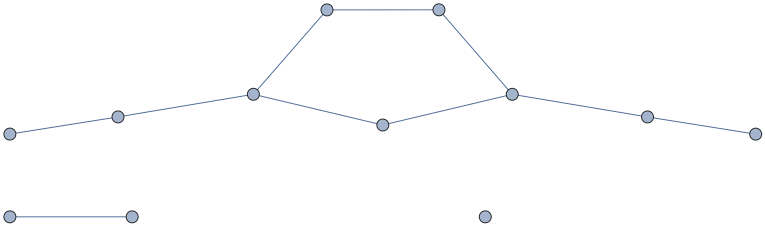
Joint work with Omer Angel and Alexander Kechris, JEMS, 2014

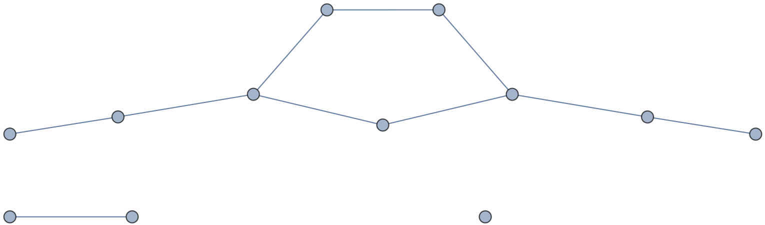


TAMU, College Station, TX 2015

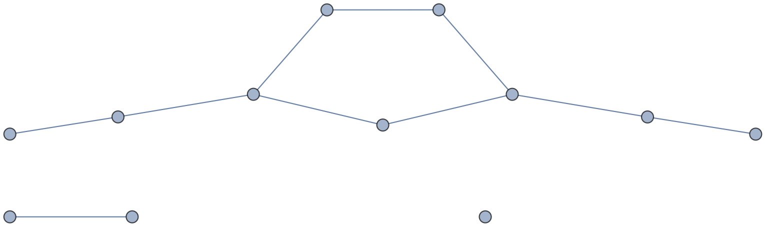
Talk will have two parts:

- a concrete part about finite graphs
- an abstract part about automorphism groups

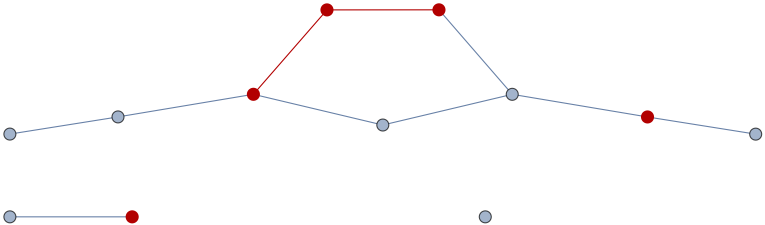




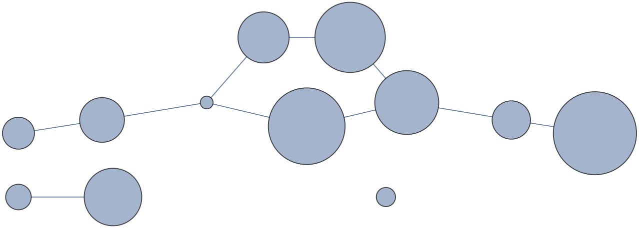
Special vertices?

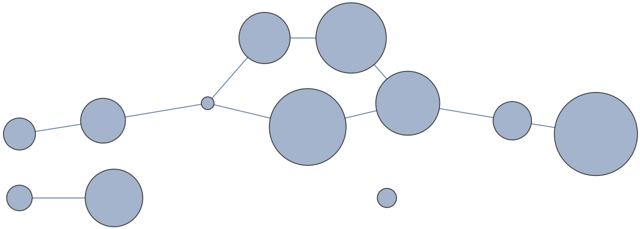


Special vertices? Preserved by isomorphism



Special vertices? Preserved by isomorphism and induced subgraphs?



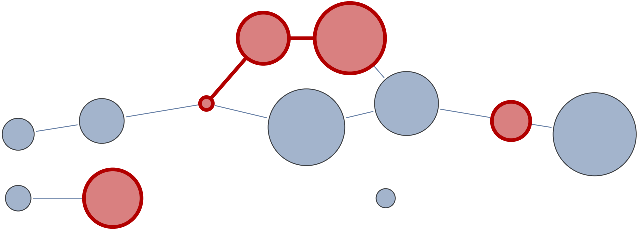


More precisely, we allow linear (total) orderings of $V(G)$ at random:

$$G = (V, E) \mapsto \mu_G \text{ on } |V|! \text{ orderings of } V$$

that are consistent:

- $\phi\colon G \to G'$ isomorphism $\Rightarrow \phi_* \mu_G = \mu_{G'}$



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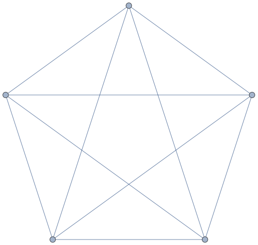

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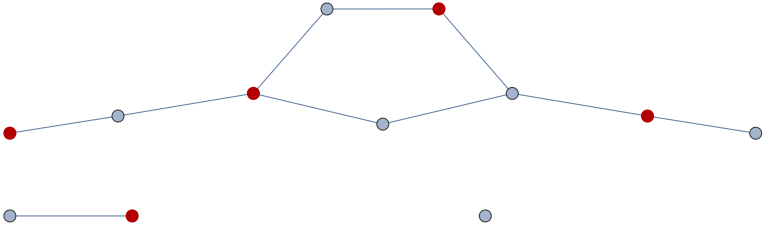

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The uniform ordering is always consistent. Is there any other?

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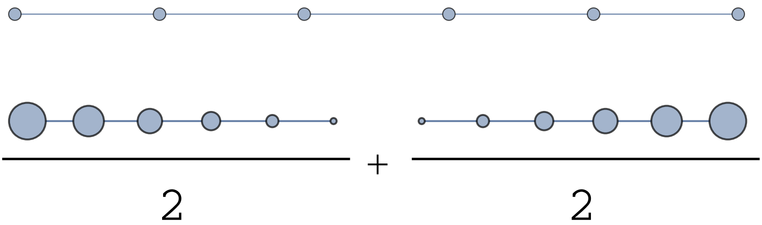

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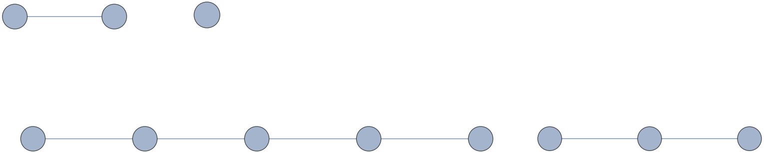

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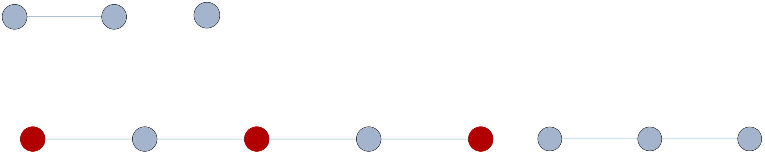

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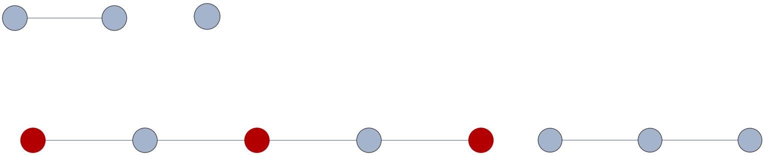

But there is a non-uniform consistent random ordering in this universe.

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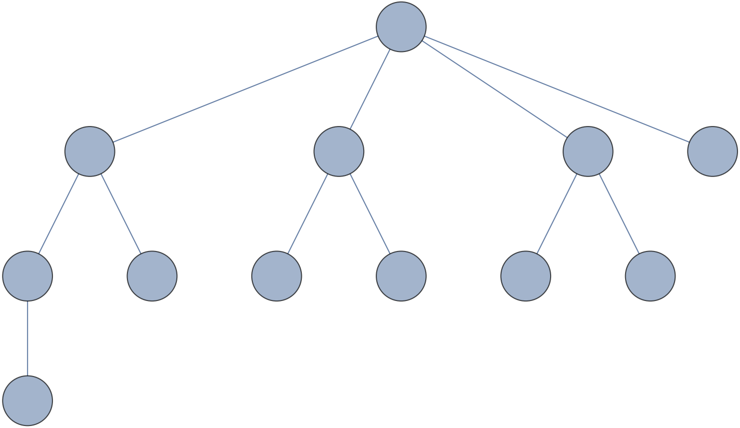

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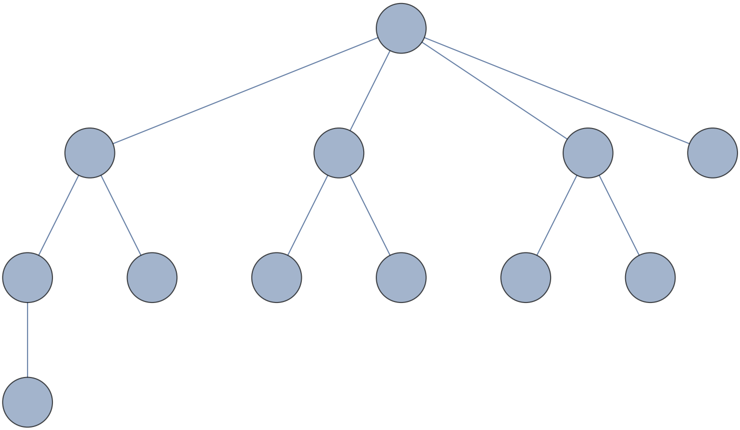

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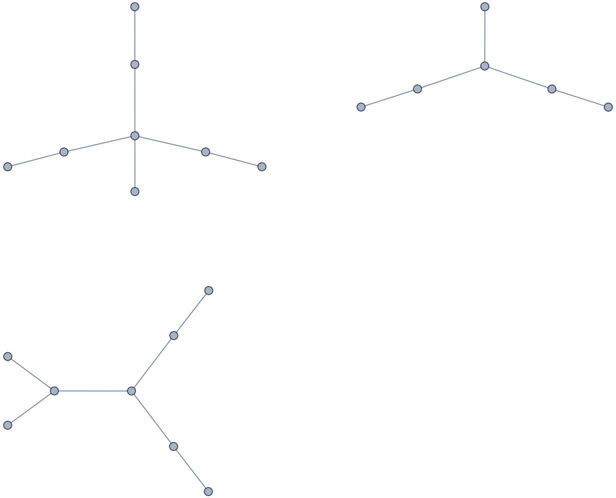

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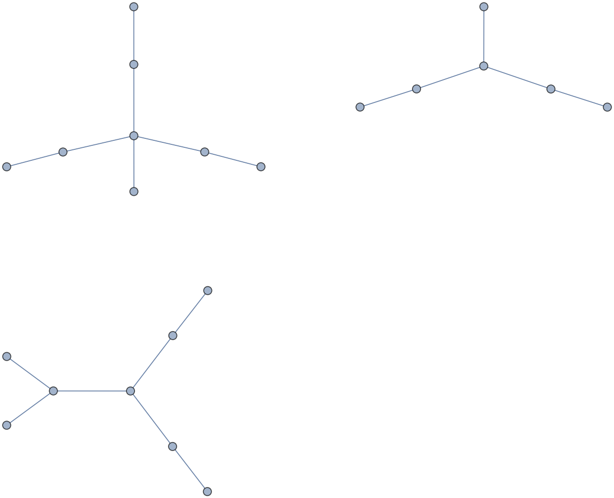

Every consistent random ordering in this universe is uniform: Balister-Bollobás-Janson (2015+).

Other structures, such as hypergraphs or metric spaces?

The answer for the universe of all finite graphs:

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We prove that graphs have only the uniform ordering as a consistent ordering.

The application:

This implies that the automorphism group of “the random graph” (an infinite graph) is uniquely ergodic, i.e., every minimal action has a unique invariant probability measure.

A quantitative version:

Theorem (Angel-Kechris-L.)

Suppose $G \mapsto \mu_G$ is a consistent ordering on graphs of size $\leq n$. Then for every H of size $k \leq n$ and for every ordering $<_H$ on $V(H)$,

$$\left| \mu_H(<_H) - \frac{1}{k!} \right| \leq C(k)\sqrt{\frac{\log n}{n}}\,.$$

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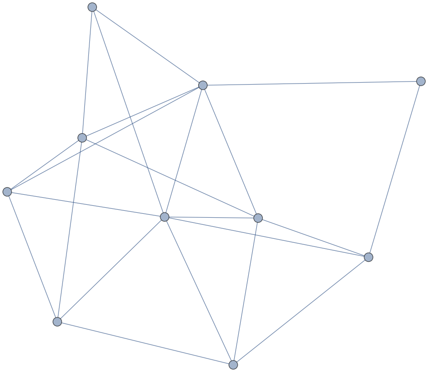

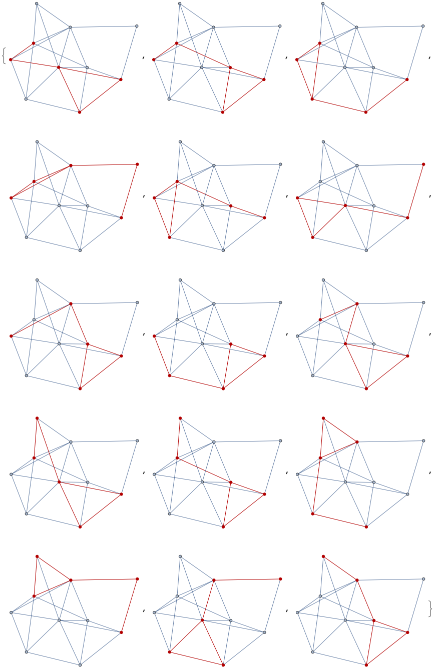

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We don't know how sharp our upper bound is. We have a lower bound that there is a consistent assignment $G \mapsto \mu_G$ on graphs of size $\leq n$ such that for all $k \in [3, n]$ there is some H of size k and some $<_H$ with

$$\left|\mu_H(<_H) - \frac{1}{k!}\right| \geq \frac{c(k)}{n}.$$

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What can we say about Γ-invariant Borel probability measures on X? A measure μ on X is Γ-invariant if $\mu(\gamma A) = \mu(A)$ for all Borel $A \subseteq X$ and all $\gamma \in \Gamma$.

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Suppose from now on that Γ is amenable. When is an invariant measure also unique? If X decomposes into invariant compact pieces, then it will not be unique. What if X is minimal, i.e., *every* Γ-orbit is dense?

Assumptions:

- X is a Γ-flow;
- Γ is amenable;
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- In fact, every countable infinite Γ has a minimal flow that has more than one invariant measure, i.e., is not uniquely ergodic.

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We need consider only one Γ-flow, the universal minimal Γ-flow, $M(\Gamma)$. Every minimal Γ-flow is a Γ-factor of $M(\Gamma)$ (i.e., there is a continuous surjection $\phi \colon M(\Gamma) \to X$ that commutes with the Γ-actions).

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We need consider only one Γ-flow, the universal minimal Γ-flow, $M(\Gamma)$. Every minimal Γ-flow is a Γ-factor of $M(\Gamma)$ (i.e., there is a continuous surjection $\phi\colon M(\Gamma) \to X$ that commutes with the Γ-actions). If Γ is compact, then $M(\Gamma) = \Gamma$.

$M(\Gamma)$ is uniquely ergodic iff *every* minimal Γ-flow is uniquely ergodic, in which case we call Γ itself uniquely ergodic.

- X is a Γ-flow;
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[These universal flows are not metrizable when Γ is locally compact but not compact.]

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Kechris-Pestov-Todorcevic (2005) gave many more examples of universal minimal flows for closed subgroups of S_∞, showing how this is related to model theory and Ramsey theory.

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This does not immediately give that $\mathrm{Aut}(\mathcal{R})$ is uniquely ergodic. In fact, that translates precisely to our theorem on uniqueness of consistent random orderings of finite graphs.

This example and others we give are the next examples of non-compact non-extremely-amenable uniquely ergodic groups.

Supplementary Material

For metric spaces, it is also true that only the uniform ordering is consistent.

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For graphs, the Ramsey property is the following theorem of Nešetřil and Rödl (1977): Consider only ordered graphs and $q \geq 1$. Suppose that K is an induced subgraph of H. Then there is a graph G containing an induced subgraph isomorphic to H such that for any coloring
$c\colon \binom{G}{K} \to \{1, \ldots, q\}$, there is $H' \in \binom{G}{H}$ such that $c\restriction\binom{H'}{K}$ is constant.

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(When all graphs are empty, this is the classical theorem of Ramsey: If $k < h$, then there is g sufficiently large such that for any coloring $c\colon \binom{[g]}{k} \to \{1, \ldots, q\}$, there is $H' \in \binom{[g]}{h}$ such that $c\restriction \binom{H'}{k}$ is constant.)