

## Bifurcation Diagram and Pattern Formation of Phase Slip Centers in Superconducting Wires Driven with Electric Currents

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We provide here new insights into the classical problem of a one-dimensional superconducting wire exposed to an applied electric current using the time-dependent Ginzburg-Landau model. The most striking feature of this system is the well-known appearance of oscillatory solutions exhibiting phase slip centers (PSC's) where the order parameter vanishes. Retaining temperature and applied current as parameters, we present a simple yet definitive explanation of the mechanism within this nonlinear model that leads to the PSC phenomenon and we establish where in parameter space these oscillatory solutions can be found. One of the most interesting features of the analysis is the evident collision of real eigenvalues of the associated  $PT$ -symmetric linearization, leading as it does to the emergence of complex elements of the spectrum.

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We consider a finite superconducting wire subjected to an electric current that is fed into one of its ends, creating a voltage difference across the wire. This is a canonical problem that has received considerable attention over the past 40 years because it involves a resistive state with the coexistence of a normal and superconducting current, yet the theoretical mechanisms at play in this physical setup have remained poorly understood. One such mechanism associated with this state is the intriguing formation of phase slip centers. These are points in space-time where the order parameter in the time-dependent Ginzburg-Landau equation (TDGL) vanishes and they can be thought of as vortices in space-time, [1]. The appearance of phase slip centers is related to oscillations found numerically through the emergence of time-periodic solutions, but another type of behavior found in the resistive state involves stationary solutions of the TDGL [2]. In this case the gauge invariant quantities reach a steady state. The phase slip centers and the associated oscillations can be indirectly observed experimentally via the appearance of steps in I–V curves ([2–4]), though we will expose hysteresis phenomena found in certain regions of parameter space that cast some doubt on previous assertions as to the equivalence of phase slip center (PSC) appearance and jumps in I–V curves.

Our goal then is to revisit this classical problem in order to provide a clean elucidation of the mechanisms leading to different patterns observed in the resistive state and to provide a “road map” in parameter (temperature or applied current) space revealing where one should expect either periodic states, stationary states, or both. In particular, we will explain why and when oscillatory solutions emerge and compute their periods as they bifurcate from the normal state. We will also consider the loss of stability of these oscillatory solutions as the temperature is lowered below a critical value that depends on the applied current  $I$ . The key idea is that the oscillations appear as a conse-

quence of a Hopf bifurcation driven by a  $PT$ -symmetric spectral problem. A crucial role in the analysis is played by the dependence of this spectrum on the applied current. An additional goal here is to elucidate the appearance of hysteresis in I–V curves in the present setup.

Our starting point is the time-dependent Ginzburg-Landau model that we write in a nondimensional form:

$$\psi_t + i\varphi\psi = \psi_{xx} + \Gamma\psi - |\psi|^2\psi. \quad (1)$$

Here  $\psi$  is the complex-valued order parameter,  $\varphi$  is the electric potential, and  $\Gamma$  is proportional to  $T_c - T$ . Conservation of the current  $I$  implies the relation

$$\frac{i}{2}(\psi\psi_x^* - \psi_x\psi^*) - \sigma\varphi_x = I, \quad (2)$$

where  $\sigma$  models the Ohmic resistivity. [In Eqs. (1) and (2) and all subsequent equations, we use a variable in subscript to denote a partial derivative]. The wire is assumed to extend along  $-L \leq x \leq L$ , and it is assumed that  $\psi(\pm L, t) = 0$ . The main conclusions below are also valid for other boundary conditions such as  $\psi_x(\pm L, t) = 0$ . In order to concentrate on the main features of the phase transition mechanism, we take  $\sigma = L = 1$ . This enables us to concentrate on the key parameters  $I$  and  $\Gamma$ . Some TDGL models include a factor  $\zeta_1/\sqrt{1 + \zeta_2|\psi|^2}$  in front of the left hand side of Eq. (1). We deal here with the small  $\zeta_2$  limit, but our essential results are valid for finite positive  $\zeta_2$ .

To understand the phase transition from the normal state to the superconducting state we linearize the TDGL (1) about the normal state  $\psi = 0$ ,  $\varphi = -Ix$ . Writing  $\psi(x, t) = u(x)e^{(\Gamma - \lambda)t}$ , we obtain for  $u(x)$  the spectral problem

$$M[u] = u_{xx} + ixIu = -\lambda u, \quad u(\pm 1) = 0. \quad (3)$$

The spectral problem (3) is called  $PT$ -symmetric, since it is invariant under the joint transformation of  $x \rightarrow -x$  (parity)

and complex conjugacy (time reversal). The normal state thus loses its stability when  $\Gamma > \text{Real}(\lambda(I))$ . However, since the spectral problem (3) is not self-adjoint, it is not clear at all that the spectrum is real. On the other hand the  $PT$  symmetry provides some useful information on the spectrum. Spectral  $PT$ -symmetric problems have attracted some interest in recent years following the numerical observation of Bender and Boettcher [5] that the spectrum of certain  $PT$ -symmetric problems is real. While Ref. [5] considered a problem on the entire line, we deal here with a problem in a finite interval. When  $I = 0$  the spectrum is of course real. The  $PT$  symmetry implies that if  $[\lambda, u(x)]$  is a spectral pair with complex  $\lambda$ , then also  $[\lambda^*, u^*(-x)]$  is a spectral pair. Since the spectrum depends smoothly on  $I$  as long as the eigenvalues remain separated [6], a real eigenvalue cannot split spontaneously into a complex pair. This implies that at least for small  $I$  all eigenvalues are real. However, when the current  $I$  is large, the lowest eigenvalues (in absolute value) are shown to satisfy  $\lambda = O(iI)$ ; namely, to leading order they are purely imaginary. This implies that eigenvalues indeed collide as  $I$  increases. Specifically we find that the first such collision occurs when the first and second eigenvalues approach each other and collide at a critical value  $I_{co} \approx 12.31$ .

At the collision point, the geometric multiplicity of the eigenvalue is 1. To find the behavior of the spectrum near  $I_{co}$  we set the current  $I$  to be  $I = I_{co} + \epsilon a$ . Here  $\epsilon$  is a small positive number, and we introduce  $a$  to determine through its sign the direction in which we move from  $I_{co}$ . We then consider an expansion of the form

$$\begin{aligned}\lambda &= \mu_0 + \epsilon^{1/2}\mu_1 + \epsilon\mu_2 + \dots, \\ u &= u_0 + \epsilon^{1/2}u_1 + \epsilon u_2 + \dots\end{aligned}\quad (4)$$

The nonanalytic nature of the expansion for  $\lambda$  is a consequence of the Jordan form of the spectral problem at the critical value  $I = I_{co}$ . The leading order term in (4) is found to be  $\mu_0 \approx 0.71$ , with an associated eigenfunction  $u_0$  that we normalize by  $u_0(0) = 1$ . The first order correction  $\mu_1$  is conveniently expressed through the auxiliary function  $K(x)$  that solves

$$K_{xx} + ixI_{co}K + \mu_0K = u_0, \quad K(\pm 1) = 0. \quad (5)$$

Writing  $u_0 = U_r + iU_i$ , and defining  $a_1 = 2 \int_{-1}^1 xU_rU_i dx$  and  $b = \int_{-1}^1 KU_0 dx$ , one obtains  $\mu_1^2 = -aa_1/b$ . A numerical integration gives  $a_1 \approx 0.29$  and  $b \approx 0.12$ . Since  $a_1/b > 0$ , we see that when  $a < 0$ , i.e., when  $I$  is a little smaller than  $I_{co}$ , there are two real solutions; these are the first two real eigenvalues just before the collision. However, for  $I$  beyond  $I_{co}$ , that is, for  $a > 0$ , the single eigenvalue  $\mu_0$  splits into a pair of complex eigenvalues. It can be further shown that  $\mu_2$  is a single real number; i.e., it is the same for both splitting eigenvalues [7]. The analysis above shows that the real part of the leading eigenvalue is not an analytic function of the current at  $I = I_{co}$ . In fact, its derivative blows up as  $I_{co}$  is approached from below. On

the other hand, the real part of the first eigenvalue (pair) is a smooth function of  $I$  just above  $I_{co}$ . This analysis holds for any later collision of real eigenvalues as well. It is in agreement with the numerical calculation presented in Fig. 1.

We computed the first few eigenvalues numerically as they increase past special collision points. Increasing  $I$  beyond  $I_{co}$ , the first two eigenvalues move as a complex pair according to the  $PT$  symmetry. The real parts of the first six eigenvalues as a function of  $I$  are plotted in Fig. 1. We see there that respective pairs of eigenvalues collide at successive critical values of  $I$ .

The normal state becomes unstable at that value of  $\Gamma$  for which  $\Gamma - \text{Real}(\lambda) = 0$ . For  $I < I_{co}$  the first eigenvalue  $\lambda(I)$  is real. When the temperature is sufficiently low, i.e., when  $\Gamma = \lambda(I)$ , the normal state loses stability. Proceeding to high order terms in the bifurcation expansion it is found that the bifurcation branch that emerges at  $\Gamma = \Gamma_1(I) = \lambda(I)$  is stable for  $I < I_k \approx 10.92$ . In this regime, i.e., when  $I < I_k$  and  $\Gamma > \Gamma(I)$ , the bifurcating solution converges to a stationary solution. By ‘‘stationary’’ here we mean that writing  $\psi = fe^{i\chi}$ , the gauge invariant quantities  $f(x, t)$ ,  $q(x, t) = \chi_x(x, t)$ , and  $\theta(x, t) = \chi_t(x, t) - \varphi(x, t)$  converge to stationary functions  $f_0(x)$ ,  $q_0(x)$ ,  $\theta_0(x)$ . Once  $I$  crosses the critical collision value  $I_{co}$  and the eigenvalue splits into a conjugate complex pair, the phase transition temperature is determined by the condition  $\Gamma = \text{Real}(\lambda(I)) = \Gamma_1(I)$ . Thus,  $I > I_{co}$  a Hopf bifurcation occurs and the solution to the full TDGL is periodic.

Consider now a current  $I > I_{co}$ . When  $\Gamma$  is below  $\Gamma_1(I)$ , the positive real part of the spectrum dominates, and the normal state is stable. Increasing  $\Gamma$  with  $I$  fixed we see that when  $\Gamma = \Gamma_1(I)$  a Hopf bifurcation into a periodic solution takes place as explained above. In addition to determining the bifurcation curve  $\Gamma = \Gamma_1(I)$ , the spectral problem (3) can also be used to compute the bifurcating branch, which is always stable, in the periodic regime. To see this, fix a

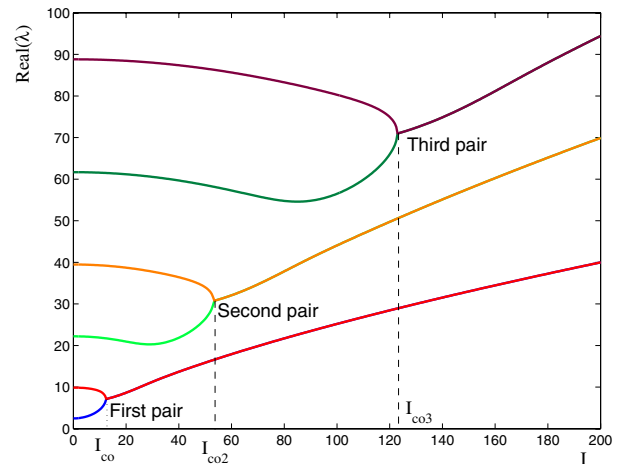


FIG. 1 (color online). The real parts of the first six eigenvalues of the  $PT$ -symmetric spectral problem (3).

current  $I$  greater than the critical value  $I_{co}$ . Let the ground state of Eq. (3) consist of the eigenvalue pair  $\lambda_r \pm i\lambda_i$ , with associated eigenfunctions  $w_1(x)$  and  $w_2(x)$  related by  $w_1(x) = w_2^*(-x)$ . We normalize both eigenfunctions by the condition  $w_i(0) = 1$ . Set the temperature to be slightly below the critical value determined by  $\Gamma = \Gamma_1(I)$ , by selecting  $\Gamma = \lambda_r + \epsilon^2$ . Neglecting a short time interval during which transients decay, the asymptotic solution of the full TDGL (1) and (2) is found to be of the form

$$u(x, t) = \epsilon A [\exp(i(\omega \epsilon^2 + \lambda_i)t) w_1(x) + \exp(-i(\omega \epsilon^2 + \lambda_i)t) w_2(x)] + O(\epsilon^3). \quad (6)$$

The amplitude  $A$  and frequency  $\omega$  are constants that are determined by the ground state  $w_1$  and  $w_2$ . They are computed numerically for each current  $I$ . For instance, when  $I = 20$  we found  $A = 0.921$ ,  $\omega = 1.8$ . One can draw a number of conclusions from the expression (6). First, the period of the oscillations is not exactly the imaginary component of the eigenvalue, but rather it is has a correction due to the nonlinear interaction of  $w_1$  and  $w_2$ . Second, the solution at  $x = 0$  is  $u(0, t) \approx 2\epsilon A \cos[(\lambda_i + \omega \epsilon^2)t]$ . Therefore we obtain a phase slip center that is periodic in time at  $x = 0$ . Ivlev and Kopnin [1] made the nice observation that a PSC can be thought of as a vortex in space-time. In this sense, the solution structure given in Eq. (6) indicates that the PSC's constitute a periodic placement of degree-one space-time vortices with period  $P = \pi/(\lambda_i + \omega \epsilon^2)$ . The curve  $\Gamma_1(I)$  is depicted by the solid line in Fig. 2.

So far we have concentrated on the smooth bifurcation of the normal state into a periodic state or into a stationary state. It turns out, though, that there are regions in the parameter plane where two metastable states coexist. The transition between them is nonsmooth, and therefore it is associated with hysteresis. We already pointed out above that for  $I_{co} > I > I_k$  the normal state bifurcates into an unstable branch. This hints that the phase transition there is nonsmooth. Indeed, we identified another curve in the phase plane, that we call  $\Gamma_2(I)$ , above which the stationary

state is stable. The curve  $\Gamma_2(I)$  is depicted as a dashed line in Fig. 2.

To understand the loss of stability of the periodic state, we recall that the Hopf bifurcation that led to it was triggered by the normal current contribution to the potential term  $i\varphi\psi$  in Eq. (1). Near the transition curve  $\Gamma = \Gamma_1(I)$  the magnitude  $|\psi|$  of the order parameter is still small, and essentially the entire current is normal. As the temperature is lowered (i.e.,  $\Gamma$  increases),  $|\psi|$  grows and so does the supercurrent, implying via (2) that the normal current decreases. This effectively returns the system to the small  $I$  regime where the bifurcation to a steady state is favored. We thus obtain a third bifurcation curve  $\Gamma_3(I)$  where the periodic state loses its stability.

At this point we make reference to Fig. 2 and consider the different regimes in the  $(I, \Gamma)$  plane. The solid curve provides the critical temperature  $\Gamma = \Gamma_1(I)$  along which the normal state loses its stability. A stable stationary state exists above the dashed line that represents a second curve  $\Gamma_2(I)$ . For  $I < I_k$  the normal state bifurcates into a stable stationary superconducting state. For  $I > I_{co}$ , on the other hand, the normal state bifurcates into a state that exhibits time-periodic oscillations. When  $I > I_{co}$ , and the temperature is further lowered ( $\Gamma$  is increased), the periodic state loses its stability at a third critical temperature  $\Gamma = \Gamma_3(I)$  represented by the dotted line in the figure. The curves  $\Gamma_2(I)$  and  $\Gamma_3(I)$  intersect at  $I_q$ . For  $I > I_q$ , the curves  $\Gamma_2(I)$  and  $\Gamma_3(I)$  coalesce. The frame on the left depicts the bifurcation curves over a large  $(I, \Gamma)$  area, while the frame on the right concentrates on the interesting area near the point  $[I_{co}, \Gamma_1(I_{co})]$ , where  $\Gamma_1(I_{co}) \approx 7.11$ . The parameter plane is partitioned into 5 domains. In domains 1, 4 and 5 there is a single stable state—the normal state in region 1, a stationary state in region 4 and a periodic state in region 5. In region 2 there are two metastable states—normal and stationary, while in region 3 a stationary and a periodic state are both metastable.

We proceed to draw two further conclusions related to the bifurcation diagram. It is useful to do so in the context of  $I$ - $V$  curves. These curves are measured or computed for

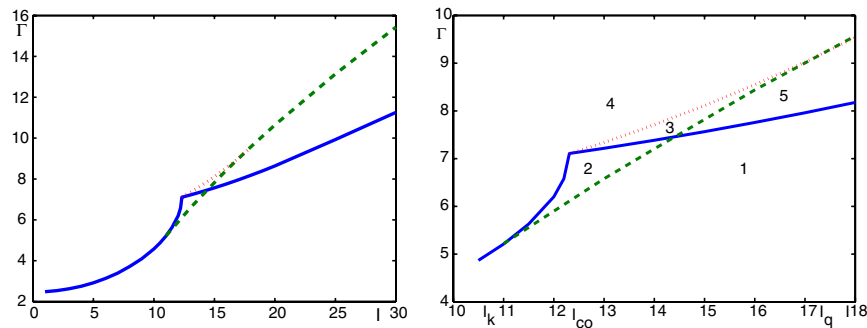


FIG. 2 (color online). The phase diagram of the different stable states in the temperature-current plane. The parameter  $\Gamma$  is proportional to  $T_c - T$ . The curves  $\Gamma_1(I)$ ,  $\Gamma_2(I)$ ,  $\Gamma_3(I)$  are drawn with solid line, dashed line, and dotted line, respectively. The meaning of the different curves and regimes is explained in detail in the text.

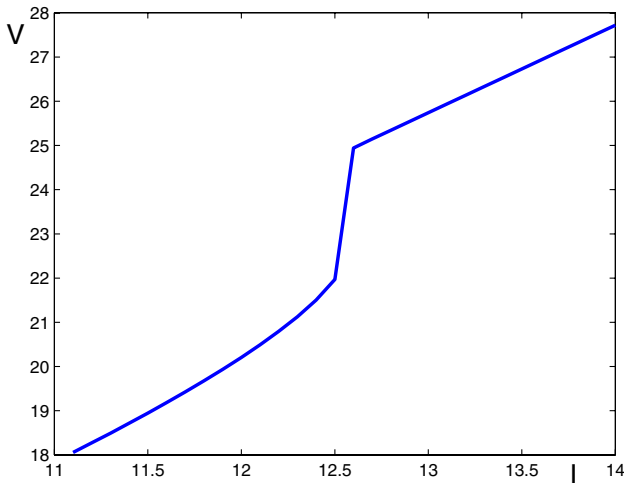


FIG. 3 (color online). The I–V curve for  $\Gamma = 6.3$  and  $I$  increasing. Notice that, although the periodic state does not exist for such temperature, and thus there is no PSC here, the I–V curve does exhibit a jump discontinuity at  $I \approx 12.57$ .

a fixed  $\Gamma$ , while the current  $I$  is raised or lowered adiabatically. When this process cuts through the metastable regions 2 and 3 in Fig. 2 a hysteresis is expected in the I–V curve. While such a hysteresis was predicted a long time ago, we point out that it is not always observed experimentally [4]. As can be seen in Fig. 2, the metastable regions are quite small, and therefore it requires careful tuning to pass through them. Another comment relates to the formation of PSC's. These points in space-time where  $|\psi|$  vanishes are often associated in the literature with jump discontinuities in the I–V curve. However, this identification works only in one way, and not all such jumps imply the presence of a PSC. For instance, we depict in Fig. 3 the I–V curve for  $\Gamma = 6.3$  and  $I$  slowly increasing. For this  $\Gamma$  one never crosses an area in the parameter plane where the periodic state is stable, and therefore there is no PSC. Nonetheless, the I–V curve exhibits a clear discontinuity at about  $I \approx 12.57$ . The actual rule for lack of smoothness in I–V curves is that a jump discontinuity indicates a nonsmooth phase transition, while a discontinuity in the derivative indicates a continuous phase transition.

To conclude, using a combination of asymptotic expansions, spectral analysis, and canonical numerical computation, we have clarified the mechanisms underlying the interesting behavior of the now classical problem of a

superconducting wire exposed to an applied electric current. In particular, retaining temperature and applied current as parameters in the time-dependent Ginzburg-Landau model, we have decomposed this two-dimensional parameter space into regions of stability of a normal, stationary and oscillatory state. The collision of real eigenvalues and the consequent emergence of complex spectrum in the associated linearized problem provides the explanation for the Hopf bifurcation leading to the appearance of the oscillatory state and the associated phase slip centers. From the theoretical standpoint, this clearly stands as a physically significant setting where  $PT$  symmetry does *not* lead to reality of the spectrum, in contrast to its common role [5,8]. The boundary of the basin of attraction of the normal state has been given precisely in terms of the real part of the leading eigenvalue in this linearized problem. The boundary between the basins of attraction of the oscillatory and stationary states has been calculated near the triple point using asymptotics, and has been computed numerically beyond this. In so doing, we have identified small regions in the temperature or current plane where hysteresis should be anticipated and where jumps in I–V curves do not necessarily correspond to the presence of PSC's. Finally, the asymptotic structure of the periodic solution bifurcating off the normal state has been developed for  $I$  above the first collision value  $I_{co}$  and for  $\Gamma$  just above the real part of the first eigenvalue. This expansion reveals the period of the oscillations and location of PSC's along the  $x$  axis.

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