7.4

Write \( u(x, y) = X(x)Y(y) \) and plug into Laplace’s equation to get

\[
\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda. \tag{0.1}
\]

Since the homogeneous boundary conditions are for \( X \), i.e. \( X(0) = 0 = X(\pi) \), the eigenvalue problem is for \( X \). As usual with homogeneous Dirichlet boundary conditions, we get

\[ X_n(x) = \sin (nx), \quad \lambda_n = n^2 \quad \text{for } n = 1, 2, \ldots. \]

(In particular, 0 is not an eigenvalue for this problem.) Then for \( Y_n \) we get

\[ Y_n(y) = A_n \cosh (ny) + B_n \sinh (ny). \]

so

\[
u(x, y) = \sum_{n=1}^{\infty} \left( A_n \cosh (ny) + B_n \sinh (ny) \right) \sin (nx).\]

Then

\[
u(x, 0) = 1 = \sum_{n=1}^{\infty} A_n \sin (nx) \tag{0.2}\]

and orthogonality implies:

\[
\int_0^\pi 1 \sin (nx) \, dx = A_n \int_0^\pi \sin^2 (nx) \, dx = A_n \frac{\pi}{2},
\]

so

\[
A_n = \frac{2}{\pi} \cos (nx) \bigg|_0^\pi = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd} \end{cases}
\]

Then \( u(x, \pi) = 1 \) implies:

\[
1 = \sum_{n=1}^{\infty} \left( A_n \cosh (n\pi) + B_n \sinh (n\pi) \right) \sin (nx), \text{ which is again } (0.2) \text{ so the quantity } C_n \text{ must equal } A_n. \text{ That is,}
\]

\[
A_n \cosh (n\pi) + B_n \sinh (n\pi) = A_n
\]

and so

\[
B_n = \frac{1 - \cosh (n\pi)}{\sinh (n\pi)} A_n.
\]
In this problem, the normal derivative is specified on the boundary of the square \( S := \{(x, y) : 0 < x < \pi, 0 < y < \pi\} \). Since \( u \) is harmonic, the Divergence Theorem tells us that:

\[
0 = \int \int_S \Delta u \, dx \, dy = \int_{\partial S} \nabla u \cdot \vec{n} \, ds = -\int_0^\pi u_y(x, \pi) \, dx
\]
since the normal derivative is zero on the other three segments comprising the boundary of \( S \). *(The minus sign in the last integral doesn’t affect the problem but it’s there because we must traverse the boundary integral in a counter-clockwise direction.)* We calculate that

\[
\int_0^\pi u_y(x, \pi) \, dx = \int_0^\pi (x^2 - a) \, dx = \pi\left(\frac{\pi^2}{3} - a\right)
\]
so there will only be a solution if \( a = \frac{\pi^2}{3} \). Picking this value for \( a \), we separate variables in Laplace’s equation and again get (0.1). This time there are three homogeneous boundary conditions \( X'(0) = X' (\pi) = Y'(0) = 0 \) so again \( X \) turns out to give the eigenvalue problem since it carries two homogeneous b.c. We find 0 is an eigenvalue with eigenfunction \( X_0(x) = 1 \) and

\[
X_n(x) = A_n \cosh (ny), \quad \lambda_n = n^2 \quad \text{for } n = 1, 2, \ldots.
\]

Then using that \( Y'(0) = 0 \) we get that \( Y_0(y) = A_0 \) while \( Y_n(y) = A_n \cosh (ny) \) for \( n = 1, 2, \ldots \) so

\[
u(x, y) = \sum_{n=0}^\infty A_n \cosh (ny) \cos (nx).
\]

We then find from the remaining boundary condition, \( u_y(x, \pi) = x^2 - \frac{\pi^2}{3} \) that

\[
x^2 - \frac{\pi^2}{3} = \sum_{n=1}^\infty n A_n \sinh (n\pi) \cos (nx),
\]
so using orthogonality we obtain,

\[
n A_n \sinh (n\pi) \int_0^\pi \cos^2 (nx) \, dx = \int_0^\pi (x^2 - \frac{\pi^2}{3}) \cos (nx) \, dx = \frac{(-1)^n 2\pi}{n^2}
\]
where the last integral is computed using two integrations by parts. Hence,

\[
A_n = \frac{(-1)^n 2\pi}{n^3 \sinh (n\pi)} \quad \text{for } n = 1, 2, \ldots
\]

Note that \( A_0 \) is not determined. This is because the Neumann problem for Laplace’s equation is only unique up to an arbitrary additive constant.
We write the Laplacian in polar coordinates and seek a solution in the form $u(r, \theta) = R(r)S(\theta)$. Separating variables as we did in class when solving Laplace’s equation in a disc, we find

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0, \quad S''(\theta) + \lambda S(\theta) = 0 \quad (0.3)$$

and the boundary conditions are:

$R(\infty) = 0, \quad S(0) = S(2\pi), \quad S'(0) = S'(2\pi).$

The eigenvalue problem for $S$ is the periodic Sturm-Liouville problem we’ve discussed in class (same as for disc) and we find that 0 is an eigenvalue with eigenfunction $S_0(\theta) = 1$ while for $n = 1, 2, \ldots$ we find

$$\lambda_n = n^2 \quad \text{with two eigenfunctions for each } n : \quad \cos(n\theta) \text{ and } \sin(n\theta). \quad (0.4)$$

Then since $R$ satisfies an Euler-type ODE, we guess $R(r) = r^\alpha$ and find two solutions: $R(r) = r^n$ or $r^{-n}$. However, the condition $R(\infty) = 0$ means we can only use $r^{-n}$. Also note that for $\lambda = 0$ we have $R_0(r) = c_1 \ln r + c_2$ but $R_0(\infty) = 0$ forces $c_1 = c_2 = 0$. This leaves us with:

$$u(r, \theta) = \sum_{n=1}^{\infty} r^{-n} (A_n \cos(n\theta) + B_n \sin(n\theta)).$$

Now we use the remaining boundary condition $u(2, \theta) = y = r \sin \theta$ to get:

$$2 \sin \theta = \sum_{n=1}^{\infty} 2^{-n} (A_n \cos(n\theta) + B_n \sin(n\theta)).$$

This equation will hold by choosing $B_1 = 4$ and all other coefficients to be zero. We get

$$u(r, \theta) = 4r^{-1} \sin \theta = \frac{4y}{x^2 + y^2}.$$

7.19

Complete answer given in back of textbook.
Here we are solving Laplace’s equation in an annulus so again we use polar coordinates and separate variables. This again leads to the O.D.E.’s (0.3). Again, the boundary conditions are periodic for \( S(\theta) \) so we once again find that (0.4) holds.

Regarding \( R \), we have again for \( \lambda = 0 \) that \( R_0(r) = A_0 + B_0 \ln r \) and since \( R(2) = 0 \) we need in particular that \( R_0(2) = 0 \) so \( R_0(r) = A_0(1 - \frac{\ln r}{\ln 2}) \). For \( n = 1, 2, \ldots \) we again have the two solutions to the Euler equation \( R(r) = r^{\pm n} \) but now the boundary condition \( R(2) = 0 \) implies that \( R_n \) is given by \( R_n(r) = \text{Const.} \left( r^n - \left( \frac{4}{r} \right)^n \right) \). Thus we obtain:

\[
u(r, \theta) = A_0(1 - \frac{\ln r}{\ln 2}) + \sum_{n=1}^{\infty} \left( r^n - \left( \frac{4}{r} \right)^n \right) (A_n \cos(n\theta) + B_n \sin(n\theta)).\]

Finally, the boundary condition \( \nu(4, \theta) = \sin \theta \) implies that

\[
\sin \theta = A_0(1 - \frac{\ln 4}{\ln 2}) + \sum_{n=1}^{\infty} (4^n - 1) (A_n \cos(n\theta) + B_n \sin(n\theta)).
\]

This can be accomplished by choosing \( B_1 = \frac{1}{3} \) and all other coefficients to be zero. Thus,

\[
u(r, \theta) = \frac{1}{3}(r - \frac{4}{r}) \sin \theta = \frac{1}{3}y - \frac{4}{3} \left( \frac{y}{x^2 + y^2} \right).
\]

**Problem A.**

Suppose \( u_1 \) and \( u_2 \) both solve this problem. Let \( v = u_1 - u_2 \). Then \( v \) solves:

\[
\Delta v = 0 \quad \text{for } a < x < b, \ c < y < d,
\]

\[
v_x(a, y) = 0 = v_x(b, y), \quad v(x, c) = 0 = v(x, d).
\]

Multiply the PDE by \( v \), integrate over the rectangle \( R := \{(x, y) : a < x < b, \ c < y < d \} \) and use Green’s identity:

\[
0 = \int_a^b \int_c^d v \Delta v \, dx \, dy = \int_{\partial R} v \nabla v \cdot \vec{n} \, ds - \int_a^b \int_c^d |\nabla v|^2 \, dx \, dy = - \int_a^b \int_c^d |\nabla v|^2 \, dx \, dy.
\]

Here we have used the boundary conditions to throw away the boundary integral since on each segment of the boundary, either \( v \) or \( \nabla v \cdot \vec{n} \) is zero. We conclude that \( v \) is constant on \( R \) and since it equals zero on the top and bottom, \( v \equiv 0 \).
Problem B. Letting \( v := u(x, y) + u(-x, y) \) we compute that

\[
v_{xx}(x, y) + v_{yy}(x, y) = u_{xx}(x, y) + u_{yy}(x, y) + u_{xx}(-x, y) + u_{yy}(-x, y),
\]

so we see that \( \Delta v = 0 \) in the rectangle. On the boundary, we have:

\[
v(x, c) = u(x, c) + u(-x, c) = g(x) + g(-x) = 0 \quad \text{and} \quad v(x, -c) = u(x, -c) + u(-x, -c) = g(x) + g(-x) = 0
\]

since \( g \) is odd, i.e. \( g(x) = -g(-x) \). Then on the other two portions of the boundary we have:

\[
v_x(a, y) = \frac{\partial}{\partial x} (u(x, y) + u(-x, y)) \bigg|_{x=a} = u_x(a, y) - u_x(-a, y) = f(y) - f(y) = 0,
\]

and

\[
v_x(-a, y) = \frac{\partial}{\partial x} (u(x, y) + u(-x, y)) \bigg|_{x=-a} = u_x(-a, y) - u_x(a, y) = f(y) - f(y) = 0.
\]

From Problem A above we know the unique solution is \( v \equiv 0 \) which means \( u(x, y) = -u(-x, y) \).