M441 - Homework #3

Solutions

October 1, 2010

A

i)
We can use D’Alembert’s formula

\[ u(x, t) = \frac{1}{2} (f(x - t) + f(x + t)) \]

where \( f(x) = (1 - x^2)^2 I_{[-1,1]}(x) \). It is easy to see that if you fix a \( x_0 \) the first part of the above formula, \( \frac{1}{2}(1 - (x_0 - t)^2)^2 I_{[-1,1]}(x_0 - t) \), will be zero when \( |x_0 - t| > 1 \); in particular, it will be zero when \( t > 1 + |x_0| \). The same reasoning can be applied to the second part: \( \frac{1}{2}(1 + (x_0 + t)^2)^2 I_{[-1,1]}(x_0 + t) \) will be zero when \( |x_0 + t| > 1 \); this means, once \( t > |x_0| + 1 \) this part will be zero. Thus, \( u(x_0, t) = 0 \) when \( t > 1 + |x_0| \). We get then

\[ \lim_{t \to \infty} u(x_0, t) = 0 \]

ii)
Analogous to the previous case.

B

The main idea here is the way we represent the functions. Note that you can rewrite \( f \) and \( g \) as

\[ f(x) = f(x)I_{[-4,-3]} + f(x)I_{[2,3]} \]

\[ g(x) = g(x)I_{[-4,-3]} + g(x)I_{[2,3]} \]

At 0, we have

\[ u(0, t) = \frac{1}{2} \left( f(-t) + f(+t) \right) + \frac{1}{2} \int_{-t}^{t} g(s) ds \]

\(^1\)The function \( I_A(x) \) is 1 when \( x \in A \), 0 otherwise; usually, it is called characteristic function.
For $0 \leq t \leq 2$, $u(0, t) = 0$, and $u(0, t)$ can be non zero only when $t \geq 2$. Note also that for $t \geq 4$, the part concerning to $f$ is zero, but the integral in $g$ is constant, because

$$
\int_{-t}^{t} g(s)ds = \int_{-4}^{4} g(s)ds \quad \text{when} \quad t \geq 4
$$

\section*{C}

The backdrop of this exercise are the concepts of domain of dependence and domain of influence. In what follows, we will assume $c > 0$. Using D’Alembert’s formula, we get

$$
u(0, t) = \frac{1}{2} \left( f(x - ct) + f(x + ct) \right) + \frac{1}{2c} \int_{x - ct}^{x + ct} g(s)ds$$

We know that if $x + ct < a$ or $b < x - ct$, all the terms in the above formula will be zero, because the support of $f$ and $g$ are contained in $(a, b)$. Since $u$ is zero when $x < a - ct$ and $x > b + ct$, $u$ only can have its support in the interval $(a - ct, b + ct)$.

\section*{D}

The main idea here was to differentiate the energy equation

$$
\frac{\partial E(t)}{\partial t} = \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} \left[ \frac{1}{2}(u_t)^2 + \frac{c^2}{2}(u_x)^2 \right] dx
$$
Due to the previous exercise, assuming $f$ and $g$ with compact supports, we get $u$ with compact support. We can differentiate then inside the integral sign.

\[ \frac{\partial E(t)}{\partial t} = \int_{-\infty}^{+\infty} \left[ u_t u_{tt} + c^2 u_x u_{xt} \right] \, dx \quad (1) \]

We know that $u_{tt} = c^2 u_{xx}$. We substitute then in the (1):

\[ \frac{\partial E(t)}{\partial t} = \int_{-\infty}^{+\infty} c^2 [u_t u_{xx} + u_x u_{xt}] \, dx \]

As $u_t u_{xx} + u_x u_{xt} = \frac{\partial}{\partial x}(u_t u_x)$, we get in (1)

\[ \frac{\partial E(t)}{\partial t} = \int_{-\infty}^{+\infty} c^2 \frac{\partial}{\partial x}(u_t u_x) \, dx \]

Since $u_t(x, t)$ and $u_x(x, t)$ are both zero when $|x|$ is big enough, the fundamental theorem of calculus shows us that the rhs in (1) zero. Then, $E$ has zero derivative on a connected interval; so, it is a constant.

Let’s start by D’Alembert’s formula, studying what happens in the “-x case”:

\[ u(-x, t) = \frac{1}{2} \left( f(-x - ct) + f(-x + ct) \right) + \frac{1}{2c} \int_{-x-ct}^{-x+ct} g(s) \, ds \]

As $f$ and $g$ are odd functions, we have

\[ u(-x, t) = \frac{1}{2} \left( -f(x + ct) - f(x - ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z) \, dz \]

where in the integral part we did a change of variables and used some integral properties\(^2\).

By $g$ oddness, we get

\[ u(-x, t) = \frac{1}{2} \left( -f(x + ct) - f(x - ct) \right) - \frac{1}{2c} \int_{x-ct}^{x+ct} g(z) \, dz = -u(x, t), \]

and we are done.

The main point here is to construct an extension to $f$ and $g$ in such a way you can apply D’Alembert’s formula while making $u(0, t) = 0$. This can be done by taking odd extensions, since then $u$ will be odd (by problem E) and so $u(0, t) = 0$. Let’s define

\[ \tilde{f}(x) = \begin{cases} f(x), & \text{if } x \geq 0 \\ -f(-x), & \text{if } x < 0 \end{cases} \]

\(^2\)Precisely, we did $z = -x$ and used $\int_{a}^{b} = -\int_{b}^{a}$
\[ \tilde{g}(x) = \begin{cases} 
\quad g(x), & \text{if } x \geq 0 \\
\quad -g(-x), & \text{if } x < 0 
\end{cases} \]

As \( \tilde{f} \) and \( \tilde{g} \) are defined on the whole line, we can apply D’Alembert’s formula to them.

\[ u(x, t) = \frac{1}{2} \left( \tilde{f}(x - ct) + \tilde{f}(x + ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} \tilde{g}(s)ds \]

In fact, we are not interested in what happens in the negative side of the string, so we must study the solution behavior only for \( x \geq 0 \). As \( t \geq 0 \), the only problem we can have is when \( x - ct < 0 \). In such a case, we have:

\[ \tilde{f}(x - ct) = -f(ct - x) \]

Further,

\[ \int_{x-ct}^{x+ct} \tilde{g}(s)ds = \int_{ct-x}^{ct} \tilde{g}(s)ds + \int_{ct-x}^{x+ct} \tilde{g}(s)ds. \]

As \( \tilde{g} \) is odd and the interval is symmetric, the first integral is zero. So, we get

\[ \int_{x-ct}^{x+ct} \tilde{g}(s)ds = \int_{ct-x}^{x+ct} \tilde{g}(s)ds = \int_{ct-x}^{ct} \tilde{g}(s)ds. \]

The solution is

\[ u(x, t) = \begin{cases} 
\frac{1}{2} \left( -f(ct - x) + f(x + ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)ds, & \text{if } x - ct < 0 \\
\frac{1}{2} \left( f(x - ct) + f(x + ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)ds, & \text{if } x - ct \geq 0 
\end{cases} \]  \hspace{1cm} (2)

Now we must study some properties of \( f \) and \( g \):

- Taking \( x = 0 \) in the above equation, we get \( u(0, 0) = f(0) = 0 \)
- \( u(0, t) = 0 \), for every \( t \geq 0 \) \( \Rightarrow \frac{\partial}{\partial t} u(0, t) = 0 \) After some calculus,

\[ \frac{\partial}{\partial t} u(0, t) = 0 = \frac{c}{2} \left( f'(ct) - f'(-ct) \right) + \frac{1}{2} (g(ct) + g(-ct))ds \]

and taking \( t = 0 \) in the above equation, we get \( g(0) = 0 \)

- Deriving \( \frac{\partial}{\partial t} u(0, t) \) in \( t \) one more time, we get \( \frac{\partial^2}{\partial t^2} u(0, t) = 0 \). Analogously to the previous case, it is not difficult to obtain \( f''(0) = 0 \).