Problem 5.5

(a) We start with a separation of variables ansatz: \( u(x, t) = X(x)T(t) \), which gives (when substituted in the pde)

\[
X(x)T'(t) = k\kappa X''(x)T(t) \quad \Rightarrow \quad \frac{X'(x)}{X(x)} = \frac{T'(t)}{kT(t)}.
\]

It’s easy to see that both sides must be constants, so we have, say:

\[
\frac{X'(x)}{X(x)} = \frac{T'(t)}{kT(t)} = -\lambda \quad \text{for some real number } \lambda.
\]

Thus, we get the following system:

\[
\begin{align*}
T' &= -\lambda kT \\
X'' &= -\lambda X
\end{align*}
\]

The boundary conditions implies then

\[
X'(0)T(t) = X'(L)T(t) = 0 \quad \Rightarrow \quad X'(0) = X'(L) = 0 \quad (\star).
\]

We must determine the the sign of \( \lambda \): If \( \lambda \) is negative, then we have

\[
X(x) = c_1 \cosh(\sqrt{-\lambda}x) + c_2 \sinh(\sqrt{-\lambda}x).
\]
Now \( X'(0) = 0 \) implies that \( c_2 = 0 \) while \( X'(L) = 0 \) implies that \( c_1 \sinh(L) = 0 \) which forces \( c_1 = 0 \). Thus, \( \lambda \) cannot be negative.

If \( \lambda = 0 \), then \( X(x) = c_1 x + c_2 \). Then the boundary conditions force \( c_1 = 0 \) but \( c_2 \) can be anything. We conclude that indeed \( \lambda = 0 \) is an eigenvalue with corresponding eigenfunction \( X_0(x) = 1 \).

Finally, we consider the case where \( \lambda \) is positive. Then solving the ODE for \( X \) gives:

\[
X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)
\]

and the boundary condition \( X'(0) = 0 \) forces \( c_2 = 0 \) while \( X'(L) = 0 \) forces

\[
c_1 \sqrt{\lambda} \sin(\sqrt{\lambda}L) = 0.
\]

We conclude that the positive eigenvalues and their corresponding eigenfunctions are given by

\[
\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad X_n(x) = \cos\left(\frac{n\pi}{L}x\right).
\]

We can then solve the ODE for \( T \) using these \( \lambda \) values to find

\[
T_n(t) = A_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \quad \text{for } n = 1, 2, \ldots
\]

while for \( \lambda = 0 \) we have simply \( T_0'(t) = 0 \) so \( T_0 \) is a constant, say \( A_0 \). Putting all this together we find

\[
u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L}x\right). \quad (\dagger)
\]

Finally, we deal with the initial condition:

\[
u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) = f(x). \quad (**)
\]

We use the orthogonality of the eigenfunctions\(^1\),

\[
\int_0^\pi \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) \, dx = 0 \quad \text{for } m \neq n, \quad \int_0^\pi \cos^2\left(\frac{n\pi}{L}x\right) \, dx = \frac{L}{2} \quad \text{for } n = 1, 2, \ldots
\]

\(^1\)Add the identities for \( \cos(A \pm B) \) to see this.
to multiply \((\ast \ast)\) by \(\cos(\frac{m\pi}{L}x)\) and integrate. We find that the coefficients in \((\dagger)\) are given by:

\[
A_0 = \frac{1}{L} \int_0^L f(x) \, dx, \quad A_n = \frac{2}{L} \int_0^L f(x) \cos(\frac{n\pi}{L}x) \, dx.
\]

(b) Taking \(L = \pi\) and \(k = 12\) in \((\dagger)\), we find

\[
u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-kn^2t} \cos(nx).
\]

For the initial condition \(f(x) = 1 + \sin^3 x\) we can use the trig identities \(\cos^2 x + \sin^2 x = 1\), \(\cos^2 x = (1 + \cos(2x))/2\) and the identity for \(\cos(A)\sin(B)\) to find that

\[
\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin (3x).
\]

Then one uses this to compute

\[
A_n = \frac{2}{\pi} \int_0^\pi \left(1 + \frac{3}{4} \sin x - \frac{1}{4} \sin (3x)\right) \cos(nx) \, dx
\]

\[
= \frac{2}{\pi} \left(\frac{3}{4(n+1)} - \frac{3}{4(n-1)} - \frac{1}{4(n+3)} + \frac{1}{4(n-3)}\right) \quad \text{if } n = 2, 4, 6, \ldots
\]

while \(A_0 = 1 + \frac{4}{3\pi}\) and \(A_n = 0\) for all \(n\) odd\(^2\).

(c) Either using the exponential decay of the terms in the infinite series \((\dagger)\) to (formally) let \(t \to \infty\) or else using the result from problem E(i) in assignment 4, we find that

\[
limit_{t \to \infty} u(x, t) = A_0 = 1 + \frac{4}{3\pi} = \text{average value of the initial data}.
\]

Physically, we find that for insulated ends with no heat source, the temperature eventually converges to the average of the initial temperature— that is, it has completely “spread out” during this diffusion process.

\(\text{A}\)

Assuming separation of variables: \(u(x, t) = X(x)T(t)\), which gives (when substituted in the pde)

\[
X(x)T''(t) = \kappa X''(x)T(t) \quad \Rightarrow \quad \frac{X'(x)}{X(x)} = \frac{T'(t)}{\kappa T(t)}.
\]

\(^2\)Do NOT worry about these integrals if you didn’t get them. I had no idea they would be this horrible when I assigned the problem!—PS
It’s easy to see that both sides must be constants. Thus, we get the following system:

\[ \begin{align*}
T' &= \lambda \kappa T \\
X'' &= \lambda X 
\end{align*} \]  

(0.1) \hspace{1cm} \text{eq1}

The boundary conditions implies then

\[ X(0)T(t) = X(\pi)T(t) = 0 \implies X(0) = X(\pi) = 0 \]  

(\ast).

We must determine the the sign of \( \lambda \). It’s not difficult to show that \( \lambda \) is negative. We will write then \( \lambda = -\mu^2 \). We can rewrite (0.1) as

\[ \begin{cases}
T' + \mu^2 \kappa T &= 0 \\ 
X'' + \mu^2 X &= 0
\end{cases} \implies 
\begin{align*}
T(t) &= A_1 e^{-\lambda \mu^2 t} \\
X(x) &= B_1 \cos(\mu x) + B_2 \sin(\mu x)
\end{align*} \]  

(0.2) \hspace{1cm} \text{eq2}

Using the boundary conditions as in (\ast), we get the \( B_1 = 0 \) in the previous formulations. Further, we get the formula \( \mu = n \), where \( n \in \mathbb{N} \), since we need \( \sin(\mu \pi) = 0 \). We can represent the solution \( u \) as

\[ u(x, t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-\kappa n^2 t}. \]  

(0.3) \hspace{1cm} \text{eq3}

We want the above function satisfying the initial data, i.e.,

\[ u(x, 0) = 3 \sin(x) + 6 \sin(5x). \]

Either using the orthogonality of \( \{\sin(n x)\}_{n \in \mathbb{N}} \) on \([0, \pi]\), or simply noting that the given initial conditions are a linear combination of eigenfunctions, we find \( B_1 = 3 \), \( B_5 = 6 \) and all other \( B_n \)'s = 0. Thus, from (0.3) we have:

\[ u(x, t) = 3 \sin(x) e^{-\kappa t} + 6 \sin(5x) e^{-\kappa 25t}. \]

Clearly, \( \lim_{t \to \infty} u(x, t) = 0. \)

\[ ^3 \text{Just note that } \lambda \int_0^{\pi} X^2 dx = \int_0^{\pi} XX \, dx = X'X \bigg|_0^\pi - \int_0^{\pi} X^2 \, dx = - \int_0^{\pi} X^2 \, dx, \text{ which means that } \lambda \leq 0. \text{ Also } \lambda = 0 \text{ is impossible since then the boundary conditions would force } X(x) = 0. \]
At the equilibrium, the pde to be solved is the following

\[ u_e(x)_{xx} = 0 \Rightarrow u_e(x) = ax + b \]

Using the boundary conditions, we get \( a = 0 \), so \( u_e(x) = b \).

Define \( e(t) = \int_0^L u(x,t)dx \). Then,

\[ e'(t) = \int_0^L u_t(x,t)dx = \int_0^L \kappa u_{xx}(x,t)dx = \kappa (u_x(\pi, t) - u_x(0, t)) = 0, \]
due to the boundary conditions. We have then that \( E(t) \) is a constant. In particular, then, \( e(\infty) = e(0) \). Thus,

\[ \int_0^L u_e(x)dx = Lb = \int_0^L u(x,0)dx = \int_0^L f(x)dx \Rightarrow u_e(x) = b = \frac{1}{L} \int_0^L f(x)dx \]

Now, we prove the second part:

\[ \int_0^L (u(x,t) - u_e(x))^2dx \rightarrow 0 \text{ as } t \rightarrow \infty. \]

Let’s define \( v(x,t) = u(x,t) - u_e(x) \). Which pde does \( v \) satisfy? After some calculus, we get

\[ \begin{align*}
  v_t &= \kappa v_{xx} \\
  v_x(0,t) &= 0 \\
  v_x(L,t) &= 0 \\
  v(x,0) &= f(x) - \frac{1}{L} \int_0^L f(x)dx \\
\end{align*} \]

Define also \( E(t) = \frac{1}{2} \int_0^L (u(x,t) - u_e(x))^2dx = \frac{1}{2} \int_0^L (v(x,t))^2dx \). We proceed as follows:

\[ E'(t) = \int_0^L u(x,t)v_t(x,t)dx = \kappa \int_0^L v(x,t)v_{xx}(x,t)dx \]

Integrating by parts, and using the boundary conditions, we obtain
\[ E'(t) = -\kappa \int_0^L (v_x(x,t))^2 \, dx \]

At this point, we can refer to the Poincaré inequality \( \int_0^L (v(x,t))^2 \, dx \leq L^2 \int_0^L (v_x(x,t))^2 \). Note that we can use this inequality since for any \( t > 0 \) we have from earlier in the problem that:

\[ \int_0^L v(x,t) \, dx = \int_0^L (u(x,t) - u_e(x)) \, dx = e(t) - e(0) = 0. \]

Thus,

\[ E'(t) \leq -\frac{\kappa}{L^2} E(t) \Rightarrow E'(t) + \frac{\kappa}{L^2} E(t) \leq 0 \Rightarrow \left( E(t) e^{\frac{\kappa}{L^2} t} \right)' \leq 0, \]

Integrating on both sides,

\[ E(t) \leq E(0) e^{-\frac{\kappa}{L^2} t} \Rightarrow \lim_{t \to \infty} E(t) = 0 \]

C

i

Consider the situation at the left endpoint \( x = 0 \). Suppose first that it’s hotter outside the left endpoint than it is right at the left endpoint, i.e. \( h_0(t) > u(0,t) \). Then the heat flux would be positive (thermal energy moving left to right is positive), so we have:

\[
\text{heat flux at } x = 0 \text{ is } > 0 \implies -\kappa u_x(0,t) > 0 \implies u_x(0,t) < 0
\]

which is consistent with the assumption \( c_0 > 0 \) since \( u_x(0,t) = c_0(u(0,t) - h_0(t)) \). The case at \( x = 0 \) when \( h_0(t) < u(0,t) \) is handled similarly, as are the two cases at \( x = L \).

ii

Suppose \( u_1 \) and \( u_2 \) are solutions of the given problem. Define \( v = u_1 - u_2 \) Due to properties of \( u_1 \) and \( u_2 \), \( v \) satisfies the following pde, boundary conditions and initial conditions:
\[
\begin{aligned}
\begin{cases}
  v_t(x, t) & = \kappa v_{xx}(x, t) \\
v_x(0, t) & = c_0 v(0, t) \quad \text{and} \quad v_x(L, t) = -c_1 v(0, t) \\
v(x, 0) & = 0
\end{cases}
\end{aligned}
\]

Thus, defining \( E(t) = \frac{1}{2} \int_0^L (v(x, t))^2 dx \), we have (using the PDE, the boundary conditions and integration by parts):

\[
E'(t) = \int_0^L v_t(x, t)v(x, t)dx = \kappa \int_0^L (v(x, t))(v_{xx}(x, t))dx = \\
= -c_1 v(L, t)^2 - c_0 v(0, t)^2 - \kappa \int_0^L (v_x(x, t))^2 dx \leq 0
\]

By construction, \( E(t) \geq 0 \). The previous result on the derivative of \( E \) shows us that \( E \) is constant. As \( E(0) = 0 \), we have that \( E(9t) \equiv 0 \) and this forces \( v(x, t) = 0 \), i.e. \( u_1 = u_2 \).