a) 

If we substitute $v(x) = \frac{\sin(\alpha \ln x)}{x^{\frac{1}{2}}}$ in the pde, we get

$$\lambda = \alpha^2 + \frac{1}{4}$$

We are looking for eigenfunctions which, in our case, must satisfy $v(b) = 0$, so

$$v(b) = \frac{\sin(\alpha \ln b)}{b^{\frac{1}{2}}} = 0 \Leftrightarrow \alpha \in \frac{\pi}{\ln(b)} Z$$

Thus,

$$\lambda_n = \left( \frac{n\pi}{\ln(b)} \right)^2 + \frac{1}{4}$$

are the eigenvalues, and the associated eigenfunctions are

$$v_n(x) = \frac{\sin\left(\frac{n\pi}{\ln(b)} \ln x\right)}{x^{\frac{1}{2}}}$$

b) 

By separation of variables, we have $u(x, t) = T(t)X(x) = TX$ which, substituting in the pde, gives

$$T'X = (x^2 X')' T \Rightarrow \left\{ \begin{array}{l} T' + \lambda T = 0 \\ (x^2 X')' + \lambda X = 0 \end{array} \right.$$  

(1)

where the second equation corresponds to the one in item a. As we have a regular Sturm Liouville problem, proposition 6.28 allows us to assert that the family $v_n$ is orthonormal and complete, hence we can write

$$f(x) = \sum a_n v_n$$

$$a_n = \frac{\int_{a}^{b} f(x) v_n(x) dx}{\int_{a}^{b} v_n^2(x) dx}$$
6.4

We proceed as in example 6.40: we know that the eigenfunction associated to
the lowest eigenvalue is the minimum of the following functional
\[
\mathcal{F}(u) = -\frac{\int_0^1 u L[w] dx}{\int_1^b u^2 dx},
\]
where \(u'(0) = u(1) = 0\), so we only have to choose a function \(f\) satisfying these
boundary conditions, because
\[
\lambda_0 = \inf \mathcal{F}(u) \leq -\frac{\int_0^1 f L[f] dx}{\int_1^b f^2 dx}
\]
We will choose \(f(x) = 1 - x^2\), but we could just as well choose for instance
\(\cos(\frac{\pi}{2} x)\)
Computing the rhs in the above expression, we get
\[
\lambda_0 \leq \mathcal{F}(f) = -\frac{\int_0^1 (1 - x^2)(-2 - x^2(1 - x^2))dx}{\int_0^1 (1 - x^2)^2 dx} = \frac{37}{14}
\]

6.6

The main idea is to use the Rayleigh quotients and the proposition 6.26; we
only need to show that the lowest eigenvalue \(\lambda_0\) is positive. To do this, let \(u_0\)
be the eigenfunction corresponding to the first eigenvalue. Then:
\[
\lambda_0 = R(u_0) = -\frac{\int_0^1 u_0 L[u_0] dx}{\int_0^1 u_0^2 dx} = -\frac{\int_0^1 u_0 (u_0'' - x^2 u_0) dx}{\int_0^1 u_0^2 dx} = \frac{uu'|_0^1 + \int_0^1 (u_0')^2 + (x u_0)^2 dx}{\int_0^1 u_0^2 dx} = \frac{\int_0^1 (u_0')^2 + (x u_0)^2 dx}{\int_0^1 u_0^2 dx},
\]
which, clearly, is non-negative. We have to prove that \(R(u_0)\) can not be
zero. Suppose, by contradiction, that \(R(u_0) = 0\). It means that
\[
\frac{\int_0^1 (u_0')^2 + (x u_0)^2 dx}{\int_0^1 u_0^2 dx} = 0.
\]
As \(u_0 \in C^2([0,1])\), the above integral asserts that \(u_0'' \equiv 0\), and then it is a
constant c in \([0,1]\). Now, aware that \(u_0 = c\), we have
\[
R(u_0) = \frac{\int_0^1 + (xc)^2 dx}{\int_0^1 c^2 dx} = \frac{1}{3} = 0,
\]
a contradiction. We have then \(R(u_0) > 0\), and so all the eigenvalues are
positive.
6.9

a) It is a calculus exercise; you have only to integrate by parts.

b) Proceed as in proposition 6.21.

c) It’s easy to show that the eigenvalues must be positives, because

\[ \lambda \int_{-1}^{1} u^2 dx = - \int_{-1}^{1} u''u = -u'u|_{-1}^{1} + \int_{-1}^{1} (u')^2 dx = -u'(1)u(1) + u'(-1)u(-1) + \int_{-1}^{1} (u')^2 dx \]

Using the boundary conditions, note that

\[ -u'(1)u(1) + u'(-1)u(-1) = (u'(1) + u'(-1))u(-1) = 0, \]

which shows that \( \lambda \geq 0 \).

It is easy to see that \( \lambda = 0 \) gives us \( u = \text{constant} \), and then the boundary condition gives that \( u \equiv 0 \). We know that \( a \cos(\sqrt{\lambda})x) + b \sin(\sqrt{\lambda})x) \) is the solution to the system. We check the compatibility of the boundary conditions with the \( \lambda \) values

\[ u(1) + u(-1) = 2a \cos(\sqrt{\lambda})) = 0 \quad \Rightarrow \quad \lambda = \left(\frac{n\pi}{2}\right)^2 \]

It is easy to check that these values are also consistent with the other b.c’s. The eigenfunctions and respective eigenvalues are then

\[ \lambda_n = \left(\frac{n\pi}{2}\right)^2 \quad \text{and} \quad v_n = a_n \cos \left(\frac{n\pi}{2} x\right) + b_n \sin \left(\frac{n\pi}{2} x\right) \]

d) The ode we have get in (6.98) has second order. So the eigenspace has maximum dimension 2; the eigenfunctions \( \cos\left(\frac{n\pi}{2} x\right) \), \( \sin\left(\frac{n\pi}{2} x\right) \) shows that the dimension indeed is two.

e) This is not a regular Sturm-Liouville sytem, because the boundary conditions are not in the form (6.21).
By separation of variables, if we proceed as in (6.9 - 6.15), we get the following system:

\[
\begin{align*}
T' + \lambda T &= 0 \\
X'' + \lambda X &= 0
\end{align*}
\] (2)

We have already solved the eigenvalue problem for \(X\) in the previous exercise: we know that \(X = a \cos(\lambda x) + b \sin(\lambda x)\) represents the eigenfunctions; then, the boundary conditions show that:

\[
X(-\pi) = X(\pi), \quad X'(\pi) = X'(\pi)
\]

\[
X(-\pi) = a \cos(\sqrt{\lambda} \pi) - b \sin(\sqrt{\lambda} \pi) = a \cos(\sqrt{\lambda} \pi) + b \sin(\sqrt{\lambda} \pi) = X(\pi) \Rightarrow \sin(\sqrt{\lambda} \pi) = 0 \Rightarrow \lambda = n^2
\]

We write an eigenfunction expansion of the solution \(u(x, t)\):

\[
u(x, t) = \sum_{n=1}^{\infty} T_n(t)(a_n(t) \cos(nx) + b_n \sin(nx))
\] (3)

where \(T_n(t) = T_n(0)e^{-n^2t}\), by (2). The constant \(T_n(0)\) was incorporated in the constants \(a_n\) and \(b_n\) above. Taking \(t=0\) in (3), we have

\[
u(x, 0) = f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))
\]

As shown in class, we can use the orthogonality of the eigenfunctions to obtain the coefficients in the previous expansion\(^1\):

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2}
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx = \frac{1}{\pi} \left( \int_{-\pi}^{0} \cos(nx) \, dx \right) = 0
\]

Analogously,

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx = \frac{1}{\pi} \left( \int_{-\pi}^{0} \sin(nx) \, dx \right) = \frac{-1 + (-1)^n}{n\pi}
\]

We can write then:

\[
u(x, t) = \frac{1}{2} - \sum_{n=0}^{\infty} \frac{2}{\pi(2n+1)} \sin((2n+1)x)e^{-(2n+1)^2t}
\]

\(^1\)As before,

\[
\int_{-\pi}^{\pi} \sin^2(nx) \, dx = \int_{-\pi}^{\pi} \cos^2(nx) \, dx = \pi
\]
A

Proceeding as in (6.9 - 6.15), we obtain the following system:

\[
\kappa(x)X''(x) - (X(x)v(x))_x + \lambda X(x) = \kappa(x)X''(x) - X'(x)v(x) + (v(x) + \lambda)X(x) = 0
\]

(4)

As we are differentiating only in \(x\), we will denote \(X''\) by \(X'\). Using remark 6.3, we can use the following integrating factor\(^2\):

\[
p(x) = e^\int_{-\infty}^{x} \frac{-v(s)}{\kappa(s)} ds
\]

If we multiply the system in (4) by \(\frac{p(x)}{\kappa(x)}\), we get:

\[
p(x)X'' - X'\frac{p(x)v(x)}{\kappa(x)} + \frac{p(x)}{\kappa(x)}(v(x) + \lambda)X(x) = p(x)X'' + p'(x)X' + \frac{p(x)}{\kappa(x)}(v(x) + \lambda)X =
\]

\[
= (p(x)X')' + \frac{p(x)}{\kappa(x)}(v(x) + \lambda)X = 0.
\]

And we are done.

B

As we saw in exercise 6.9, the eigenfunctions associated to this problem are \(\cos(\lambda x)\) and \(\sin(\lambda x)\). Considering boundary conditions, it is not hard to prove that:

\[
\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad v_n = \sin\left(\frac{n\pi}{L}x\right),
\]

for \(n = 1, 2, \ldots\)

By definition 6.36, we have a Rayleigh quotient \(R(u)\) associated to the Sturm-Liouville problem \(u'' + \lambda u = 0, u(0) = u(L) = 0\).

\[
R(u) = -\frac{\int_0^L uL[u]dx}{\int_0^L u^2dx} = \frac{-uu'|_0^L + \int_0^L (u')^2dx}{\int_0^L u^2dx} = \frac{\int_0^L (u')^2dx}{\int_0^L u^2dx}
\]

We know that the infimum of the above functional is realized when \(u\) is the first eigenfunction of the Sturm-Liouville system (namely, the eigenfunction \(u_0\) associated with the minimum eigenvalue \(\lambda_0\)). Thus, for every function \(f\) satisfying the boundary conditions, we have:

\[
R(u_0) = \lambda_0 \leq R(f)
\]

As \(\lambda_0 = \left(\frac{\pi}{L}\right)^2\), the previous inequality shows that

\[
\int_0^L (f)^2dx \leq \left(\frac{L}{\pi}\right)^2 \int_0^L (f')^2dx.
\]

\(^2\)The chain rule gives then \(p'(x) = \frac{-v(x)}{\kappa(x)} p(x)\).