Let $g(x)$ be the Fourier sine series of $f$:

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right)$$  \hspace{1cm} (1)$$

where the $b_n$'s are defined by the following integral

$$b_n = \frac{2}{L} \int_{0}^{L} f(s) \sin\left(\frac{n\pi}{L} s\right) ds$$

The main idea is to rewrite each term in the expression (1), which can be done using integration by parts:

$$b_n = \frac{2}{L} \int_{0}^{L} f(s) \sin\left(\frac{n\pi}{L} s\right) ds = -\frac{2}{n\pi} f(s) \cos\left(\frac{n\pi}{L} s\right) \bigg|_{0}^{L} + \frac{2}{n\pi} \int_{0}^{L} f'(s) \cos\left(\frac{n\pi}{L} s\right) ds =$$

$$= \frac{2L}{(n\pi)^2} f'(s) \sin\left(\frac{n\pi}{L} s\right) \bigg|_{0}^{L} - \frac{2L}{(n\pi)^2} \int_{0}^{L} f''(s) \cos\left(\frac{n\pi}{L} s\right) ds = -\frac{2L}{(n\pi)^2} \int_{0}^{L} f''(s) \cos\left(\frac{n\pi}{L} s\right) ds.$$

Note that the values of $f$ on the boundary were used. Now, considering $f \in C^2$, we know that $f''$ is bounded in a compact interval $[0, L]$, i.e., $|f''| \leq M$.

We have then

$$|b_n \sin\left(\frac{n\pi}{L}\right)| \leq |b_n| = \left| \frac{2L}{(n\pi)^2} \int_{0}^{L} f''(s) \cos\left(\frac{n\pi}{L} s\right) ds \right| \leq \frac{2L}{(n\pi)^2} \int_{0}^{L} |f''(s) \cos\left(\frac{n\pi}{L} s\right)| ds$$

and as $|\cos()| \leq 1$ and $|f''| \leq M$,

$$|b_n| \leq \frac{2L}{(n\pi)^2} \int_{0}^{L} M ds = \frac{2ML^2}{(n\pi)^2} \sim \frac{constant}{n^2}.$$

We know that the series $\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)$ converges, which means that the series $g$ is uniformly convergent, due to Weierstrass M-test\(^1\).

\(^1\)Be careful with this statement. It doesn’t say that the series is converging uniformly to $f$, only that it is converging uniformly
B

First, we must check the compatibility conditions:

\[ u(x, 0) = f(x) \quad \Rightarrow \quad u(0, 0) = f(0) = u(L, 0) = f(L) = 0 \]

\[ u(0, t) = u(L, t) = 0 \quad \Rightarrow \quad u_t(0, t) = u_t(L, t) = 0 \quad \Rightarrow \quad u_t(0, 0) = g(0) = u_t(L, 0) = g(L) = 0 \]

Now, we write down the formula (3):

\[ \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right) \right] \sin\left(\frac{n\pi x}{L}\right) \]

We can find the coefficients \( a_n \) by evaluating the above expression when \( t = 0 \), setting it equal to \( f(x) \) and using orthogonality. Then by differentiating the series termwise in \( t \), plugging in \( t = 0 \) and setting it equal to \( g(x) \), we can similarly find the \( b_n' \)'s. Thus, we can write:

\[ a_n = \frac{2}{L} \int_{0}^{L} f(s) \sin\left(\frac{n\pi}{L} s\right) ds \]

and

\[ b_n = \frac{2}{n\pi} \int_{0}^{L} g(s) \sin\left(\frac{n\pi}{L} s\right) ds \]

Now, to prove that the series converges we must proceed as in exercise A: integrating by parts and using the fact that \( f \in C^2 \) and that \( g \in C^1 \). For the case \( g \), we would have:

\[ b_n = \frac{2}{n\pi} \int_{0}^{L} g(s) \sin\left(\frac{n\pi}{L} s\right) ds = -\frac{2L}{(n\pi)^2} g(0) \cos\left(\frac{n\pi}{L} s\right) \bigg|_{0}^{L} + \frac{2L}{(n\pi)^2} \int_{0}^{L} g'(s) \cos\left(\frac{n\pi}{L} s\right) ds \]

\[ = \frac{2L}{(n\pi)^2} \int_{0}^{L} g'(s) \cos\left(\frac{n\pi}{L} s\right) ds. \]

Thus, \( |b_n| \leq \text{Const.}/n^2 \). The case for estimating the \( a_n' \)'s is exactly as in exercise A.

C

i)

To write \( f \) as a cosine series we must extend the function periodically, in such a way that it is even. Here, we will extend \( f \) as follows:

\[ \tilde{f}(x) = \begin{cases} 
  f(x) & \text{if } x \in [0, 1] \\
  f(-x) & \text{if } x \in [-1, 0] 
\end{cases} \]

Now, the Fourier expansion for \( \tilde{f} \) doesn’t have the sin terms:
\[\hat{f} = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)\]

and now we can determine the coefficients\(^2\)

\[a_0 = \frac{9}{16}\]

\[a_n = \int_{-1}^{1} \hat{f} \cos\left(\frac{n\pi x}{2}\right) = 2 \int_{0}^{1} f(x) \cos\left(\frac{n\pi x}{2}\right) = 2 \left( \int_{0}^{1/2} (x + 1) \cos\left(\frac{n\pi x}{2}\right)dx + \int_{1/2}^{1} \cos\left(\frac{n\pi x}{2}\right)dx \right) = 2 \int_{0}^{1/2} x \cos\left(\frac{n\pi x}{2}\right)dx = \sin\left(\frac{n\pi}{4}\right) \left[ \frac{1}{n\pi} - \left( \frac{2}{n\pi} \right)^2 \right]\]

ii)

To write \(f\) as a sin series, we must extend \(f\) with an odd extension:

\[\tilde{f} = \begin{cases} f(x) & \text{if } x \in [0,1] \\ -f(-x) & \text{if } x \in [-1,0) \end{cases}\]

We have \(\tilde{f} = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\), and now we proceed as in the case (i). The final answer is

\[b_n = \frac{(-\sin\left(\frac{n\pi}{2}\right))[1 + 2n\sin\left(\frac{n\pi}{2}\right)]}{n^2 \pi}\]

D

i)

As in example 5.4, we assume that the solution \(u\) can be represented as a summation of eigenfunctions

\[u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin(nx) \quad (*)\]

Here, the eigenfunctions and respective eigenvalues associated to the homogeneous problem are

\[X_n = \sin(nx) \quad \lambda_n = n^2\]

we substitute (*) in the pde, obtaining

\[\sum_{n=1}^{\infty} (T_n' + n^2 T_n) \sin(nx) = e^{-t} \sin(3x),\]

and, by uniqueness of representation of a function as a Fourier series, the coefficients must be same. Thus, we obtain the following system:

\(^2\)Note, now we are working on the domain [-1,1].
\[
\begin{cases}
T_n'(t) + n^2 T_n(t) = 0 & \text{when } n \neq 3 \\
T_3'(t) + 9T_3(t) = e^{-t}
\end{cases}
\]

We can solve the second system in (3) using a integrant factor (in our case, \(e^{9t}\)). We get
\[
\begin{align*}
T_n &= T_n(0)e^{-n^2 t} & \text{when } n \neq 3 \\
T_3(t) &= T_3(0)e^{-9t} + \frac{1}{8}(e^{-t} - e^{-9t})
\end{align*}
\]

And then
\[
u(x, t) = \sum_{1}^{\infty} T_n(0)e^{-n^2 t} \sin(nx) + \frac{1}{8}(e^{-t} - e^{-9t}) \sin(3x) \tag{4}
\]

ii) Setting \(t = 0\) in (4), we obtain
\[
u(x, 0) = f(x) = \sum_{1}^{\infty} T_n(0) \sin(nx),
\]
which means that the r.h.s in the above equality is the fourier representation of the function \(f\). We must proceed as in the exercise C, extending \(f\) in an odd way.

\[
\tilde{f} = \begin{cases} 
 f(x) & \text{if } x \in [0, \pi] \\
 f(-x) & \text{if } x \in [-\pi, 0)
\end{cases}
\]

and now, we can obtain the coefficients of the above fourier expansion:
\[
T_n(0) = \frac{1}{\pi} \int_{\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \int_{\pi}^{\pi} x \sin(x) \sin(nx) dx
\]
\[
= \frac{1}{\pi} \int_{\pi}^{\pi} x (\cos((n-1)x) - \cos((n+1)x)) dx
\]

which gives
\[
T_n(0) = \begin{cases} 
0, & \text{if } n \text{ is odd} \\
\frac{-2}{\pi} \left( \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} \right), & \text{if } n \text{ is even}
\end{cases}
\]

iii) Just plug the values of \(T_n(0)\) we found above in (4) and substitute in the pde.