ON THE BEHAVIOR OF A SUPERCONDUCTING WIRE SUBJECTED TO A CONSTANT VOLTAGE DIFFERENCE *

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Abstract. We investigate a version of the time-dependent Ginzburg-Landau system that models a thin superconducting wire subjected to an applied voltage. Using a mixture of rigorous analysis, formal asymptotics and numerics, we analyze the behavior of solutions as the physical parameters of wire length, voltage and temperature are varied. Stable periodic solutions are shown to exist exhibiting phase slip centers (zeros of the order parameter), with period-doubling, period-tripling and chaotic behavior emerging in certain length/voltage regimes.

Key words. time-dependent Ginzburg-Landau equations, superconductivity, applied voltage

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1. Introduction. In this article, we will study the behavior of a thin superconducting wire subjected to a constant voltage difference $V$ across its ends. Earlier works by physicists, partly experimental, partly computational, have revealed an array of anomalous behavior ranging from ‘S’-shaped I-V curves to period doubling to the emergence of quasi-periodic states, cf. [6, 7, 11]. Our aim here is to initiate a systematic analysis of a version of the time-dependent Ginzburg-Landau model (TDGL), [5], adapted to this 1-d, constant voltage setting, as was used in [6, 7, 11]. Our techniques will include rigorous proof, formal asymptotic expansions and computational methods.

Physically, this problem is closely related to the situation in which a thin superconducting wire is subjected to a constant applied current. The applied current problem can also be modeled using a version of time-dependent Ginzburg-Landau, see e.g. [2], [3], [4], [9], [10]. A key feature shared by these two problems is the presence, in certain parameter regimes, of a so-called “resistive superconducting state” in which supercurrent and normal ohmic current co-exist. Another salient feature in common is the emergence of stable periodic states characterized by the periodic appearance of phase slip centers (PSC’s), i.e. zeros of the Ginzburg-Landau order parameter, at distinct locations on the wire. What is not shared by these two problems, among other things, is the rationale for these periodic solutions having PSC’s. In [12], [13] it was shown that the periodic solutions for the applied current problem arise through a Hopf bifurcation from the normal (non-superconducting) state as the temperature is lowered past some critical value depending on the current. There it was also shown that stable stationary superconducting states exist in certain parameter regimes.

For the constant voltage problem to be considered in this article, the normal state does not represent an equilibrium solution and indeed no stationary states can exist. Therefore, the periodicity here has nothing to do with a Hopf bifurcation but rather is driven by the $2\pi/V$ periodic boundary conditions (see 2.6). One thing that

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makes the problem so intriguing, however, is that by moving about in the parameter domain (voltage, wire length, temperature) one observes the disappearance of the stable $2\pi/V$-periodic solutions in favor of $2\pi k/V$-periodic solutions with $k \geq 2$, and ‘quasi-periodic’ solutions, with the number and location of the PSC’s within a period depending sensitively on the parameter values.

In the next section, we describe how one arrives at the time-dependent Ginzburg-Landau model for this setting. We also lay out some basic properties of the solution. Section 3 contains most of the rigorous mathematics and in it we establish the existence, uniqueness and global asymptotic stability of the period $2\pi/V$ solution to the problem when the wire (half)length $L$ lies below some critical value. In Section 4, we study the small voltage regime using the method of formal asymptotic expansions with multiple time-scales. In Section 5 we construct a formal boundary layer solution that captures the large voltage regime. In Section 6 we give a simple rigorous argument for the existence of at least one phase slip center per period. Then we present some of the numerical evidence for period doubling, period tripling and more anomalous behavior when $L$ and the voltage $V$ are increased. In particular, we focus on the variety of ways in which these PSC’s can appear.

2. Explanation of the model and basic properties. To describe the modeling consider first a standard 3-d TDGL system in which the superconducting portion of the wire occupies a thin cylindrical region $D$ with axis of length $2L$ centered on the $x$-axis. This thin region $D$ is assumed to bridge two bulk superconducting samples occupying regions $\Omega_L$ and $\Omega_R$, to the left and right of the wire, respectively. Thus, we have

$$
\psi_t + i\phi \psi = (\nabla - iA)^2 \psi + (\Gamma - |\psi|^2)\psi \quad \text{in} \ D \cup \Omega_L \cup \Omega_R, \\
\nabla \times \nabla \times A = -\sigma (A_t + \nabla \phi) + \frac{i}{2}(\psi \nabla \psi^* - \psi^* \nabla \psi) - |\psi|^2 A \quad \text{in} \ D \cup \Omega_L \cup \Omega_R,
$$

where $\psi : D \cup \Omega_L \cup \Omega_R \times [0, \infty) \rightarrow \mathbb{C}$ is the Ginzburg-Landau order parameter with the density of superconducting electrons given by $|\psi|^2$. $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the magnetic potential so that $\nabla \times A$ is the effective magnetic field, and $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ is the scalar electric potential whose negative gradient $-\nabla \phi$ represents the electric field. The parameter $\Gamma$ is proportional to $T_c - T$, where $T$ is the actual temperature and $T_c$ is the phase transition temperature in the absence of external currents or magnetic fields. In this investigation, we take the temperature, hence $\Gamma$, to be uniform throughout $D \cup \Omega_L \cup \Omega_R$. These equations are supplemented by Maxwell’s equations in the exterior of $D \cup \Omega_L \cup \Omega_R$. We note that under the gauge transformation $(\psi, \phi, A) \rightarrow (e^{i\chi}, \phi - \chi_t, A + \nabla \chi)$ for any smooth function $\chi$, the TDGL system is invariant.

Following [6, 7, 11], within the region $D$ occupied by the thin wire, one adopts a one-dimensional ansatz, i.e. we take $\psi = \psi(x, t)$, $\phi = \phi(x, t)$ and $A = A(x, t)$ for $-L < x < L$. Because the current carried by the thin wire is small and the induced magnetic field is also very small, one ignores the external problem off of the wire in this model. Then by taking $\chi$, i.e. choosing a gauge, such that $A + \chi_x = 0$ we can eliminate the magnetic potential. Under these gauge choices and assumptions, we replace (2.1)-(2.2) with the simpler TDGL system :

$$
\psi_t + i\phi \psi = \psi_{xx} + (\Gamma - |\psi|^2)\psi \quad \text{for} \ -L < x < L, \ t > 0, \\
\phi_{xx} = \frac{i}{2\sigma}(\psi \psi^*_x - \psi^* \psi_x)_x \quad \text{for} \ -L < x < L, \ t \geq 0,
$$

where the equation (2.4) arises as the one-dimensional consequence of conservation of
the total current, i.e. \( \text{div} \, J = 0 \). More precisely, the current total \( J \) obeys \( \frac{d}{dx} J = 0 \)

\[
J := -\sigma \phi_x + \frac{i}{2} (\psi \psi_x^* - \psi_x^* \psi) = \text{normal current} + \text{supercurrent}.
\]

The system (2.3)-(2.4) must be supplemented with boundary conditions. These are in some sense inherited from the conditions within the left and right bulk samples. To this end, one assumes that within the bulk samples \( \Omega_L \) and \( \Omega_R \), the superconducting state is homogeneous in the sense that \( |\psi|^2 = \Gamma \) and \( \psi = \psi(t) \) only, while at the same time the bulk samples are maintained at two different potentials, say \( \phi = 0 \) in \( \Omega_L \) while \( \phi = V \) in \( \Omega_R \). Applying (2.1) in \( \Omega_L \), one finds \( \psi_t \equiv 0 \) so that, up to a constant rotation, one has \( \psi = \sqrt{\Gamma} e^{-iVt} \). In summary, we arrive at the following boundary conditions for the problem (2.3)-(2.4):

\[
\psi(-L, t) = \sqrt{\Gamma}, \quad \psi(L, t) = \sqrt{\Gamma} e^{-iVt}, \quad \phi(-L, t) = 0, \quad \phi(L, t) = V \quad \text{for} \ t \geq 0.
\]

Finally, we specify initial conditions for the order parameter

\[
\psi(x, 0) = \psi_0(x) \quad \text{for} \quad -L \leq x \leq L.
\]

Our investigation focuses on the system (2.3)-(2.4), (2.6)-(2.7). In what follows we will set the conductivity \( \sigma = 1 \) and study an array of behaviors arising in the solution depending on the values of the temperature-dependent parameter \( \Gamma \), the wire’s length \( 2L \) and the voltage difference \( V \).

We point out that using (2.4) and the boundary conditions (2.6) one can solve for \( \phi \) in terms of \( \psi \) to find

\[
\phi = \phi[\psi] := \int_{-L}^{x} \text{Im}(\psi^* \psi_x)dx' + \frac{V}{2L} (x + L) - \frac{(x + L)}{2L} \int_{-L}^{x} \text{Im}(\psi^* \psi_x)dx'.
\]

In light of this, it will sometimes be convenient to write (2.3)-(2.4) simply as one nonlocal equation for \( \psi \), i.e.

\[
\psi_t = \psi_{xx} + (\Gamma - |\psi|^2) \psi - i\phi[\psi] \psi \quad \text{for} \quad -L < x < L, \; t > 0.
\]

This is in particular a relatively easy way to check that given smooth initial data \( \psi_0 \) compatible with the boundary data, there exists a unique smooth solution \( \psi \) to (2.9),(2.6),(2.7), since (2.9) is amenable to a standard fixed point argument.

Another easy property to check is that

\[
|\psi(x, t)| \leq \sqrt{\Gamma}
\]
provided that $|\psi_0| \leq \sqrt{\Gamma}$, which we shall assume throughout. This follows from the maximum principle applied to the PDE satisfied by $|\psi|^2$.

Before beginning our analysis we also point out that if one defines an energy via

$$E(\psi) := \int_{-L}^{L} \left( \frac{1}{2} |\psi_x|^2 + \frac{1}{4} (|\psi|^2 - \Gamma)^2 \right) dx$$

then any $\psi$ satisfying (2.3)-(2.4), (2.6)-(2.7) will obey the rule

$$\frac{d}{dt} E(\psi) = - \int_{-L}^{L} \left\{ |\psi_t + i \phi_\psi|^2 + |\phi_x|^2 \right\} dx - V \cdot J(t)$$

where the current $J$ is given by (2.5). The identity (2.12) is derived by direct calculation, see e.g. [8], p. 23 for a similar calculation. Note that (2.12) does not imply energy dissipation since in light of the boundary conditions (2.6), one expects $\phi_x > 0$ and so from (2.5), $J(t)$ tends to be negative.

3. Behavior for ‘short’ wires. The rigorous analysis of the problem will focus on the parameter regime where the length of the wire is not too large, that is, $L$ lies below some critical value. Our main result in this section, Theorem 3.5, asserts the existence of a unique, globally asymptotically stable solution having period $2\pi/V$, provided $L$ is less than a critical value depending only on $\Gamma$. Throughout this section, we will take the conductivity $\sigma$ to be 1 for simplicity of presentation only.

Theorem 3.1. Let $\psi_1$ and $\psi_2$ be any two solutions of the system (2.3)-(2.4), (2.6). Then there exists a value $L_0 = L_0(\Gamma)$ such that for all $V > 0$ and all $L \leq L_0$,

$$\int_{-L}^{L} |\psi_1(x,t) - \psi_2(x,t)|^2 dx \leq \left( \int_{-L}^{L} |\psi_1(x,0) - \psi_2(x,0)|^2 dx \right) e^{-t} \text{ for all } t > 0.$$  

Proof. Let $\phi_j(x,t) := \phi[j]$ where $\phi[j]$ is defined through (2.8) and let $w(x,t) := \psi_1(x,t) - \psi_2(x,t)$. Then $w(\pm L,t) = 0$ for all $t \geq 0$, and using equation (2.3) we find

$$w_t = (\psi_1)_t - (\psi_2)_t,$$

$$= w_{xx} + (\Gamma - |\psi_1|^2) w - i (\phi_1 - \phi_2) \psi_1 - i \phi_2 w - (|\psi_1|^2 - |\psi_2|^2) \psi_2.$$  

This implies

$$\int_{-L}^{L} w_t^2 dx = \int_{-L}^{L} (w^* w_1 + w w_1^*) dx$$

$$= -2 \int_{-L}^{L} |w_x|^2 dx + 2 \int_{-L}^{L} (\Gamma - |\psi_1|^2) |w|^2 dx + i \int_{-L}^{L} (\phi_1 - \phi_2) ((\psi_1)^* w - (\psi_1) w^*) dx$$

$$+ \int_{-L}^{L} ((\psi_2)^2 - |\psi_1|^2) ((\psi_2)^* w + (\psi_2) w^*) dx \quad \text{(by integration by parts)}$$

$$:= -2 \int_{-L}^{L} |w_x|^2 dx + I + II + III.$$  

First note that

$$I = 2 \int_{-L}^{L} (\Gamma - |\psi_1|^2) |w|^2 dx \leq 2\Gamma \int_{-L}^{L} |w|^2 dx.$$
Next, using (2.10) we see that

\[ II = i \int_{-L}^{L} \text{Im} \left\{ \int_{-L}^{x} (-w_{x}^{*} \psi_{1} + (\psi_{2})^{*} w_{x}) \, dx' \right\} \{ (\psi_{1})^{*} w - (\psi_{1}) w^{*} \} \, dx \]

\[ - i \int_{-L}^{L} \text{Im} \left\{ \frac{1}{2L} (x + L) \int_{-L}^{L} (-w_{x}^{*} \psi_{1} + (\psi_{2})^{*} w_{x}) \, dx' - w^{*} \psi_{1} \right\} \{ (\psi_{1})^{*} w - (\psi_{1}) w^{*} \} \, dx \]

(by integration by parts)

\[ \leq \int_{-L}^{L} \left( 4\sqrt{\Gamma} \int_{-L}^{L} |w_{x}| \, dx' + \sqrt{\Gamma} |w| \right) \left( 2\sqrt{\Gamma} |w| \right) \, dx \]

\[ \leq 8L \Gamma \int_{-L}^{L} |w_{x}|^{2} \, dx + 2\Gamma(4L + 1) \int_{-L}^{L} |w|^{2} \, dx. \]

The penultimate inequality follows simply from repeated use of the triangle inequality and (2.10).

Finally, turning to the term of \( III \) in equation (3.3), we estimate that

\[ III = \int_{-L}^{L} |\psi_{1}|^{2} |\psi_{2}| \left( |\psi_{2}| - |\psi_{1}| \right) \left( (\psi_{2})^{*} w + (\psi_{2}) w^{*} \right) \, dx \]

\[ \leq 4 \Gamma \int_{-L}^{L} ||\psi_{2}|| - |\psi_{1}||w| \, dx \leq 4 \Gamma \int_{-L}^{L} |w|^{2} \, dx \]

Now, let us require that \( L \leq L_{0} := \min \{ \frac{1}{8 \Gamma}, \frac{\pi}{2 \sqrt{2 + 8 \Gamma}} \} \). Note that this condition implies

\[ 2(4L \Gamma - 1) \leq -1 \quad \text{and} \quad -\frac{\pi^{2}}{4L^{2}} + 1 + 8 \Gamma \leq -1. \]

Applying our estimates for \( I, II \) and \( III \) and (3.4) into equation (3.3), we obtain through an appeal to the Poincaré inequality that

\[ \frac{d}{dt} \int_{-L}^{L} |w|^{2} \, dx \leq - \int_{-L}^{L} |w_{x}|^{2} \, dx + (1 + 8 \Gamma) \int_{-L}^{L} |w|^{2} \, dx, \]

\[ \leq \left( -\left( \frac{\pi}{2L} \right)^{2} + 1 + 8 \Gamma \right) \int_{-L}^{L} |w|^{2} \, dx \leq - \int_{-L}^{L} |w|^{2} \, dx. \]

That is,

\[ \frac{d}{dt} \int_{-L}^{L} |w|^{2} \, dx \leq - \int_{-L}^{L} |w|^{2} \, dx \quad \text{if} \quad L \leq L_{0} := \min \{ \frac{1}{8 \Gamma}, \frac{\pi}{2 \sqrt{2 + 8 \Gamma}} \}. \]

Thus we have

\[ \int_{-L}^{L} |w(x, t)|^{2} \, dx \leq \left( \int_{-L}^{L} |w(x, 0)|^{2} \, dx \right) e^{-t}, \]

which is equation (3.1). \( \Box \)

Now, we will use the energy inequality (2.12) to get a preliminary bound on \( \psi_{x} \) in the following Lemma. Later we will improve on this bound.
Lemma 3.2. Let $\psi$ be any solution of the system (2.3)-(2.4), (2.6). Then

\[ \int_{-L}^{L} |\psi_x(x,t)|^2 dx \leq \left( tV \sqrt{\frac{T}{2L}} + \sqrt{2E(\psi(x,0))} \right)^2 \text{ for all } t > 0. \]

Proof. Before we start, we note that the total current is

\[ J(t) = \frac{1}{2L} \int_{-L}^{L} J(t) dx = -\frac{V}{2L} + \frac{i}{4L} \int_{-L}^{L} (\psi_x^* - \psi^* \psi_x) dx \]

and using the Cauchy-Schwarz inequality

\[ V^2 = \left( \int_{-L}^{L} \phi_x dx \right)^2 \leq 2L \int_{-L}^{L} |\phi_x|^2 dx. \]

From (2.12), (3.6) and (3.7), we have

\[ \frac{d}{dt} E(\psi) = -\int_{-L}^{L} |\psi_x + i\phi|^2 + |\phi_x|^2 dx - V \cdot J(t) \leq -\int_{-L}^{L} |\phi_x|^2 dx - V \cdot J(t) \]

\[ \leq -\frac{V^2}{2L} - V \left( -\frac{V}{2L} + \frac{i}{4L} \int_{-L}^{L} (\psi_x^* - \psi^* \psi_x) \right) \leq \frac{V \sqrt{T}}{2L} \int_{-L}^{L} |\psi_x| dx. \]

Integrating equation (3.8) with respect to $t$ from 0 to $T$, we have

\[ E(\psi(x,T)) - E(\psi(x,0)) \leq \frac{V}{2L} \sqrt{T} \int_{0}^{T} \int_{-L}^{L} |\psi_x| dx dt, \]

and as a consequence, we obtain the estimate

\[ \frac{1}{2} \int_{-L}^{L} |\psi_x(x,t)|^2 dx \leq E(\psi(x,0)) + \frac{V}{2L} \sqrt{T} \int_{0}^{t} \int_{-L}^{L} |\psi_x(x, s)| dx ds. \]

That is,

\[ \int_{-L}^{L} |\psi_x(x,t)|^2 dx \leq 2E(\psi(x,0)) + V \sqrt{\frac{2T}{L}} \int_{0}^{t} \left( \int_{-L}^{L} |\psi_x(x, s)|^2 dx \right)^{\frac{1}{2}} ds. \]

To solve this inequality, let $f(t) = \int_{0}^{t} \left( \int_{-L}^{L} |\psi_x(x, s)|^2 dx \right)^{\frac{1}{2}} ds$. Then equation (3.9) takes the form

\[ (f'(t))^2 \leq 2E(\psi(x,0)) + V \sqrt{\frac{2T}{L}} f(t). \]

Solving this differential inequality for $f$ yields

\[ f(t) \leq \frac{1}{V} \sqrt{\frac{L}{2T}} \left( \frac{TV^2}{2L} t^2 + 2tV \sqrt{\frac{T}{L} E(\psi(x,0))} \right). \]
That is,
\[
\int_0^t \left( \int_{-L}^L |\psi_x(x,s)|^2 dx \right)^{\frac{3}{2}} ds \leq \frac{1}{V} \sqrt{\frac{L}{2T}} \left( \frac{GV^2}{2L} t^2 + 2tV \sqrt{\frac{E(\psi(x,0))}{L}} \right).
\]

Applying this inequality to equation (3.9), we obtain
\[
\int_{-L}^L |\psi_x(x,t)|^2 dx \leq 2E(\psi(x,0)) + V \sqrt{\frac{2T}{L/V}} \left( \frac{GV^2}{2L} t^2 + 2tV \sqrt{E(\psi(x,0))} \right)
\]
\[
= \left( tV \sqrt{\frac{\Gamma}{2L}} + \sqrt{2E(\psi(x,0))} \right)^2.
\]
which is (3.5).

**Proposition 3.3.** Let \( \psi_1 \) and \( \psi_2 \) be any two solutions of the system (2.3)-(2.4), (2.6). Then there exists a positive value \( L_0 = L_0(\Gamma) \) such that for all \( V > 0 \) and all \( L \leq L_0 \),
\[
\int_{-L}^L |(\psi_1)_x(x,t) - (\psi_2)_x(x,t)|^2 dx \leq \left( \int_{-L}^L |(\psi_1)_x(x,0) - (\psi_2)_x(x,0)|^2 dx \right) e^{-t} + c_0 t^3 e^{-t}
\]
(3.10) for some \( c_0 = c_0(L, \Gamma, V) > 0 \).

**Proof.** As before, let \( \phi_j(x,t) := \phi(\psi_j) \), cf. (2.8), and let \( w(x,t) := \psi_1(x,t) - \psi_2(x,t) \). By equation (3.2), direct calculation and an integration by parts one finds that
\[
\frac{d}{dt} \int_{-L}^L |w_x|^2 dx = - \int_{-L}^L w_{xx} w_x^* + w_1 w_{xx}^* dx
\]
\[
= -2 \int_{-L}^L |w_{xx}|^2 dx - \int_{-L}^L (|\psi_1|^2 - |\psi_2|^2) (w^* w_{xx} + w_{xx} w^*) dx - i \int_{-L}^L \phi_2 (w_{xx} w^* - w_{xx}^* w) dx
\]
\[
+ \int_{-L}^L (|\psi_1|^2 - |\psi_2|^2) (w_{xx}^* w^* + w^* w_{xx}) dx - i \int_{-L}^L (\phi_1 - \phi_2) (w_{xx}^* - w_{xx}) \psi_1 dx
\]
\[
= -2 \int_{-L}^L |w_{xx}|^2 dx + P + Q + R + S.
\]
As a consequence of Theorem 3.1 we find
\[
P \leq 2\Gamma \int_{-L}^L |w| w_{xx} dx \leq \frac{1}{8} \int_{-L}^L |w_{xx}|^2 dx + 8\Gamma^2 \int_{-L}^L |w|^2 dx
\]
\[
\leq \frac{1}{8} \int_{-L}^L |w_{xx}|^2 dx + 8\Gamma^2 c_1 e^{-t} \quad \text{for some} \quad c_1 > 0.
\]

Next, to estimate \( Q \) we note that
\[
|\phi(\psi)| \leq 2\sqrt{T} \int_{-L}^L |\psi_x| dx + V \leq 2\sqrt{2\Gamma L} \left( \int_{-L}^L |\psi_x|^2 dx \right)^{\frac{1}{2}} + V
\]
\[(3.12) \quad \leq 2\sqrt{2TL} \left( tV\sqrt{\frac{\Gamma}{2L}} + \sqrt{2E(\psi(x,0))} \right) + V \quad \text{(by Lemma 3.2)} \]
\[= 2\Gamma t + 4\sqrt{\Gamma LE(\psi(x,0))} + V. \]

Then we see from (3.12) and Theorem 3.1 that
\[Q \leq 2 \left( 2\Gamma t + 4\sqrt{\Gamma LE(\psi(x,0))} + V \right) \int_{-L}^{L} |w_{xx}| |w| dx \]
\[\leq 8 \left( 2\Gamma t + 4\sqrt{\Gamma LE(\psi(x,0))} + V \right)^2 c_2 e^{-t} + \frac{1}{8} \int_{-L}^{L} |w_{xx}|^2 dx. \]

Turning to the term \( R \) in equation (3.11), we again invoke Theorem 3.1 to estimate that
\[R \leq \int_{-L}^{L} \left| \psi_2 \right| + \left| \psi_1 \right| |\psi_2| - \left| \psi_1 \right| |(\psi_2)^*w_{xx} + (\psi_2)w_{xx}^*| dx \]
\[\leq 4\Gamma \int_{-L}^{L} \left| \psi_2 \right| - \left| \psi_1 \right| |w_{xx}| dx \leq 4\Gamma \int_{-L}^{L} |w||w_{xx}| dx \]
\[\leq 32\Gamma^2 \int_{-L}^{L} |w|^2 dx + \frac{1}{8} \int_{-L}^{L} |w_{xx}|^2 dx \leq 32\Gamma^2 c_3 e^{-t} + \frac{1}{8} \int_{-L}^{L} |w_{xx}|^2 dx \text{ for some } c_3 > 0. \]

Finally, we estimate \( S \) in a manner similar to that used to estimate \( II \) in the proof of Theorem 3.1:
\[S \leq \int_{-L}^{L} \left( 4\sqrt{\Gamma} \int_{-L}^{L} |w_x| dx + \sqrt{\Gamma} |w| \right) \left( 2\sqrt{\Gamma} |w_{xx}| \right) dx \]
\[\leq 16\Gamma^2 \left( \int_{-L}^{L} |w_x| dx \right)^2 + \left( \int_{-L}^{L} |w_{xx}| dx \right)^2 + 8\Gamma^2 \int_{-L}^{L} |w|^2 dx + \frac{1}{8} \int_{-L}^{L} |w_{xx}|^2 dx \]
\[\leq 32\Gamma^2 L \int_{-L}^{L} |w|^2 dx + 2L \int_{-L}^{L} |w_{xx}|^2 dx + 8\Gamma^2 c_4 e^{-t} + \frac{1}{8} \int_{-L}^{L} |w_{xx}|^2 dx \text{ for some } c_4 > 0. \]

Now we require that
\[(3.13) \quad L \leq L_0 := \min \left\{ \frac{1}{4}, \frac{1}{8\Gamma}, \frac{\pi}{2\sqrt{1 + 4\Gamma}} \right\}, \]
which in particular implies that
\[(3.14) \quad -\frac{3}{2} + 2L \leq -1, \quad \text{and} \quad -\frac{\pi^2}{4L^2} + 4\Gamma \leq -1. \]

Applying our estimates for \( P, Q, R, \) and \( S \) and (3.13)-(3.14) to equation (3.11), we obtain
\[\frac{d}{dt} \int_{-L}^{L} |w_x|^2 dx \leq \left( -2 + 2L + \frac{1}{2} \right) \int_{-L}^{L} |w_{xx}|^2 dx + 32\Gamma^2 L \int_{-L}^{L} |w_x|^2 dx + ct^2 e^{-t} \]
\[\leq -\int_{-L}^{L} |w_{xx}|^2 dx + 4\Gamma \int_{-L}^{L} |w_x|^2 dx + ct^2 e^{-t} \text{ for some } c := c(L, \Gamma, V) > 0. \]
We may then invoke the Poincaré inequality since ∫_{-L}^{L} w_{x} dx = 0, along with (3.14) to find that
\[ \frac{d}{dt} \int_{-L}^{L} |w_{x}|^2 dx \leq -\frac{\pi^2}{4L^2} \int_{-L}^{L} |w_{x}|^2 dx + 4\Gamma \int_{-L}^{L} |w_{x}|^2 dx + ct^2 e^{-t} \]
\[ \leq -\int_{-L}^{L} |w_{x}|^2 dx + ct^2 e^{-t}. \]

Finally, using Grönwall’s inequality, we have
\[ \int_{-L}^{L} |w_{x}(x, t)|^2 dx \leq \left( \int_{-L}^{L} |w_{x}(x, 0)|^2 dx \right) e^{-t} + ct^3 e^{-t} \quad \text{for some} \quad c_0 = c_0(L, \Gamma, V) > 0. \]

**Corollary 3.4.** Let ψ₁ and ψ₂ be any two solutions of the system (2.3)-(2.4), (2.6). Then there exists a value L₀ = L₀(Γ) such that for all V > 0 and all L ≤ L₀,
\[ (3.15) \]
\[ \int_{-L}^{L} |\psi_{1}(x, t) - \psi_{2}(x, t)|^2 dx + \int_{-L}^{L} |(\psi_{1})_{x}(x, t) - (\psi_{2})_{x}(x, t)|^2 dx \]
\[ \leq \left( \int_{-L}^{L} |\psi_{1}(x, 0) - \psi_{2}(x, 0)|^2 dx + \int_{-L}^{L} |(\psi_{1})_{x}(x, 0) - (\psi_{2})_{x}(x, 0)|^2 dx \right) e^{-t} + ct^3 e^{-t} \]
\[ \quad \text{for some} \quad c_0 = c_0(L, \Gamma, V) > 0. \]

**Proof.** This follows from Theorem 3.1 and Proposition 3.3. □

We now use Theorem 3.1 and Corollary 3.4 to establish the existence, uniqueness, and asymptotic stability of a \( \frac{2\pi}{V} \)-time periodic solution to our problem when \( L \) is sufficiently small.

**Theorem 3.5.** For all \( L \leq L_0(\Gamma) \) and all \( V > 0 \), there exists exactly one time-periodic solution of the system (2.3)-(2.4), (2.6). This solution has period \( \frac{2\pi}{V} \). Moreover, this solution is globally asymptotically stable in \( H^1((-L, L)) \).

**Proof.** We begin with the proof of existence. To this end, let \( \psi \) be a solution (2.3)-(2.4), (2.7) where \( \psi_0 \) is an arbitrary smooth initial condition compatible with the boundary conditions. For all \( x \in [-L, L] \) and for all \( j = 0, 1, 2, \ldots \), set \( \psi^{(j)}(x) := \psi(x, \frac{2\pi j}{V}) \). Applying Corollary 3.4 with \( \psi_{1} = \psi \) and \( \psi_{2} = \psi(x, t + \frac{2\pi}{V}) \), there exists \( L_0 = L_0(\Gamma) \) such that for all \( V > 0 \) and all \( L \leq L_0 \), \( \|\psi^{(j+1)} - \psi^{(j)}\|_{H^1((-L, L))} \leq c_0 e^{-\frac{2\pi}{V} j} \) for some constant \( c > 0 \). Thus, \( \{\psi^{(j)}\} \) is a Cauchy sequence in \( H^1((-L, L)) \), so that \( \psi^{(j)} \) converges strongly in \( H^1((-L, L)) \) to a limit \( \psi_{\infty} \). Moreover, since in one dimension the sup-norm is dominated by the \( H^1 \)-norm, the sequence also converges uniformly on \([-L, L]\). Thus \( \psi_{\infty} \) is continuous and it satisfies the boundary conditions satisfied by \( \psi^{(j)} \). That is, \( \psi_{\infty}(\pm L) = \sqrt{\Gamma} \).

Now, let \( \psi_{\infty} \) be the solution of (2.3)-(2.4), (2.6) having the initial condition \( \psi_{\infty}(x, 0) = \psi_{\infty}(x) \). Then \( \psi_{\infty}(x, \frac{2\pi}{V}) = \psi_{\infty}(x) \), since by Theorem 3.1
\[ \|\psi_{\infty}(\cdot, \frac{2\pi}{V}) - \psi_{\infty}^2\|_{L^2(-L, L)} \leq \|\psi_{\infty}(\cdot, \frac{2\pi}{V}) - \psi^{(j+1)}\|_{L^2(-L, L)}^2 + \|\psi^{(j+1)} - \psi_{\infty}\|_{L^2(-L, L)}^2 \]
\[ \leq \|\psi_{\infty} - \psi^{(j)}\|_{L^2(-L, L)}^2 e^{-\frac{2\pi}{V} j} + \|\psi^{(j+1)} - \psi_{\infty}\|_{L^2(-L, L)}^2 \].
and these norms approach zero as $j \to \infty$ in light of the $L^2$-convergence of the sequence $\{\psi^{(j)}\}$. By the same argument, we have $\psi^{\infty}(x, \frac{\tau_2}{\tau_1}j) = \psi^{\infty}_0(x)$ for all $j$. That is, there exists a $\frac{\tau_2}{\tau_1}$-time periodic solution of the system (2.3)-(2.4), (2.6).

For the uniqueness argument, let $\psi_1$ and $\psi_2$ be any two time-periodic solutions of the system (2.3)-(2.4), (2.6). Then by Corollary 3.4 we have

\begin{equation}
(3.16)
\int_{-L}^{L} |\psi_1(x, t) - \psi_2(x, t)|^2 + |(\psi_1)_x(x, t) - (\psi_2)_x(x, t)|^2 dx \\
\leq \left( \int_{-L}^{L} |\psi_1(x, 0) - \psi_2(x, 0)|^2 + |(\psi_1)_x(x, 0) - (\psi_2)_x(x, 0)|^2 dx \right) e^{-t} + c_0 t^3 e^{-t}.
\end{equation}

Thus, the difference approaches zero as $t \to \infty$. But the boundary conditions (2.6) imply that $\psi_1 - \psi_2$ must be periodic with period $\frac{\tau_2}{\tau_1}k$ for some integer $k$ and so this is only possible if $\psi_1 = \psi_2$.

To establish the global asymptotic stability of the time-periodic solution $\psi^{\infty}$, let $\tilde{\psi}$ be any other solution of (2.3)-(2.4), (2.6) subject to an arbitrary initial condition $\tilde{\psi}_0$. Again applying Corollary 3.4, we have

\begin{align*}
\int_{-L}^{L} |\psi^{\infty}(x, t) - \tilde{\psi}(x, t)|^2 + |\psi^{\infty}_x(x, t) - \tilde{\psi}_x(x, t)|^2 dx \\
\leq \left( \int_{-L}^{L} |\psi^{\infty}(x, 0) - \tilde{\psi}_0(x)|^2 + |\psi^{\infty}_x(x, 0) - \tilde{\psi}_0(x)|^2 dx \right) e^{-t} + c_0 t^3 e^{-t} \\
\to 0 \text{ as } t \to \infty.
\end{align*}

4. The small voltage regime. In an effort to further explore how the solution depends upon the physical parameters of problem (2.3)-(2.4), (2.6)-(2.7), we investigate next the asymptotic regime in which the voltage $V$ is assumed to be small. Below we present conclusions based on formal asymptotics using a multiple time-scale expansion. In future work, we hope to rigorize these results.

To this end, let us write $V = \epsilon$ in this section, where we assume $0 < \epsilon \ll 1$ and let us also introduce a “slow” time-scale $\tau$ via $\tau := ct$. Note that in this notation, the boundary conditions (2.6) can be written as

\begin{equation}
(4.1) \quad \psi(-L, \tau) = \sqrt{\Gamma}, \quad \psi(L, \tau) = \sqrt{\Gamma} e^{-i\tau}, \quad \phi(-L, \tau) = 0, \quad \phi(L, \tau) = \epsilon.
\end{equation}

In seeking an asymptotic expansion for the solution, one might be tempted to look for $\psi$ and $\phi$ as functions of $x$ and $\tau$ only, but this approach would fail to account for a brief—compared with the $\tau$ time-scale— transient period in which the solution accommodates a general initial condition (2.7). Thus we are led to pursue a solution to (2.3)-(2.4), (4.1), (2.7) taking the form

\begin{equation}
(4.2) \quad \psi \sim \psi^0(x, t, \tau) + \epsilon \psi^1(x, \tau) + \ldots, \quad \phi \sim \phi^0(x, t, \tau) + \epsilon \phi^1(x, \tau) + \ldots
\end{equation}

Inserting (4.2) into the system, the leading order problem is

\begin{equation}
(4.3) \quad \psi^{0}_t + i \phi^0 \psi^0 = \psi^{0}_{xx} + (\Gamma - |\psi^0|^2) \psi^0,
\end{equation}
We find voltage is very small, one should expect the emergence of a solution of period 2 in the small voltage regime brings the solution to the constant state initial transition layer described earlier that takes place on the pre-determined to be one cannot specify an initial condition for (4.11) or (4.12) as the solution at again \( \psi \) and these serve as initial conditions in capturing the leading order behavior on the slow time-scale.


Let us first analyze the behavior of \( \psi \) and \( \phi \) in the initial stage of the evolution in which \( \tau \approx 0 \), that is, on a time interval that is \( O(\frac{1}{\epsilon}) \). Within this initial period, the boundary conditions for \( \psi \) become simply

\[
\psi^0(\pm L, 0, \tau) = \sqrt{\Gamma}.
\]

One readily checks that problem (4.3)-(4.4), (4.6)-(4.8) is dissipative. Indeed, a calculation similar to that used to derive (2.12), yields the fact that the energy (2.11) will obey the rule

\[
\frac{d}{dt} E(\psi^0)(t, \tau = 0) = - \int_{-L}^{L} \left\{ |\psi^0_1(x, t, 0) + i\phi^0(x, t, 0)\psi^0(x, t, 0)|^2 + |\phi^0_0(x, t, 0)|^2 \right\} dx.
\]

Consequently, one can argue that in the regime \( 1 \leq t \leq 1 \), \( \psi^0 \) and \( \phi^0 \) must tend towards a stable equilibrium of the system. Since the time integral of the right-hand side of (4.9) must be finite, one finds that \( \phi_0 \) must tend to zero (at least along a subsequence) and so by the homogeneous boundary conditions (4.6), \( \phi^0(x, t, \tau) \) must tend to a zero equilibrium state for \( t \gg 1 \). On light of (4.9), note that \( \psi^0 \) should then tend to zero as well. We then expect \( \psi^0 \) to tend to the global energy minimizer of (2.11) satisfying (4.8), namely the constant function \( \sqrt{\Gamma} \). In summary, over the initial transient period we find

\[
\lim_{t \to \infty} \psi^0(x, t, 0) = \sqrt{\Gamma} \quad \text{and} \quad \lim_{t \to \infty} \phi^0(x, t, 0) = 0
\]

and these serve as initial conditions in capturing the leading order behavior on the slow \( \tau \) time-scale.

Having dispensed with this early transition layer, we can solve (4.3)-(4.6), (4.10) by taking \( \phi^0(x, \infty, \tau) \equiv 0 \) and by taking \( \psi^0(x, \infty, \tau) \) as the solution to

\[
\psi^0_{xx} + (\Gamma - |\psi^0|^2) \psi^0 = 0, \quad \psi^0(-L, \infty, \tau) = \sqrt{\Gamma}, \quad \psi^0(L, \infty, \tau) = \sqrt{\Gamma} e^{-i\tau}.
\]

Equivalently, one can exploit the variational characterization of (4.11) and take \( \psi^0(x, \infty, \tau) \) as the solution to

\[
\inf \left\{ E(\psi): \psi(-L, \tau) = \sqrt{\Gamma}, \quad \psi(L, \tau) = \sqrt{\Gamma} e^{-i\tau} \right\},
\]

where again \( E \) is given by (2.11) and where \( \tau \) appears simply as a parameter evolving the solution periodically through the boundary condition at \( x = L \). Note as such that one cannot specify an initial condition for (4.11) or (4.12) as the solution at \( \tau = 0 \) is pre-determined to be \( \psi^0(x, \infty, \tau) \equiv \sqrt{\Gamma} \). This once again highlights the need for the initial transition layer described earlier that takes place on the \( t \) time-scale and which brings the solution to the constant state \( \sqrt{\Gamma} \) when \( t \gg 1 \) yet \( \tau \ll 1 \).

This completes a description of the leading order behavior of the solution \( (\psi, \phi) \) in the small voltage regime \( V = \epsilon \ll 1 \). The main conclusion is that when the applied voltage is very small, one should expect the emergence of a solution of period \( 2\pi/V \).
with the evolution of the order parameter $\psi$ governed approximately by (4.11) or (4.12)—in particular, there should be no period-doubling solution. Of course, to this order, the electric potential is asymptotically zero, while for the actual solution, the electric potential is clearly nontrivial due to the applied voltage. However, since the voltage difference in this regime is $O(\epsilon)$, this correction is a lower order effect and we do not pursue it further here.

As a check on the validity of our expansion, we took $\Gamma = 1$, $L = 1$ and $V = 0.01$ and at various times we computed the difference in the spatial Sobolev norm $H^1$ between the numerically computed solution to (2.3)-(2.4), (2.6) with constant initial condition 1 and the solution to (4.11). After sampling various times during a period, we found good agreement, with the $H^1$-difference not exceeding 0.006 when using a time resolution of 0.002 and spatial resolution of 0.01.

5. The large voltage regime. We now turn to an analysis of the solution to (2.3)-(2.4), (2.7) in the setting where the voltage difference $V$ is large. For large values of the voltage, one expects that the current will be largely normal, with the supercurrent suppressed. However, the boundary conditions (2.6) prevent the modulus of the order parameter from being uniformly small, resulting in the formation of boundary layers near the left and right endpoints of the wire.

As was done in Section 4, we will pursue a description of the leading order behavior of the solution via formal matched asymptotic expansions. Unlike the previous section, however, here we will only use one temporal scale but will use two spatial scales with a stretched variable introduced to capture the boundary layer effect. For simplicity of presentation only, we will set $\Gamma = 1$ for the remainder of this section.

To begin the analysis, we set $\epsilon := 1/V$ so that the large voltage assumption reads as $\epsilon \ll 1$, we rescale time via $\tau := t/\epsilon$ and rescale the electric potential via $\tilde{\phi} = \phi/\epsilon$.

Then (2.3)-(2.4), (2.6) transform to:

\begin{align}
(5.1) & \quad \psi_\tau + i \tilde{\phi} \psi = \epsilon \left( \psi_{xx} + (1 - |\psi|^2) \psi \right) \quad \text{for } -L < x < L, \quad \tau > 0, \\
(5.2) & \quad \tilde{\phi}_{xx} = \epsilon i \frac{1}{2} \left( \psi \psi_x^* - \psi^* \psi_x \right)_x \quad \text{for } -L < x < L, \quad \tau \geq 0,
\end{align}

with the boundary conditions now taking the form

\begin{align}
(5.3) & \quad \psi(-L, \tau) = 1, \quad \psi(L, \tau) = e^{-i \tau}, \quad \tilde{\phi}(-L, \tau) = 0, \quad \tilde{\phi}(L, \tau) = 1 \quad \text{for } \tau \geq 0.
\end{align}

To these conditions we add the initial condition

\begin{align}
(5.4) & \quad \psi(x, 0) = \psi_0(x) \quad \text{for } -L \leq x \leq L,
\end{align}

which we take to be compatible with the boundary conditions.

Referring back to (2.8), we prefer here solve for $\tilde{\phi}$ as a function of $\psi$ and so to replace (5.2) and (5.3) with

\begin{align}
(5.5) & \quad \tilde{\phi}(x, \tau) = \Phi_0(x) + \epsilon i \left\{ \int_{-L}^{x} (\psi \psi_x^* - \psi^* \psi_x) \, dx' - \frac{(x + L)}{2L} \int_{-L}^{x} (\psi \psi_x^* - \psi^* \psi_x) \, dx' \right\},
\end{align}

where we have introduced the linear profile

\begin{align}
(5.6) & \quad \Phi_0(x) := \frac{(x + L)}{2L}.
\end{align}
An advantage of this formulation is that we will only need to pursue an expansion for the one unknown $\psi$ and then an expansion for $\tilde{\phi}$ will automatically be induced by (5.5). A considerable disadvantage, however, is that since the expression for $\tilde{\phi}$ involves integration over the entire interval $-L \leq x \leq L$, the correction to $\Phi_0$ on the right-hand side of (5.5) will involve consideration of both the outer and inner (boundary layer) solutions for $\psi$. Our reason for taking this tack then is not so much for simplicity but out of necessity: we seek a (formal) argument that justifies the claim that the electric potential is close to the linear profile with an error small enough to ignore at various junctures in our presentation.

Our asymptotic solution will consist of a transient solution valid up to any time $\tau$ satisfying $0 \leq \tau \ll \frac{1}{\epsilon}$ so that in particular, terms of order $O(\epsilon \tau)$ remain $O(1)$. The boundary layer solutions we will consider, both in the transient and steady-state parts of the evolution, will be of width $\epsilon^{1/3}$. The reason behind the choice of the power $\frac{1}{3}$ will become apparent once we construct the boundary layer solutions.

Outer transient solution (valid for $-L + \epsilon^{1/3} < x < L - \epsilon^{1/3}$ and for $0 \leq \tau \ll \frac{1}{\epsilon}$): Before proceeding with this expansion, we find it convenient to recast (5.1) in terms of the function $\Psi$ given by

$$
\psi(x, \tau) = \Psi(x, \tau)e^{-i\Phi_0(x)\tau}
$$

rather than in terms of $\psi$. Then (5.1) takes the form, for $-L < x < L$, $\tau > 0$,

$$
\Psi_x + i(\tilde{\phi} - \Phi_0)\Psi = \epsilon \left( \Psi_{xx} - \frac{i\tau}{L} \Psi_x - \frac{\tau^2}{4L^2} \Psi + (1 - |\Psi|^2) \Psi \right).
$$

and (5.5) takes the form

$$
\tilde{\phi}(x, \tau) - \Phi_0(x) = \frac{\epsilon^{1/2}}{2} \left\{ \int_{-L+\epsilon^{1/3}}^x \left( \Psi\Psi_x - \Psi^*\Psi_x^* - i\frac{\tau}{L} |\Psi|^2 \right) dx' 
- \frac{(x + L)}{2L} \int_{-L+\epsilon^{1/3}}^{L-\epsilon^{1/3}} \left( \Psi\Psi_x - \Psi^*\Psi_x^* - i\frac{\tau}{L} |\Psi|^2 \right) dx' \right\} + \text{contribution from the boundary layers}.
$$

After constructing the boundary layer solutions in the next step of the calculation, we will argue a posteriori that in fact the boundary layer contribution to (5.9) is $O(\epsilon)$ and therefore of no consequence in our leading order solution.

We supplement (5.8)-(5.9) with the initial condition (5.4)

$$
\Psi(x, 0) = \psi_0(x) \quad \text{for} \quad -L + \epsilon^{1/3} < x < L - \epsilon^{1/3}
$$

and begin the construction of the transient outer solution by assuming an expansion of the form

$$
\Psi = \Psi_0(x, \tau, \epsilon) + \epsilon \Psi_1(x, \tau, \epsilon) + \cdots.
$$

Substituting this into (5.8) and collecting the leading order terms gives

$$
(\Psi_0)_x = -\frac{\epsilon \tau^2}{4L^2} \Psi_0.
$$

Here we have in particular used (2.10) in ignoring the nonlinear terms coming from the right-hand side of (5.8) and we have used the assumption that $0 \leq \tau \ll \frac{1}{\epsilon}$ to ignore
the term of order $O(\epsilon \tau)$ coming from (5.9). Using (5.10), we can then immediately solve to find

$$
\Psi_0(x, \tau, \epsilon) = \psi_0(x) e^{-\frac{\epsilon}{2L} \tau^3}
$$

or, going back to (5.7),

$$
\psi \sim \psi_0(x) e^{-\frac{\epsilon}{2L} \tau^3} e^{-i \frac{(x+L)}{2L} \tau}, \quad \text{along with } \tilde{\phi} \sim \Phi_0(x) = \frac{(x + L)}{2L}.
$$

This completes the leading order description of the outer solution in the transient phase where $\tau \ll \frac{1}{\epsilon}$.

So far, we have ignored the boundary conditions (5.3) to be satisfied by $\psi$ and consideration of (5.11) reveals, not surprisingly, that the outer solution fails to satisfy these conditions at $x = \pm L$. This will force us to match these solutions to boundary layer expansions holding near the endpoints.

Transient inner solution near $x = -L$: Now we will match this transient outer solution to inner solutions valid near the endpoints and for $0 \leq \tau \ll \frac{1}{\epsilon}$ so as to accommodate the boundary conditions (5.3). We begin with the introduction of a stretched variable

$$
y = \frac{(L + x)}{\epsilon^{1/2}}
$$

and look for a solution $(\psi, \tilde{\phi})$ to (5.1), (5.5) valid near the left endpoint $x = -L$ where $\psi$ takes the form

$$
\psi = \psi_0^I(y, \tau) + O(\epsilon^{1/2}).
$$

The boundary conditions for $\psi_0^I$ come from (5.3) and from matching to the outer solution (5.11) namely,

$$
\psi_0^I(0, \tau) = 1, \quad \psi_0^I(\infty, \tau) = e^{-\frac{\epsilon}{2L} \tau^3}.
$$

The initial condition coming from (5.4) is simply

$$
\psi_0^I(y, 0) = 1 \quad \text{for} \quad 0 \leq y < \infty.
$$

Note that by (5.6) and (5.12), $\Phi_0 = \frac{\epsilon^{1/2}}{2L} y$ in terms of $y$. Again we will argue a posteriori that the contribution to $\tilde{\phi}$ in (5.9) from the transient boundary layer constructions, i.e. the contributions involving integrals over the intervals $-L < x < -L + \epsilon^{1/3}$ and $L - \epsilon^{1/3} < x < L$ are $O(\epsilon)$ are therefore negligible. As before, we use that $0 \leq \tau \ll \frac{1}{\epsilon}$ to argue then that $\tilde{\phi} = \Phi_0 + O(1)$ and plug (5.13) into (5.1).

Collecting $O(1)$ terms yields

$$
(\psi_0^I)_\tau + \imath \frac{\epsilon^{1/2}}{2L} y \psi_0^I = (\psi_0^I)_y \quad \text{for} \quad 0 < y < \infty, \quad t > 0.
$$

Note that the $\epsilon^{1/2}$-term appears on the left in (5.16) since we are matching this to the outer solution in the region and for large $y$-values this term becomes significant.

Though one could solve the system (5.16), (5.14), (5.15), we will not do so here. More interesting is the nature of the steady state solution which we will pursue a bit
later. We limit our analysis here to justification of the claim made below (5.9). To see this, note simply that a further change of variables

\[ s = \left( \frac{\sqrt{\tau}}{2L} \right)^{1/3} y, \quad \eta = \left( \frac{\sqrt{\tau}}{2L} \right)^{2/3} \tau \]

converts this system into one that is independent of \( \epsilon \), namely

\[ (\psi_0^l)_{\eta} + i s \psi_0^l = (\psi_0^l)_{ss} \]

for \( 0 < s < \infty, \ \eta > 0 \), \( \psi_0^l(0, \eta) = 1, \ \psi_0^l(\infty, \eta) = e^{-\frac{\sqrt{3}}{2}}, \ \text{and} \ \psi_0^l(s, 0) = 1 \).

Hence it is clear that the solution \( \psi_0^l \) in these new variables \( s \) and \( \eta \) has an \( s \)-derivative that is uniformly bounded. Converting this statement back into a \( y \)-derivative via (5.17) and then to an \( x \)-derivative via (5.12) we find that the \( x \)-derivative is of order \( O(\epsilon^{-1/3}) \). Consequently, the contribution to \( \tilde{\phi} \) as calculated in (5.9) from this transient boundary layer holding for \( -L \leq x \leq -L + \epsilon^{1/3} \) is indeed \( O(\epsilon) \) as claimed.

**Transient inner solution near** \( x = L \) : In a similar manner we introduce a stretched variable \( y' \) to capture the boundary layer solution valid for \( 0 \ll \tau \ll 1/\epsilon \) at the right endpoint via

\[ y' = \frac{(L-x)}{\epsilon^{1/2}}, \]

and look for a solution \((\psi, \tilde{\phi})\) valid near \( x = L \) where \( \psi \) takes the form

\[ \psi = \psi_0^r(y', \tau) + O(\epsilon^{1/2}). \]

As before, the boundary conditions for \( \psi_0^r \) come from (5.3) and from matching to the outer solution (5.11). Consequently, we have

\[ \psi_0^r(0, \tau) = e^{-\tau}, \ \psi_0^r(\infty, \tau) = e^{-\frac{\sqrt{3}}{12L^2} \tau^2} \]

and the initial conditions are again

\[ \psi_0^r(y', 0) = 1 \quad \text{for} \quad 0 \leq y' < \infty \]

in light of (5.4).

Note that by (5.20), \( \Phi_0 = 1 - \frac{\epsilon^{1/2}}{2\tau} y' \) in terms of \( y' \) and as before, we argue a posteriori through (5.9) that \( \tilde{\phi} = \Phi_0 + O(1) \). Plugging (5.21) into (5.1) and collecting \( O(1) \) terms yields

\[ (\psi_0^r)' + i \left( 1 - \frac{\epsilon^{1/2}}{2L} y' \right) \psi_0^r = (\psi_0^r)_{y'} \quad \text{for} \quad 0 < y' < \infty, \ \tau > 0. \]

Now we introduce the function \( f(y', \tau) \) via

\[ f(y', \tau) = \psi_0^r(y', \tau) e^{i\tau} \]

and then again make the change of variables (5.17) to find that the system (5.24),(5.22), (5.23) transforms to

\[ f_{\eta} - i s f = f_{ss} \quad \text{for} \quad 0 < s < \infty, \ \eta > 0, \]

\[ f(0, \eta) = 1, \quad f(\infty, \eta) = e^{-\frac{\sqrt{3}}{2}}, \ \text{and} \quad f(s, 0) = 1. \]
As before, we do not stop here to discuss an explicit solution to this system governing the transient behavior near the right endpoint. We simply note that since the system is independent of $\epsilon$, the function $f$ will have a bounded $s$-derivative, leading via (5.17) and then (5.20) once again to justification of the claim appearing below (5.9).

Outer steady-state solution (valid away from $x = \pm L$ and for $\tau \gg 1/\sqrt[3]{\epsilon}$): In view of (5.11) we see that once $\tau$ reaches values such that $\tau \gg 1/\sqrt[3]{\epsilon}$, the transient solution $\Psi_0$ becomes exponentially small. From this point on, then, the leading order solution to the outer problem, valid away from $x = \pm L$, is just the purely normal solution to (5.1)-(5.2) given by

$$
\psi \sim 0, \quad \phi \sim \Phi_0(x) = \frac{(x + L)}{2L} \quad \text{for } \tau \gg \frac{1}{\epsilon^{1/3}}.
$$

Steady-state inner solution near $x = -L$ : Passing to the regime where $\tau \gg 1/(\epsilon^{1/3})$ in the boundary layer, we find that $\psi_0^l$ will approach a steady-state solution to (5.16), i.e. for $0 < y < \infty$,

$$
(\psi_0^l)_{yy} - i\left(\frac{1/2}{2L}\right) y \psi_0^l = 0 \quad \text{with } \psi_0^l(0) = 1, \quad \psi_0^l(\infty) = 0,
$$

where the second boundary condition matches to the outer solution. (Note that $\tau \gg 1/\sqrt[3]{\epsilon}$ means that $\eta \gg 1$, cf. (5.17)). Replacing the variable $y$ by $s$ as in (5.17) we obtain

$$
\psi_{ss} = \frac{s\psi}{s} = 0 \quad \text{with } \psi(0) = 1, \quad \psi(\infty) = 0.
$$

Now let $z = -is$, to obtain the Airy equation

$$
\psi_{zz} - z\psi = 0 \quad \text{with } \psi(0) = 1, \quad \psi(-i\infty) = 0.
$$

We recall that both $\{Ai(z), Ai(ze^{2\pi i/3})\}$, and $\{Ai(z), Ai(ze^{-2\pi i/3})\}$ form pairs of linearly independent solutions to the Airy equation $w'' - zw = 0$. Here $Ai(z)$ is the Airy function and satisfies the asymptotic formula

$$
Ai(z) \sim \frac{1}{2\pi} z^{-1/2} e^{-\frac{2}{3}z^{3/2}} \quad \text{for } |z| \gg 1, \quad |\arg z| < \pi,
$$

cf. [1], pg. 448.

One can check using (5.30) that $Ai(ze^{2\pi i})$ decays along the negative imaginary axis, so the solution of (5.29) is

$$
\psi_0^l(y) = \frac{Ai((\frac{1/2}{2\pi} L^{1/3} ye^{\frac{2\pi i}{3}}))}{Ai(0)}.
$$

See [2, 8] for a similar analysis of (5.29).

Steady-state inner solution near $x = L$ : In a similar manner, we see that the transient boundary layer solution $\psi_0^r$ valid near $x = L$ settles into a steady-state solution to (5.24), (5.22) once $\tau \gg 1/(\epsilon^{1/3})$. In view of (5.26)-(5.27), this corresponds to a solution $f = f(s)$ for $0 < s < \infty$ to the system

$$
f_{ss} + is f = 0 \quad \text{with } f(0) = 1, \quad f(\infty) = 0.
$$

Letting $z = is$ one again arrives at the Airy equation $f_{zz} - zf = 0$ but now the boundary conditions are $f(0) = 1$ and $f(i\infty) = 0$. This time an appeal to (5.30)
reveals that $Ai(ze^{-\frac{2\pi i}{3}})$ is the solution decaying along the positive imaginary axis so that

$$f(y') = \frac{Ai((\frac{1}{2L})^{1/3}y' e^{-\frac{2\pi i}{3}})}{Ai(0)}$$

and so $\psi_0(y', \tau) = \frac{Ai((\frac{1}{2L})^{1/3}y' e^{-\frac{2\pi i}{3}})}{Ai(0)} e^{-i\tau}$.

This completes a derivation of the leading order behavior of the solution to our problem for large $V$. We have found that after a transient period, the solution settles into a $2\pi$-periodic profile with an exponential suppression of the order parameter away from the boundary. We also remark that what one can observe numerically to be a rather broad width to the boundary layer can be explained by the nature of the dependence of the solution on $\epsilon = 1/V$ in (5.31) and (5.32), namely as a factor of $\epsilon^{1/6}$ inside the Airy function. Note also that in light of the exponential decay of the two Airy functions used in the boundary layers, we can in fact write down a formula (written in terms of the original variables) that holds uniformly throughout the interval $-L \leq x \leq L$:

$$\psi(x, t) \sim \frac{Ai\left(\frac{V^{1/3}(L+x)}{(2L)^{1/3}} e^{\frac{2\pi i}{3}}\right)}{Ai(0)} + \frac{Ai\left(\frac{V^{1/3}(L-x)}{(2L)^{1/3}} e^{-\frac{2\pi i}{3}}\right)}{Ai(0)} e^{-iV t},$$

valid for $t = \epsilon \tau \gg \epsilon^{2/3}$. Comparison of this solution to the actual numerically computed $2\pi/V$-periodic solution to (2.3)-(2.4), (2.6) reveals very good agreement. For example, for the case of $L = 4$ and $V = 80$ one can see a comparison of the graphs of the moduli of the two solutions taken at the time $t_0 = 20.74$ (when a phase slip center appears) in Figure 5.1 below. Comparisons at other times reveal a similarly close fit.

6. Phase slip centers, period doubling and beyond. In the previous three sections we presented arguments, some rigorous, some formal, for the dominance of a $P$-periodic solution, where $P := 2\pi/V$ when $L$ is small, $V$ is small or $V$ is large, respectively. We turn our focus now to the rich nature of phase slip centers (PSC’s) – that is, zeros of the order parameter $\psi$ – for this flow, and to the emergence of more complicated solutions including period doubling, period tripling and what one might
call quasi-periodic dynamics when one explores other regions in the $L - V$ plane. After presenting a rigorous result on the appearance of PSC’s, the remainder of this section will involve numerical experiments.

Certainly, one salient feature of the periodic solutions to this voltage-driven Ginzburg-Landau model is the presence of PSC’s. Given that the boundary conditions (2.6) imply $|\psi(\pm L, t)|^2 = \Gamma > 0$, it is perhaps not so obvious that the solution $\psi$ to (2.3)-(2.4), (2.6), mapping $[-L, L] \times [0, \infty)$ into the complex plane should vanish at all in the $xt$-plane. Here we make an observation, based on a simple topological argument, that any periodic solution to (2.3)-(2.4), (2.6) always has at least one phase slip center in each period.

**Proposition 6.1.** Assume $\Psi$ is a $\frac{2\pi k}{V}$-time periodic solution of (2.3)-(2.4), (2.6) for some positive integer $k$. Then there exists $(x_0, t_0) \in (-L, L) \times [0, \frac{2\pi k}{V}]$ such that $\Psi(x_0, t_0) = 0$, and hence by the periodicity, $\Psi(x_0, t_0 + \frac{2\pi k}{V} j) = 0$ for all $j = 0, 1, 2, \cdots$.

**Proof.** Let $\psi(x, \cdot) := \Psi(x, \cdot)$ for each $x \in [-L, L]$. Then $\psi^{-L}$ and $\psi^L$ are in particular continuous functions from $[0, \frac{2\pi k}{V}]$ to $\mathbb{C}$, and $\Psi : [-L, L] \times [0, \frac{2\pi k}{V}] \to \mathbb{C}$ is of course a continuous map as well. Thus $\Psi$ is a homotopy between $\psi^L(t) = \sqrt{\Gamma} e^{-iVt}$ and $\psi^{-L}(t) = \sqrt{\Gamma}$. That is, $\Psi$ represents a continuous deformation of the map $\psi^L$ to the map $\psi^{-L}$ as $x$ ranges from $L$ to $-L$. As there is no way to continuously deform a circle $\psi^L$ that contains the origin in its interior into the point $\psi^{-L} = \sqrt{\Gamma}$ without at some value $x = x_0$, $\psi^{x_0}$ passing through the origin, there necessarily exists $(x_0, t_0) \in (-L, L) \times [0, \frac{2\pi k}{V}]$ such that $\Psi(x_0, t_0) = 0$. $\square$

So far in this article, we have only discussed parameter regimes where solutions have period $P$, that is, where $k = 1$ in the notation of Proposition 6.1. However, much more complicated solutions exist. Indeed, period doubling was first observed numerically for this system in [11], where the authors found that by increasing $L$, they could produce solutions that oscillated with a frequency higher than $2\pi/V$. Our numerical experiments also reveal period $2P$ solutions, along with, for example $3P$, $4P$, $8P$, etc. as well as ‘quasi-period’ solutions, that is, highly oscillatory solutions for which periodicity is lost. Our code uses a Crank Nicolson scheme to solve the evolution equation (2.3) for $\psi$, where a numerical quadrature is applied to equation (2.9) to integrate $\phi$.

In Figure 6.1 we present a few such examples. Here and for the remainder of the graphs in this section, we fix $\Gamma = 1$. We have chosen to depict the quantity $|\psi(0, t)|^2$ as a function of time in these graphs because PSC’s often occur at $x = 0$. However, this is not always their location. For example, when $L = 10$ and $V = 0.8$ (top right graph in Figure 6.1), one finds a $3P$-periodic solution with only 1 PSC per period located at the origin while two other PSC’s occur at other times at locations near $x = \pm 4$. The PSC occurring near $x = 4$ is depicted in Figure 6.2.

Note that Proposition 6.1 guarantees the existence of at least one PSC per period, but one might reasonably expect that, for instance, in the period-doubling regime where $k = 2$, one should see two PSC’s within a given period. Our numerical experiments show that this is sometimes but not always the case, so that the proposition as stated is in this sense sharp.

For example, in Figure 6.1(top left), where $L=10$ and $V=0.22$, one finds a period $2P$-solution which indeed has two PSC’s per period $4\pi/V \approx 57.12$, with the two PSC’s occurring at $x = 0$ at different times. This perhaps more typical behavior can also be understood topologically by sketching the closed loops $\psi^L$ used in the proof of Proposition 6.1. See Figure 6.3(A). However, as an example of a period-doubling solution with only one PSC per period, consider the case where $L = 5$ and where the
behavior of a superconducting wire subjected to a voltage difference

![Graphs showing amplitude vs. time for different voltages and periods.](image)

Fig. 6.1. (Top left) Period 2P solution. (Top right) Period 3P solution. (Bottom left) Period 8P solution. (Bottom right) Quasi-periodic solution.

Fig. 6.2. Square modulus of 3P-periodic solution at time of PSC near $x = 4$.

Voltage is taken to be $V = 1.16$. See Figure 6.3(B). Here we find only one phase slip center per period, located at the origin. In this case, one again finds that for positive $x$-values, the immersed curves $\psi^x$ wind twice around the origin each period while for negative $x$-values, the curves do not wind around the origin at all, but now the curve containing the PSC, namely $\psi^0$, forms a cusp passing through the origin only once.

Finally, we comment on the stability of the period 2P solutions obtained numerically. Unlike the situation for small $L$ where it was shown in Section 3 that the $P$-periodic solutions enjoy global asymptotic stability, the same cannot be said of the period doubling solutions. This is due to the fact that a time-shift by $2\pi/V$ produces a different $4\pi/V$ periodic solution with the same parameter values, the point being that the shifted function still satisfies the boundary conditions (2.6). Nonetheless, solving the problem under a fairly wide variety of initial conditions revealed a rather
large basin of attraction for the $4\pi/V$ periodic solutions.

7. Conclusions. Through a mixture of rigorous analysis, formal expansions and numerics, we have explored the behavior of the solution to the one-dimensional time-dependent Ginzburg-Landau model modified to account for an applied voltage $V$. We find rigorously that when the length $2L$ of the wire is below some critical value, there is a unique globally asymptotically stable solution of period $2\pi/V$ having one phase slip center per period. For larger $L$ values, however, we find a much more complicated picture as we let $V$ range from small values to larger ones. For small $V$, we formally predict that there is again a $2\pi/V$-periodic solution which to leading order in $V$ is described by the solution to a minimization problem with periodic boundary conditions. As $V$ is increased, we find that there is numerical evidence of period-doubling, along with period-tripling and quasi-periodicity. Finally, for large enough values of $V$, we find formally that a stable $2\pi/V$-periodic solution re-emerges, though now it comes with boundary layers at the two endpoints and an exponential suppression of the order parameter in the interior. We summarize the situation with the phase diagram depicted in Figure 7.1, in which we again take $\Gamma = 1$.

We also find numerically that in the parameter domain possessing more complicated solutions, there is an assortment of phase slip center behavior including (i) $2\pi k/V$-periodic solutions with $k$ PSC’s all occurring at $x = 0$ (ii) $2\pi k/V$-periodic solutions with $k$ PSC’s where some occur at the origin, while others do not (iii) $2\pi k/V$-periodic solutions with less than $k$ PSC’s per period (iv) $2\pi k/V$-periodic solutions with PSC’s that appear to the left and then the right of the origin alternating in time (at least for certain $\Gamma \neq 1$).
Behavior of a superconducting wire subjected to a voltage difference

Fig. 7.1. Phase diagram in the $L-V$ plane. Here $P_1$ denotes stability of period $P = 2\pi/V$ solutions, $P_2$ denotes stability of $2P$-periodic solutions and $C$ includes emergence of various $kP$-periodic solutions for $k \geq 3$ along with quasi-periodic solutions. The arrows in the figure key correspond to the nature of the transition as voltage is increased.

REFERENCES