# THE RESISTIVE STATE IN A SUPERCONDUCTING WIRE: BIFURCATION FROM THE NORMAL STATE

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Abstract. We study formally and rigorously the bifurcation to steady and time-periodic states in a model for a thin superconducting wire in the presence of an imposed current. Exploiting the PT-symmetry of the equations at both the linearized and nonlinear levels, and taking advantage of the collision of real eigenvalues leading to complex spectrum, we obtain explicit asymptotic formulas for the stationary solutions, for the amplitude and period of the bifurcating periodic solutions and for the location of their zeros or "phase slip centers" as they are known in the physics literature. In so doing, we construct a center manifold for the flow and give a complete description of the associated finite-dimensional dynamics.

## 1. Introduction

One of the natural applications of superconducting materials is to exploit their infinite conductivity to transmit electric currents. The goal of this paper is to analyze a number of asymptotic problems that arise in the study of such current transmission through a wire. We consider a simple canonical problem, in which the superconducting portion of the wire is of a finite extent $-L \leq x \leq L$. It is assumed that a normal current $I$ is fed into the the wire at its left end. It is known that under the right circumstances, for example for a temperature $T$ that is sufficiently small, the current in the wire itself will be in part normal and in part superconducting. This coexistence of two types of currents in the wire is called *a resistive state*.

The resistive state in superconducting wires received some attention by physicists who observed a variety of phenomena that are unique to this situation. To review these observations, which involve oscillatory, that is to say, inherently time-varying behavior, it is standard practice to use the time-dependent Ginzburg-Landau model (TDGL). For a three-dimensional wire

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occupying say a thin cylindrical region $D$ with axis of length $2L$ centered on the $x$-axis, this system can be written in non-dimensional form as

$$\psi_t + i\phi\psi = (\nabla - iA)^2\psi + (\Gamma - |\psi|^2)\psi \text{ for } (x,y,z) \in D, \tag{1.1}$$

$$\nabla \times \nabla \times A = -\sigma(A_t + \nabla\phi) + \frac{i}{2}\left(\psi\psi_x^* - \psi_x\psi^*\right) - |\psi|^2 A \text{ for } (x,y,z) \in D, \tag{1.2}$$

(cf. [17]) where $\psi : D \to \mathbb{C}$ is the Ginzburg-Landau parameter whose square modulus measures the density of superconducting electrons, $A : \mathbb{R}^3 \to \mathbb{R}^3$ is the magnetic potential whose curl measures the effective magnetic field and $\phi : \mathbb{R}^3 \to \mathbb{R}$ is the scalar electric potential whose gradient represents the electric field. The parameter $\Gamma$ is proportional to $T_c - T$, where $T$ is the actual temperature, and $T_c$ is the phase transition temperature in the absence of external currents (i.e. in the case $I = 0$). In (1.2), $*$ denotes complex conjugation and the right-hand side represents the sum of normal current (with associated ohmic conductivity $\sigma$) and supercurrent. We note that the TDGL is invariant under the gauge transformation

$$(\psi, \phi, A) \to (\psi e^{ig}, \phi - g_t, A + \nabla g) \quad \text{for any smooth scalar function } g = g(x,y,z,t).$$

To pursue an appropriate three-dimensional analysis of the problem of forcing an applied current into a wire, one would then have to impose inhomogeneous boundary conditions on the normal component of the normal current and couple the system above to a Maxwell system on the exterior of $D$. This is not the direction we will follow; instead we adopt the model favored for many years in the physics literature on the subject, e.g. [8], and view the wire as a one-dimensional object. Before leaving the higher dimensional setting we should comment, however, that an interesting two-dimensional study of the stability of the normal state with an applied electric current can be found in [1].

In such a one-dimensional model, the exterior problem is typically ignored as a lower-order effect and so all unknowns are taken simply to be defined along the wire as functions of $x$ and $t$ only. Furthermore, through the gauge choice $g_x = -A$, one can eliminate the magnetic potential $A$ completely from the system. Then using the remaining freedom in the $t$-dependence of $g$, one can insist on the convenient normalization

$$\phi(0,t) = 0 \quad \text{for all } t \geq 0. \tag{1.3}$$

Under these assumptions and gauge choice, (1.1) reduces to

$$\psi_t + i\phi\psi = \psi_{xx} + \Gamma\psi - |\psi|^2\psi \quad \text{for } -L < x < L,\ t > 0. \tag{1.4}$$

Regarding the fate of the second equation (1.2), note that necessarily the divergence of the total current, that is, the right-hand side, must vanish. In one dimension, this condition however implies simply that the total current is a constant. Therefore, since we are specifying that the current at the endpoints of the wire is purely normal and equal to $I$, we arrive at the relation

$$\frac{i}{2}\left(\psi\psi_x^* - \psi_x\psi^*\right) - \sigma\phi_x = I \quad \text{for } -L < x < L,\ t \geq 0. \tag{1.5}$$

Since the natural setup is for the wire to be connected at its endpoints to a metal exhibiting normal conductivity, we supplement the system (1.3)–(1.5) with homogeneous Dirichlet boundary conditions on the order parameter

$$\psi(\pm L, t) = 0, \tag{1.6}$$

along with initial conditions on $\psi$. The general nature of our results apply also to other homogeneous boundary conditions, including in particular the case of homogeneous Neumann boundary conditions.

As long as the temperature is sufficiently high, that is $\Gamma$ is low, the wire is in the normal state. This means that $\psi = 0$, and the current in the wire is purely ohmic, i.e.

$$\phi_x = -I/\sigma. \tag{1.7}$$

As the temperature $T$ is lowered, one reaches a critical value, determined by a curve $\Gamma = \Gamma_1(I)$ where the normal state loses its stability and a nontrivial superconducting state might emerge. Two such states were discovered in the early 1980's. The first one is a stable stationary solution, reported by Langer and Baratoff [9]. The notion of 'stationarity' requires some care here. Let us express the order parameter in polar form $\psi = fe^{i\chi}$; then the gauge invariant quantities $f(x,t)$, $q(x,t) := \chi_x(x,t)$ and $\theta(x,t) := \chi_t(x,t) - \phi(x,t)$ converge to stationary functions $f_0(x)$, $q_0(x)$, $\theta_0(x)$. On the other hand, Langer and Ambegaokar ([11] (see also Ivlev and Kopnin [8]) found in their numerical simulations that for some values of the parameters

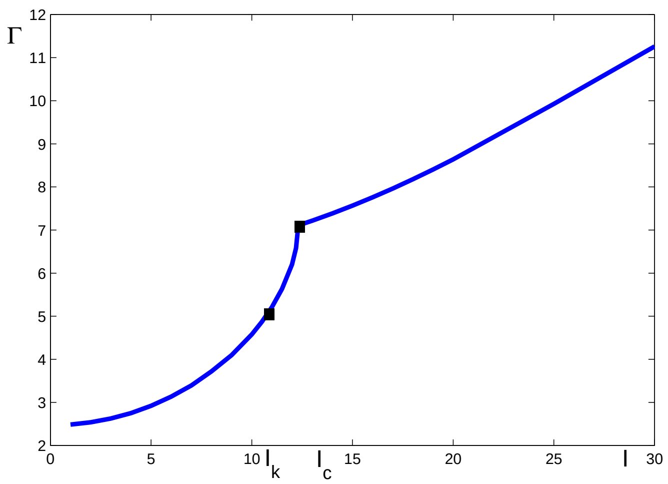


FIGURE 1

$(I, \Gamma)$, the normal state bifurcates into a state where the order parameter oscillates in time. This periodic behavior is particularly interesting in light of the dissipative nature of the TDGL model (1.4), but it is made possible by the presence of the applied current which effectively disrupts the gradient-flow structure of the system.

For the duration of this investigation, we fix $L = 1$ so that the wire occupies the interval $[-1, 1]$. We also set the conductivity $\sigma = 1$ in order to focus on the different kinds of states that emerge at different points in the $(I, \Gamma)$ plane.

In a recent work [13] the authors used numerical simulations and some analytical arguments to identify a more elaborate phase transition picture. In particular the curve $\Gamma_1(I)$ was shown to be associated with an interesting spectral problem. It was also shown that there exist two critical currents $I_k$ and $I_c$ that play an important role in the system behavior. The curve $\Gamma_1(I)$ is depicted in Figure 1. Specifically, the normal state (N) is only stable for $\Gamma < \Gamma_1(I)$. When the temperature is decreased and $\Gamma$ increases past $\Gamma_1(I)$, the (N) state become unstable. If $I < I_k \approx 10.92$, then the (N) state bifurcates into a stationary (S) state. On the other hand, the bifurcation branch to a stationary state is unstable for $I_k < I < I_c$, while if $I > I_c \approx 12.31$, the (N) state bifurcates to a stable time-periodic (P) state.

To understand the phase transitions described above, and in general the solution to equations (1.4)-(1.5), it is useful to study a number of mathematical problems:

(1) What is the meaning of the special current value $I_c$ where the bifurcating state switches from a stationary one to a periodic one?

(2) From the viewpoint of self-adjoint operators arising in the time-independent case, the answer to the first question involves an unusual spectral problem involving the collision of two eigenvalues. Moreover, near the critical value $I_c$, the spectrum of the underlying operator changes its nature, from real to complex. Therefore it is desirable to understand this spectrum near the special value $I_c$.

(3) What is the nature of the bifurcating branch near the curve $\Gamma_1(I)$? In the language of dynamical systems, we ask what is the geometry of the center manifold there? This question, in fact, involves a number of issues. For instance, is the bifurcation branch stable and what is its shape? In addition, we point out that in the periodic case there are points in space-time where the order parameter $\psi$ vanishes. Such points are called phase slip centers (PSC's) since the phase 'exploits' the vanishing of the amplitude to have a discontinuity there, thus relaxing large accumulated phase. Thus one of the relevant questions would be to identify these points.

The loss of stability of the normal state is studied through the linearization of equations (1.3)-(1.6) around the normal state

$$\psi \equiv 0, \quad \phi = -Ix. \tag{1.8}$$

It is convenient to express the solution $\psi$ of the linearized equation in the form $\psi(x,t) = u(x)e^{(\Gamma-\lambda)t}$; then we obtain that $u(x)$ is the solution of the following spectral problem

$$Mu := u_{xx} + ixIu = -\lambda u, \quad u(\pm 1) = 0. \tag{1.9}$$

Clearly the stability of the normal state is determined by whether $\Gamma$ is larger or smaller than $\operatorname{Re}\lambda_1$ where $\lambda_1$ refers to the eigenvalue of the operator $M$ having smallest real part. In the next section we shall therefore consider this eigenvalue and examine some of its properties for small $I$ and for large $I$. The results of this section will help us in identifying the critical value $I_c$. Then, in Section 3, we will examine in more detail the leading eigenvalue of $M$ for $I$ values near $I_c$. In Section 4 we construct stationary solutions to (1.4)-(1.6) for $I < I_c$ using formal asymptotic expansions and multiple time-scales. In Section 5, we construct periodic solutions using the same methods in the regime $I > I_c$ where the spectrum of $M$ has become complex. We then make these calculations rigorous in Section 6 by constructing the center manifold for the solution immediately after bifurcation and studying the O.D.E.'s which govern the flow on the center manifold.

A novel aspect of the rigorous analysis in Section 6 is that both the linearized and the full nonlinear system admit what is called PT-symmetry; namely an invariance under the joint transformation of $x \to -x$ and complex conjugation. This type of symmetry has been the focus of a number of recent investigations (see e.g. [2, 4, 5, 12]). In particular, in the analysis of periodic bifurcation we make strong use of this symmetry to reduce the dimension of what turns out to be a four-dimensional phase space to a planar system exhibiting a standard Hopf bifurcation. Furthermore, we are able to then go further and describe bifurcation for the full four-dimensional system involving possibly non-PT-symmetric solutions, proving that, at least in the vicinity of the normal solution $\psi \equiv 0$ and for values of the bifurcation parameter that we study, solutions generically converge, up to fixed complex rotation, to the manifold of solutions exhibiting PT-symmetry, and thereafter to the stable, PT-symmetric periodic solutions arising through Hopf bifurcation within that manifold. However (Theorem 6.11), we also exhibit through direct calculation the existence of unstable, non-PT-symmetric periodic solutions in the same vicinity, i.e., persistent solutions that do not converge to the manifold of PT-symmetry. This shows that the observed generic convergence to PT-symmetry is the result of detailed local dynamics on the center manifold about the normal state and not, say, a global principle associated with a decreasing Lyapunov functional. In particular, there might exist attracting steady or periodic states far from the normal state that are not PT-symmetric, an intriguing possibility to keep in mind in further investigations.

Finally, in Section 7, we show that our rigorous construction of periodic solutions leads to a proof of the appearance of the phase slip centers, that is, periodically appearing zeros of the order parameter.

## 2. The spectrum of the canonical PT-symmetric problem for small $I$ and for large $I$

In this section we examine the spectrum of $M$ defined in equation (1.9). The operator $M$ is not self-adjoint, of course. On the other hand it enjoys a symmetry that is called PT. The letter P stand for parity, i.e. transforming $x \to -x$, while the letter $T$ stands for time reversal, i.e. complex conjugacy. One readily checks that under this pair of operations, $M$ is unchanged.

PT-symmetric spectral problems seem to have been little studied until quite recently. In one of the earliest works on this subject (in a physics context), Bender and Boettcher [2] considered a canonical PT-symmetric operator on the entire real line and observed through numerical simulations that the spectrum is real. In the case of a finite interval as in equation (1.9) above, the situation is more involved. In particular, we will formally demonstrate the appearance of complex eigenvalues for $I$ large with corresponding eigenfunctions possessing an internal layer.

We look first in the case of small $I$. When $I = 0$, the spectrum is of course real, and can, in fact, be written down explicitly

$$\lambda_k(I = 0) = (\pi k/2)^2 , \quad k = 1, 2, ... \tag{2.1}$$

The PT-symmetry of $M$ implies that if $(\lambda, u(x))$ is a spectral pair of eigenvalue and eigenfunction, then $(\lambda^*, u^*(-x))$ is also a spectral pair. Langer and Tretter [12] have shown that, as long as the eigenvalues of a PT-symmetric problem are simple, the spectrum is a smooth function of the problem's parameters. In our case it implies that since the eigenvalues are well-separated at $I = 0$, they are smooth functions of $I$ at least for $I$ small. However, this implies that the eigenvalues must remain real also for $I$ positive (but small), since a real eigenvalue can become complex only by splitting into a pair of eigenvalues (by the PT-symmetry).

What happens when $I$ is increased? As long as the eigenvalues do not collide, they remain real. We now show formally that in fact the eigenvalues must collide for some $I$ and establish an asymptotic formula for the leading (complex) eigenvalue when $I$ is large. Precise and rigorous asymptotics for this spectral problem were carried out by Shkalikov [14]. His work (see also [15]) was performed in the context of the Orr-Somerfeld equation in hydrodynamic stability theory and makes use of a change of variables leading to an Airy-type equation. We present the following formal calculation with the hope that it makes the emergence of the complex spectrum more understandable. Note also that the formal method given here applies in more general circumstances, whereas the exact solution of [14] is specific to the exact form of the equations under consideration.

To study the spectrum as $I$ becomes large, it is convenient to introduce a small positive parameter $\varepsilon$, and then write $I = \varepsilon^{-2}$. It is clear that the eigenvalues must also be large to balance the large 'potential' $i\varepsilon^{-2}x$. We therefore write to leading order

$$\lambda = \varepsilon^{-2}(\alpha + i\beta) + o(\varepsilon^{-2}). \tag{2.2}$$

The eigenvalue problem (1.9) can be written to leading order as $u_{xx} + \varepsilon^{-2}\mathcal{Q}u = 0$, where $\mathcal{Q} := x+\beta-i\alpha$. Crudely setting $\varepsilon = 0$, we obtain a formal eigenvalue problem $(ix-(\alpha+i\beta))\psi = 0$ for the multiplication operator $ix$, which evidently has only pure imaginary, essential spectrum. From this we may conjecture that the spectrum of $L$ becomes complex as $I \to \infty$ ($\varepsilon \to 0$); however, this limit is very singular and must be examined in more detail (indeed, on the whole line, the spectrum is real for $I$ large [8].)

Consider first the case where the potential $\mathcal{Q}$ does not vanish for $x$ in the interval $[-1, 1]$. For example, this would occur if $\alpha \neq 0$. If $\mathcal{Q} \neq 0$ then there is no turning point in a standard JBKW [1] expansion of equation (1.9). Therefore, we seek in this case an asymptotic expansion of the form

$$u(x) = e^{iS(x)/\varepsilon}. \tag{2.3}$$

Substituting (2.3) and (2.2) into (1.9) we get to leading order

$$S_x = \pm i^{1/2}\,(x + \beta - i\alpha)^{1/2}\,. \tag{2.4}$$

[1]We use the term JBKW instead of WKB since this expansion method was introduced by Jeffries in 1923, three years before it was rediscovered by Wentzel, Kramers and Brillouin, who are also now ordered alphabetically.

Integrating the last equation from $-1$ to $x$ gives

$$S^{\pm} = \pm\frac{2}{3}i^{1/2}\left((x+\beta-i\alpha)^{3/2} - (-1+\beta-i\alpha)^{3/2}\right). \tag{2.5}$$

For future reference we introduce the notation

$$S^{+}(1) = \operatorname{Re} S + i\operatorname{Im} S. \tag{2.6}$$

The general solution to the equation (1.9) is (to leading order)

$$u(x) \sim Ae^{iS^{+}(x)/\varepsilon} + Be^{iS^{-}(x)/\varepsilon}. \tag{2.7}$$

Substituting this solution into the boundary conditions at $\pm 1$, and seeking a pair of nontrivial coefficients $A, B$ leads to the following complex-valued equation

$$e^{iS^{+}(1)/\varepsilon} = e^{iS^{-}(1)/\varepsilon}. \tag{2.8}$$

In particular the magnitudes of the two sides of equation (2.8) must be the same. Equating the absolute values, and using the notation (2.6) gives

$$e^{\operatorname{Im} S/\varepsilon} = e^{-\operatorname{Im} S/\varepsilon}. \tag{2.9}$$

Therefore, a regular JBKW expansion without a turning point is feasible only if $\operatorname{Im} S = 0$.

Recalling the definition of $\operatorname{Im} S$, the last condition on it implies (for some real number $\chi$)

$$(1+\beta-i\alpha)^{3/2} - (-1+\beta-i\alpha)^{3/2} = \chi i^{-1/2}. \tag{2.10}$$

We now show that equation (2.10) holds for any $\alpha$ if $\beta = 0$. Setting $\beta = 0$, we write

$$-1 - i\alpha = \rho e^{i(-\pi+\mu)}, \quad 1 - i\alpha = \rho e^{i(-\mu)}. \tag{2.11}$$

Here $\rho = \sqrt{1+\alpha^2}$. Substituting this into (2.10) with $\beta = 0$, and defining $\gamma = 3\mu/2$ gives

$$\rho^{3/2} i^{-1/2} G = \chi i^{1/2}, \quad G = \sqrt{2}\left(\cos\gamma + \sin\gamma\right). \tag{2.12}$$

This proves the assertion above with

$$\chi = \sqrt{2}(1+\alpha^2)^{3/2}\left(\cos\gamma + \sin\gamma\right). \tag{2.13}$$

It remains to show that there exist values of $\alpha$ for which equation (2.8) is solvable. Since equation (2.8) holds if and only if $\operatorname{Im} S = 0$ and $\sin(\operatorname{Re} S) = 0$, we obtain the condition

$$\operatorname{Re} S = \varepsilon n \pi, \tag{2.14}$$

where $n$ is an integer. The calculation above for $\chi$ together with equation (2.14) imply

$$\frac{2\sqrt{2}}{3}\left(1+\alpha^2\right)^{3/2}(\cos\gamma + \sin\gamma) = \varepsilon n \pi. \tag{2.15}$$

Consider the left-hand side as a function of $\alpha$ (recall that $\gamma$ also depends on $\alpha$ through the relation (2.11)). When $\alpha$ tends to $\infty$ then $\gamma$ tends to $3\pi/4$, and the term $\cos\gamma + \sin\gamma$ on left-hand side approaches zero. However, the $\alpha^3$ term on the left hand side grows faster, and therefore the entire left-hand side goes to $\infty$. On the other hand, when $\alpha = 0$, the left-hand side becomes $2\sqrt{2}/3$. Therefore, varying $\alpha$ the left-hand side obtains all the values in the interval $(2\sqrt{2}/3, \infty)$. This means that for any fixed $\varepsilon$ there exist infinitely many $n$ values for which equation (2.15) has a solution $\alpha_n$. Consequently there are infinitely many real eigenvalues of order $O(\varepsilon^{-2})$.

The JBKW expansion above captured the real eigenvalues. They are all of $O(\varepsilon^{-2})$. However, this expansion fails when $\alpha = 0$ and $-1 \leq \beta \leq 1$ since in this case there is a turning point. We shall now construct a solution for such a case. If we order the eigenvalues by their real part, the eigenvalues we shall now construct come before those derived above. Typically a JBKW expansion captures the large eigenvalues and the associated oscillatory behavior of the eigenfunction. The lower eigenfunctions tend to oscillate less. Since we are now looking into the possible construction of a complex eigenvalue, we need to recall that they come in pairs of conjugate numbers. Geometrically it means that we anticipate two eigenfunctions, related by PT-symmetry, and therefore we anticipate in one case a turning point in $[-1, 0)$ and in another case a turning point in $(0, 1]$, i.e. the turning point is at $x = \pm\beta$. While we seek now an eigenvalue that to leading order is purely imaginary, we shall be able to obtain also a lower order correction for it that will have real and imaginary parts. Therefore we assume that the eigenvalue is of the form

$$\lambda \sim i\varepsilon^{-2}\beta_0 + \varepsilon^{-\nu}(\alpha_1 + i\beta_1), \tag{2.16}$$

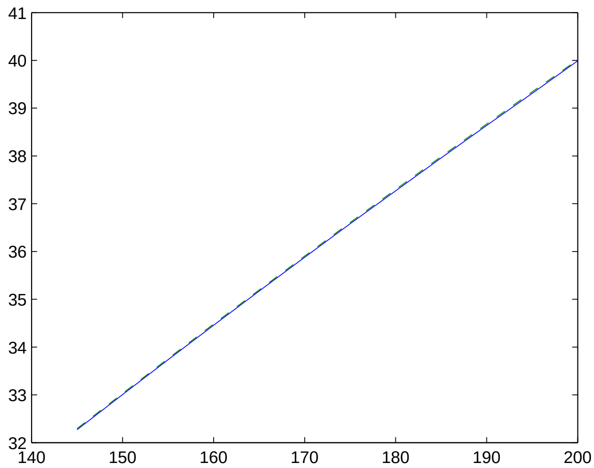


FIGURE 2. The (numerically computed) real part of the leading eigenvalue (solid line) compared to the the asymptotic expansion (2.22 (dashed line). The horizontal axis in this figure is the current $I$, and the vertical axis is the real component of the eigenvalue.

where the exponent $\nu$ is still to be determined.

Consider, then, without loss of generality, the case where the turning point is in the interior of the interval $(-1, 0)$, i.e. $-1 < \beta_0 < 0$. To leading order the potential is balanced exactly at the point $x = -\beta_0$. We therefore expect the eigenfunction to be supported in a small neighborhood of this point, and to decay away from it. Hence, we construct an internal layer around this point by defining an inner variable $y$ through

$$x = -\beta_0 + \varepsilon^\gamma y \tag{2.17}$$

for some $\gamma > 0$. Substituting the transformation (2.17) into equation (1.9) we see that in order to balance the different terms in the equation we must set $\gamma = 2/3$ and $\nu = 4/3$. We thus obtain on the $y$ scale the balanced equation

$$u_{yy} + iyu + (\alpha_1 + i\beta_1)\, u = 0. \tag{2.18}$$

While equation (2.18) describes the internal layer form of the eigenfunction $u$, the outer solution is of course $u \equiv 0$. Therefore, matching the inner and outer solutions implies that we should consider equation (2.18) over the entire real line with the conditions

$$u(y = \pm\infty) = 0. \tag{2.19}$$

The eigenvalue problem (2.18)-(2.19) can be solved explicitly in terms of Bessel (or Hankel) functions. This was done by Ivlev and Kopnin [8] who concluded that this problem has no solution with finite $L_2$ norm. This means that our assumption that $\beta_0 < 1$ is not consistent. In other words, the concentration cannot occur in the interior of $[-1, 1]$. We therefore consider now the last remaining case in which $\beta_0 = 1$. The scaling (2.17) of the internal variable $y$ is the same, except that now it is more appropriate to call it the 'boundary layer' variable. Thus, the boundary layer equation (2.18) is considered over the half line $y \in [0, \infty)$. The Dirichlet condition and the matching to the outer solution together imply the condition

$$u(0) = u(\infty) = 0. \tag{2.20}$$

The half-line eigenvalue problem (2.18), (2.20) was also studied in [8]. In this case the eigenfunction has finite $L_2$ norm. The authors computed the leading eigenvalue to be approximately

$$\alpha_1 + i\beta_1 = 1.17 - 2.02i. \tag{2.21}$$

Returning to the original notation for the current, we obtain the eigenvalue

$$\lambda \sim 1.17 I^{2/3} + i\left(I - 2.02 I^{2/3}\right) \quad \text{for} \quad I \gg 1. \tag{2.22}$$

In Figure 2 we depict the asymptotic expansions for the real part of the leading eigenvalue (dashed line) and the numerically computed real part (solid lines). Similarly, we depict in Figure 3 the asymptotic imaginary part (dashed line) and the actual imaginary part (solid line). In both cases the curves are very close to each other (the error is roughly 0.02).

The eigenvalues are arranged by their real part in an ascending order. It is found numerically that as $I$ increases towards the critical value of $I_c \approx 2.27$ (for $L = 1$ and Neumann boundary conditions) and at the critical value $I_c \approx 12.31$ (for $L = 1$ and Dirichlet boundary conditions), the first and second eigenvalue collide. These results are consistent with the bounds of ref. [12]. In the Neumann case their estimate for $I$ below which the entire spectrum is real is $I < \pi^2/8$, while the corresponding Dirichlet estimate is $I < 3\pi^2/8$.

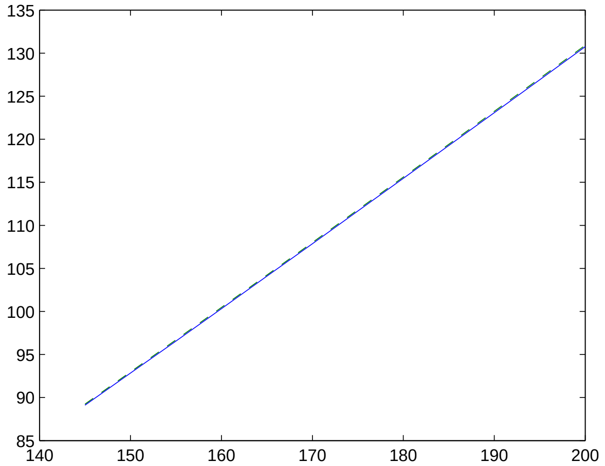


FIGURE 3. The (numerically computed) imaginary part of the leading eigenvalue (solid line) compared to the asymptotic expansion (2.22 (dashed line). The horizontal axis in this figure is the current $I$, and the vertical axis is the imaginary component of the eigenvalue.

## 3. Eigenvalue collision in PT-symmetric problems

In this section we look in some detail at the collision process of two eigenvalues. Our goal is to derive an asymptotic expansion for the eigenvalues and eigenfunctions near collision, using $I - I_c$ as our small parameter. The analysis in this section will again be formal.

For this purpose let $I$ be near a critical value $I_c$ the first two eigenvalues (ordered by their real parts) $\lambda_1$ and $\lambda_2$ collide; that is, the eigenvalues are real for $I$ just below $I_c$ but coincide at $I = I_c$. Let the associated eigenfunctions be $u_1$ and $u_2$, respectively. We should note that the analysis we will present below is valid near the collision of *any* two eigenvalues but we focus on the first collision since this is most relevant to the stable bifurcation picture to be presented subsequently.

It follows from equation (1.9) that they satisfy

$$\int_{-1}^{1} u_1(x)u_2(x)\, dx = 0. \tag{3.1}$$

We term this property as $PT - orthogonality$. It is different, of course, from usual orthogonality. As long as the eigenvalues are real, their associated eigenfunctions are (up to a normalization) PT-symmetric. Therefore PT-orthogonality is the same as orthogonality in the Krein inner product: $[f, g] = \int f(x)g^*(-x)\, dx = 0$ [12].

Now, as $I$ approaches $I_c$, we assume that $\lambda_i$ and $\lambda_{ii}$ approach a common value $\lambda^{(0)}$. At the same time, $u_1$ and $u_2$ also approach a common function that we call $u^{(0)}$. Notice, that this statement is made after some proper normalization, since the problem is linear. We really mean that the ratio $u_1/u_2$ approaches a complex constant. To see why, assume to the contrary that there are two independent eigenfunctions $u_1$ and $u_2$ associated with the real eigenvalue $\lambda^{(0)}$. Equation (1.9) is of second order, and therefore its solution space is spanned by two independent functions. However, $u_1$ and $u_2$ cannot form such a basis, since they both satisfy homogeneous Dirichlet boundary condition, while clearly there are solutions of (1.9) that do not satisfy such conditions.

Having established that the collision eigenvalue $\lambda^{(0)}$ has an algebraic multiplicity 2, but a geometric multiplicity 1 (and thus we can say that the operator $M$ is Jordan at $I_c$), we proceed to study what happens when $I$ increases past $I_c$.

We first write the eigenvalue problem for the operator $M$ near $I_c$ in the form

$$u_{xx} + i(I_c + \varepsilon a)xu + \lambda u = 0, \tag{3.2}$$

where the sign of the parameter $a$ determines if we move up or down from $I_c$. Because of the singular nature of $M$ at $I_c$, it turns out that the perturbation scheme is not analytic. Rather, we need to expand the eigenfunction $u$ and the eigenvalue $\lambda$ in powers of $\varepsilon^{1/2}$:

$$u = u^{(0)} + \varepsilon^{1/2}u^{(1)} + \varepsilon u^{(2)} + ...., \quad \lambda = \lambda^{(0)} + \varepsilon^{1/2}\lambda^{(1)} + \varepsilon\lambda^{(2)} + .... \tag{3.3}$$

At the first order we find of course

$$\mathcal{L}u^{(0)} := u^{(0)}_{xx} + ixI_c u^{(0)} + \lambda^{(0)}u^{(0)} = 0, \tag{3.4}$$

where we used this opportunity to introduce the operator notation $\mathcal{L}$. It is important to note that the PT-orthogonality noted above implies

$$\int_{-1}^{1} u^{(0)}(x)^2 \, dx = 0. \tag{3.5}$$

At the $O(\varepsilon^{1/2})$ level we get

$$\mathcal{L}u^{(1)} = u^{(1)}_{xx} + iI_c x u^{(1)} + \lambda^{(0)} u^{(1)} = -\lambda^{(1)} u^{(0)}. \tag{3.6}$$

Multiplying the last equation by $u^{(0)}$, integrating over the interval $[-1, 1]$, and using the PT-orthogonality (3.5) we see that equation (3.6) is solvable. However, unlike the case of regular eigenvalue perturbation schemes, we gain no information on $\lambda^{(1)}$ at this level. We therefore need to proceed to the $O(\varepsilon)$ level:

$$\mathcal{L}u^{(2)} = u^{(2)}_{xx} + iI_c x u^{(2)} + \lambda^{(0)} u^{(2)} = -\lambda^{(1)} u^{(1)} - \lambda^{(2)} u^{(0)} - iaxu^{(0)}. \tag{3.7}$$

To get a solvability condition we multiply both sides by $u^{(0)}$, integrate over the interval and use (3.5) to find

$$-\lambda^{(1)} \int_{-1}^{1} u^{(1)}(x) u^{(0)}(x) \, dx = ia \int_{-1}^{1} x u^{(0)}(x)^2 \, dx \tag{3.8}$$

It is convenient at this point to introduce some notation. First, we set

$$u^{(0)} = \operatorname{Re} u^{(0)}(x) + i \operatorname{Im} u^{(0)}(x). \tag{3.9}$$

Using the PT symmetry of $u^{(0)}$, we choose a normalization in which $\operatorname{Re} u^{(0)}$ is even, while $\operatorname{Im} u^{(0)}$ is odd. Then we define the real parameter $a_1$ through

$$a_1 = -i \int_{-1}^{1} x u^{(0)}(x)^2 \, dx = 2 \int_{-1}^{1} x \operatorname{Re} u^{(0)}(x) \operatorname{Im} u^{(0)}(x) \, dx \tag{3.10}$$

Next, let $K(x)$ be the solution of the nonhomogeneous ODE

$$\mathcal{L}K = K_{xx} + iI_c x K + \lambda^{(0)} K = u^{(0)}, \quad K(\pm 1) = 0. \tag{3.11}$$

The identity (3.5) ensures that equation (3.11) is solvable. Using this canonical function $K$, we express $u^{(1)}$ as

$$u^{(1)}(x) = -\lambda^{(1)} K(x). \tag{3.12}$$

Finally, we define

$$b = \int_{-1}^{1} K(x)u^{(0)}(x)\,dx. \tag{3.13}$$

Notice that $K$ is also PT-symmetric, i.e. $K(x) = \bar{K}(-x)$. In particular $\operatorname{Re} K$ is even while $\operatorname{Im} K$ is odd. A numerical integration of $K$, and a numerical evaluation of the functionals in (3.10) and (3.13) gives

$$a_1 \approx 0.29, \quad b \approx 0.12. \tag{3.14}$$

We use the notation above to derive from equation (3.6) the relation

$$\lambda_1^2 = -aa_1/b \approx -2.42a. \tag{3.15}$$

When $a < 0$, $I$ is just below $I_c$ and there are two real solutions. The negative one corresponds to the first eigenvalue, and the positive one corresponds to the second eigenvalue. On the other hand, when $I$ increases past $I_c$, i.e. when $a > 0$, there is a complex pair of conjugate solutions. This implies that the critical eigenvalue $\lambda^{(0)}$ splits into a complex conjugate pair with the $\mathcal{O}(\varepsilon^{1/2})$ correction $\lambda^{(1)}$ being purely imaginary. We remark that the same analysis applies to any collision of real eigenvalues. In Figure 4 we compare the asymptotic expansion (3.3) of the imaginary part of the first eigenvalue (dashed line) with the numerically computed value (solid line).

To find the next term $\lambda^{(2)}$ in the eigenvalue expansion, we proceed further to the $O(\varepsilon^{3/2})$ level:

$$\mathcal{L}u^{(3)} = -\lambda^{(2)}u^{(1)} - \lambda^{(1)}u^{(2)} - \lambda^{(3)}u^{(0)} - iaxu^{(1)}. \tag{3.16}$$

Multiply equation (3.16) by $u^{(0)}$ and integrate by parts over $[-1, 1]$ to get

$$\lambda_2 \int_{-1}^{1} u^{(1)}u^{(0)}\,dx + \lambda^{(1)} \int_{-1}^{1} u^{(2)}u^{(0)}\,dx + ia \int_{-1}^{1} xu^{(1)}u^{(0)}\,dx = 0. \tag{3.17}$$

Substituting the relation (3.12) into equation (3.17) and dividing by $\lambda^{(1)}$ gives

$$-b\lambda^{(2)} + \int_{-1}^{1} u^{(2)}u^{(0)}\,dx - ia \int_{-1}^{1} xK(x)u^{(0)}\,dx = 0. \tag{3.18}$$

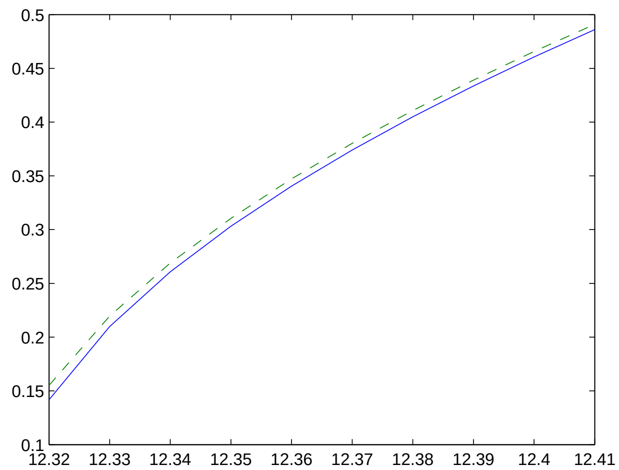


FIGURE 4. The (numerically computed) imaginary part of the leading eigenvalue (solid line) compared to the leading order term in the asymptotic expansion (3.3 (dashed line). The horizontal axis in this figure is the current $I$, and the vertical axis is the imaginary component of the eigenvalue.

It is useful at this point to introduce additional canonical functions and functionals in the spirit of $K$, $a_1$ and $b$ defined above. Thus we define two canonical functions $\zeta$ and $w$ through:

$$\mathcal{L}\zeta = K(x) - \theta_1. \quad \zeta(\pm 1) = 0. \tag{3.19}$$

Here $\theta_1$ is a constant chosen such that equation (3.19) is solvable. Namely

$$\theta_1 = \int_{-1}^{1} K(x)u^{(0)}(x)dx \Big/ \int_{-1}^{1} u^{(0)}(x)dx. \tag{3.20}$$

The next canonical function $w(x)$ is defined by

$$\mathcal{L}w = -ixu^{(0)} - \theta_2, \quad w(\pm 1) = 0, \tag{3.21}$$

with

$$\theta_2 = -i\int_{-1}^{1} xu^{(0)}(x)^2\, dx \Big/ \int_{-1}^{1} u^{(0)}(x)dx. \tag{3.22}$$

Using $\zeta$, $w$, and $K$ we can write

$$u^{(2)}(x) = (\lambda^{(1)})^2\zeta(x) + aw(x) + \lambda^{(2)}K(x). \tag{3.23}$$

We further define the functionals

$$d_1 = \int_{-1}^{1} \zeta u^{(0)}\,dx,\quad d_2 = \int_{-1}^{1} w u^{(0)}\,dx,\quad d_3 = i\int_{-1}^{1} xK(x)u^{(0)}(x)\,dx, \tag{3.24}$$

A numerical computation gives

$$d_1 \approx -0.014,\quad d_2 \approx -0.02,\quad d_3 \approx -0.02. \tag{3.25}$$

Using these functionals and formula (3.23) for $u^{(2)}$ in equation (3.18), and using (3.15) to eliminate the contribution of the coefficients $\theta_1, \theta_2$, gives the following expression for $\lambda_2$:

$$\lambda^{(2)} = \left(ad_3 - ad_2 + (\lambda^{(1)})^2 d_1\right)/2b. \tag{3.26}$$

We can conclude now an interesting fact. The eigenvalue $\lambda^{(0)}$ splits into two eigenvalues as the current is varied away from $I_c$. For $I < I_c$ we obtain a real pair, while for $I > I_c$ we obtain a complex pair. This splitting manifests itself in the two values for $\lambda^{(1)}$ obtained from equation (3.15). We can also see how the single eigenfunction $u^{(0)}$ splits into two eigenfunctions through equation (3.6). On the other hand, $\lambda^{(1)}$ appears in equation (3.26) only through its square. Therefore equation (3.26) implies that $\lambda^{(2)}$ is unique and real. In particular, if we draw the real part of the colliding eigenvalues as a function of $I$ near the collision, we obtain that the function is not analytic at $I_c^-$. In fact $d\lambda_0/dI$ blows up as we approach $I_c$ from below, due to the $\mathcal{O}(\varepsilon^{1/2})$ contribution to the expansion (3.3) coming from (3.15). Yet due to the fact that $\lambda^{(1)}$ is purely imaginary for $I$ just above $I_c$, we see that the graph of $\mathrm{Re}\,\lambda_1(I)$ $(= \mathrm{Re}\,\lambda_2(I))$ is differentiable from the right at $I = I_c$. This analytical conclusion is verified in the numerical solution.

In Figure 5 we compare the asymptotic expansion (3.3) of the real component of the first eigenvalue (dashed line) with the numerically computed value (solid line). The two lines are almost indistinguishable, and the error is $O(0.001)$.

## 4. The bifurcation from (N) to (S)

In the next two sections we study the shape and the stability of the bifurcation branch from the normal (N) state to either the stationary (S) or periodic (P) state using formal asymptotic

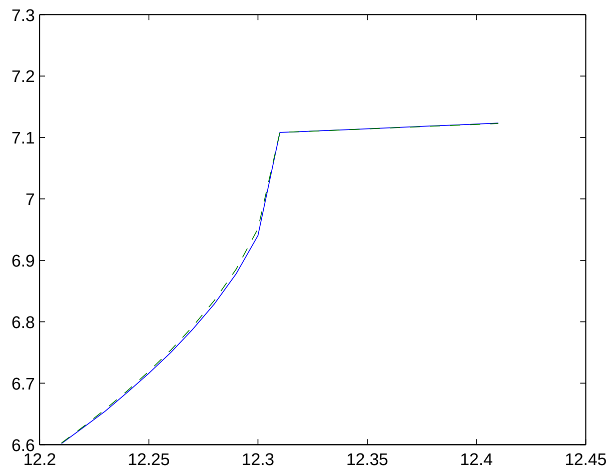


FIGURE 5. The (numerically computed) real part of the leading eigenvalue (solid line) compared to the leading order term in the asymptotic expansion (3.3 (dashed line). The horizontal axis in this figure is the current $I$, and the vertical axis is the real component of the eigenvalue.

expansions and multiple time-scales. Later, in Section 6, we will present a rigorous justification of these calculations by appealing to center manifold theory.

For this purpose, it is convenient to rewrite the system (1.4)-(1.5) as a single nonlocal complex equation by first solving for the electric potential $\phi$ in (1.5), thereby obtaining

$$\phi = -Ix + \frac{i}{2}\int_0^x \left(\psi\psi_x^* - \psi^*\psi_x\right)\, dx'. \tag{4.1}$$

Then we substitute this into (1.4) to obtain

$$\psi_t = \psi_{xx} + ixI\psi + \Gamma\psi + \mathcal{N}[\psi], \tag{4.2}$$

where we have introduced notation for the cubic nonlinearity

$$\mathcal{N}[\psi] := -\left|\psi\right|^2\psi + \frac{1}{2}\,\psi\int_0^x \left(\psi\psi_x^* - \psi^*\psi_x\right)\, dx'. \tag{4.3}$$

As always, this equation is augmented with Dirichlet boundary conditions at $x = \pm 1$ and initial conditions.

We work in this section with a fixed current $I$ in the regime $I < I_c$ where the leading eigenvalue $\lambda_1$ of the operator $M$ is real (as is the entire spectrum) and we expect a bifurcation to a stationary state. The transition takes place exactly when $\Gamma$ crosses the value $\lambda_1$. This defines the curve $\Gamma = \Gamma_1(I)$.

We denote the leading eigenfunction by $u_1$, and normalize it as usual by $u_1(0) = 1$. Recall that $u_1$ satisfies the equation

$$L_1 u_1 := (u_1)_{xx} + ixIu_1 + \lambda_1 u_1 = 0. \tag{4.4}$$

To find the solution to (4.2) just above the curve $\Gamma_1(I)$, we set $\Gamma = \lambda_1 + \varepsilon$, so that (4.2) takes the form

$$\psi_t = L_1\psi + \varepsilon\psi + \mathcal{N}[\psi]. \tag{4.5}$$

Anticipating the contribution of the nonlinear terms in the forthcoming expansion, we seek a solution that is proportional to leading order to $u_1$ plus a small perturbation:

$$\psi(x,t) \sim \varepsilon^{1/2}\alpha(\tau)u_1(x) + \varepsilon^{3/2}\psi_1(x,\tau) + ... \tag{4.6}$$

Since $\Gamma$ is a small perturbation of $\lambda_1$, we expect the time evolution to be slow, hence we have introduced the time-scale $\tau = \varepsilon t$. Our goal here is to compute the function $\alpha(\tau)$ and thus to obtain completely the leading order term in the expansion.

Substituting the ansatz (4.6) into (4.5), we see through (4.4) that the $O(\varepsilon^{1/2})$ terms are balanced by the choice above for $\psi$, with the function $\alpha(\tau)$ not yet determined. Proceeding the $O(\varepsilon^{3/2})$ level, we obtain

$$-L_1\psi_1 = (-\alpha_\tau + \alpha)\, u_1 + \mathcal{N}[\alpha u_1]. \tag{4.7}$$

To obtain a solvability condition for $\psi_1$, we multiply equation (4.7) by $u_1(x)$ and integrate over $[-1, 1]$. We obtain the following equation for $\alpha(\tau)$:

$$\alpha_\tau = \alpha + \frac{\int_{-1}^{1} \mathcal{N}[\alpha u_1]u_1\, dx}{\int_{-1}^{1} u_1^2\, dx} = \alpha + \chi_{11}|\alpha|^2\alpha. \tag{4.8}$$

After a lengthy calculation, the coefficient $\chi_{11}$ is found to be given by

$$\chi_{11} = \left(\frac{1}{2}c_{1111} - \gamma_{11}\right)/\beta, \tag{4.9}$$

where

(4.10)

$$\beta = \int_{-1}^{1} u_1^2\,dx, \ \ \gamma_{11} = \int_{-1}^{1} |u_1|^2 u_1^2, \ \ c_{1111} = \int_{-1}^{1} u_1^2\theta_{11}\,dx, \ \ \theta_{11} = \int_{0}^{x} u_1(u_1)_x^* - u_1^*(u_1)_x\,dx'.$$

The stability of the bifurcation branch depends on the sign of $\chi_{11}$. Examination of (4.8) reveals that there is stable branch of equilibria if $\chi_{11} < 0$, with $|\alpha| = \frac{1}{\sqrt{-\chi_{11}}}$. We note that in light of the rotational invariance of the whole problem (4.2), there is in fact an entire circle of equilibria with $\alpha$ given by $\frac{1}{\sqrt{-\chi_{11}}}e^{i\theta_0}$, $\theta_0 \in [0, 2\pi)$.

On the other hand, if $\chi_{11} > 0$ then an unstable branch of equilibria exists for $\varepsilon$ small and negative wherein (4.8) is replaced by

$$\alpha_\tau = -\alpha + \chi_{11}|\alpha|^2\alpha$$

and the equilibrium value of $\alpha$ is given by $1/\sqrt{\chi_{11}}$. It turns out that both signs can occur, depending on the current $I$. For instance, when $I = 7$, we get

$$\beta(7) = 0.785, \ \gamma_{11}(7) = 0.652, \ c_{1111}(7) = 0.375, \ \ \chi_{11}(7) = -0.592,$$

while

$$\beta(11) = 0.403, \ \gamma_{11}(11) = 0.449, \ c_{1111}(11) = 0.924, \ \ \chi_{11}(11) = 0.03.$$

A careful computation shows that $\chi_{11}$ vanishes at $I \approx 10.93$. We denote this critical value by $I_k$ and conclude that the bifurcation from (N) to (S) is stable (type II) for $I < I_k$ and unstable for $I_k < I < I_c$. Finally we point out that the expansion in this section breaks down for $I$ near $I_k$, and one needs to proceed to higher order terms there.

## 5. The bifurcation from (N) to (P)

In this section we compute asymptotic approximations for the solution in the oscillatory (P) state. In this state the solution $\psi(x,t)$ is time-periodic. We assume in this section that the current $I$ is fixed in the regime $I > I_c$. The transition to the (P) state takes the form of a Hopf bifurcation; namely, the real part of the spectrum is zero, and the bottom of the spectrum consists of a conjugate pair of purely imaginary eigenvalues. We point out that the spectrum of the operator $M$ (cf. (1.9)) *does* have a nonzero real part, but this real part is exactly balanced at the transition curve by our choice in this section of $\Gamma = \Gamma_1(I) + \varepsilon = \operatorname{Re}\lambda_1 + \varepsilon$ with $\varepsilon > 0$ in (4.2). Hence, if we extend our definition (4.4) of the linear operator $L_1$ to include the case where $\lambda_1$ is complex via

$$L_1 u := u_{xx} + ixIu + (\operatorname{Re}\lambda_1)u, \tag{5.1}$$

then it is $L_1$ that possesses a pair of purely imaginary eigenvalues with corresponding eigenfunctions $u_1$ and $u_2$ satisfying

$$L_1 u_1 = -i(\operatorname{Im}\lambda_1)u_1 \quad \text{and} \quad L_1 u_2 = i(\operatorname{Im}\lambda_1)u_2. \tag{5.2}$$

We choose to normalize the eigenfunctions so that $u_j(0) = 1,\ j = 1, 2$, and we assume $\lambda_1$ is defined so that $\operatorname{Im}\lambda_1 > 0$. With the above choice of $\Gamma$ and definition of $L_1$, we again find that (1.4)-(1.5) takes the form (4.5).

At leading order, we expect the solution to be comprised of a linear combination of solutions to the equation $\psi_t = L_1\psi$. This leads us to seek a periodic solution to (4.5) of the form

$$\psi(x,t) \sim \varepsilon^{1/2}\left(\alpha_1(\tau)e^{-i\operatorname{Im}\lambda_1 t}u_1(x) + \alpha_2(\tau)e^{i\operatorname{Im}\lambda_1 t}u_2(x)\right) + \varepsilon^{3/2}\psi_1(x,t) + \ldots \tag{5.3}$$

Here $\psi_1$ is assumed to be a periodic function of $t$ with period $p_\varepsilon = 2\pi/\operatorname{Im}\lambda_1 + \mathcal{O}(\varepsilon)$ and $\alpha_1, \alpha_2$ are coefficients that we expect to evolve slowly in time. We thus have set $\alpha_i = \alpha_i(\tau)$, where, just as before, $\tau = \varepsilon t$. Our goal here is to compute the functions $\alpha_i(\tau)$ and thus to obtain completely the leading order term in the expansion.

Substituting the ansatz (5.3) into equations (4.5) shows that the $O(\varepsilon^{1/2})$ terms are balanced by the choice above for $\psi$, with the coefficients $\alpha_i(\tau)$ not yet determined. Proceeding to the $O(\varepsilon^{3/2})$ level, we obtain

$$(\psi_1)_t - L_1\psi_1 = \quad e^{-i\operatorname{Im}\lambda_1 t}u_1\left(\alpha_1 - \alpha_{1\tau}\right) + e^{i\operatorname{Im}\lambda_1 t}u_2\left(\alpha_2 - \alpha_{2\tau}\right) \tag{5.4}$$

$$+ \quad \mathcal{N}[\alpha_1 e^{-i\operatorname{Im}\lambda_1 t}u_1 + \alpha_2 e^{i\operatorname{Im}\lambda_1 t}u_2], \tag{5.5}$$

cf. (4.3).

To obtain a first solvability condition for $\psi_1$, we multiply equation (5.5) by $e^{i\operatorname{Im}\lambda_1 t}u_1(x)$ and integrate over $[-1,1] \times [0, 2\pi/\operatorname{Im}\lambda_1]$. A second solvability condition is obtained by integrating similarly against the function $e^{-i\operatorname{Im}\lambda_1 t}u_2(x)$. We note that through (5.2) and the assumed periodicity of $\psi_1$, the first integration against the left-hand side of (5.5) yields, after an integration by parts:

$$\int_{-1}^{1}\int_{0}^{2\pi/\mathrm{Im}\,\lambda_1} \left((\psi_1)_t - L_1\psi_1\right) e^{i\mathrm{Im}\,\lambda_1 t} u_1\, dt\, dx$$
$$= -i\mathrm{Im}\,\lambda_1 \int_{-1}^{1}\int_{0}^{2\pi/\mathrm{Im}\,\lambda_1} \psi_1 e^{i\mathrm{Im}\,\lambda_1 t} u_1\, dt\, dx - \int_{-1}^{1}\int_{0}^{2\pi/\mathrm{Im}\,\lambda_1} \psi_1 e^{i\mathrm{Im}\,\lambda_1 t} L_1 u_1\, dt\, dx = \mathcal{O}(\varepsilon).$$

Similarly, the left-hand side in the second integration vanishes to leading order.

After a lengthy but straight-forward calculation, these two integrations then give rise to a pair of equations for the coefficients $\alpha_i(\tau)$, namely

$$(\alpha_1)_\tau = \alpha_1 + \left(\chi_{11}|\alpha_1|^2 + \chi_{12}|\alpha_2|^2\right)\alpha_1, \tag{5.6}$$

$$(\alpha_2)_\tau = \alpha_2 + \left(\chi_{11}^*|\alpha_2|^2 + \chi_{12}^*|\alpha_1|^2\right)\alpha_2. \tag{5.7}$$

Here the coefficient $\chi_{11}$ is again given by (4.9) while $\chi_{12}$ is defined by through:

$$\chi_{12} = \left(\frac{1}{2}c_{1122} + \frac{1}{2}c_{1212} - 2\gamma_{21}\right)/\beta, \tag{5.8}$$

where

(5.9)

$$\beta := \int_{-1}^{1} u_1^2\, dx,\ \gamma_{ij} = \int_{-1}^{1} |u_i|^2 u_j^2\, dx,\ c_{ijkl} = \int_{-1}^{1} u_i u_j \theta_{kl}\, dx,\ \theta_{kl} = \int_0^x \left(u_k (u_l)_x^* - u_l^*(u_k)_x\right)\, dx'.$$

We note that the notation above is consistent with (4.10).

To analyze the evolution of the $\alpha_i's$, we note that from (5.6)-(5.7) it is easy to derive the system

$$(|\alpha_1|)_\tau = |\alpha_1|(1 + (\mathrm{Re}\,\chi_{11}|\alpha_1|^2 + \mathrm{Re}\,\chi_{12}|\alpha_2|^2), \tag{5.10}$$

$$(|\alpha_2|)_\tau = |\alpha_2|(1 + (\mathrm{Re}\,\chi_{11}|\alpha_2|^2 + \mathrm{Re}\,\chi_{12}|\alpha_1|^2), \tag{5.11}$$

governing the evolution of the moduli of the $\alpha_i$. Now if the initial conditions for (4.5) are taken such that $\alpha_1(0) \neq 0$ and $\alpha_2(0) \neq 0$, then it follows from (5.10)-(5.11) that $\alpha_1$ and $\alpha_2$ are non-zero for all future times. In this case, we introduce the Ricatti transform $r := |\alpha_1|/|\alpha_2|$.

From (5.10)-(5.11) we obtain that

$$\begin{aligned} r' &= \frac{|\alpha_2||\alpha_1|' - |\alpha_2|'|\alpha_1|}{|\alpha_2|^2} \\ &= \frac{|\alpha_1||\alpha_2|(\operatorname{Re}\chi_{11} - \operatorname{Re}\chi_{12})(|\alpha_1|^2 - |\alpha_2|^2)}{|\alpha_2|^2} \\ &= (\operatorname{Re}\chi_{11} - \operatorname{Re}\chi_{12})|\alpha_2|^2 r(r^2-1) = -(\operatorname{Re}\hat{\chi})\,|\alpha_2|^2 r(r^2-1), \end{aligned} \tag{5.12}$$

where $\cdot'$ denotes $\frac{d}{d\tau}$ and we have introduced the complex constant

$$\hat{\chi} := \chi_{12} - \chi_{11}. \tag{5.13}$$

Provided that $\operatorname{Re}\hat{\chi} > 0$, we learn from (5.12) that $r \to 1$ at an exponential rate as $\tau \to \infty$. One can indeed check numerically that for $I > I_c$, the inequality $\operatorname{Re}\hat{\chi} > 0$ holds. See Figure 6.

Returning to the system (5.10)-(5.11) with this information, we can determine the asymptotic value of the modulus of both $\alpha_1$ and $\alpha_2$ to be

$$|\alpha_1(\tau)| \sim |\alpha_2(\tau)| \sim \sqrt{\frac{-1}{(\operatorname{Re}\chi_{11} + \operatorname{Re}\chi_{12})}} = \sqrt{\frac{1}{\operatorname{Re}\tilde{\chi}}} \quad \text{for } \tau >> 1$$

where we have introduced another complex constant

$$\tilde{\chi} := -(\chi_{11} + \chi_{12}). \tag{5.14}$$

Numerical calculation reveals that $\operatorname{Re}\tilde{\chi} > 0$ for $I > I_c$ as well. Again, see Figure 6.

Substitution into (5.6)-(5.7) then yields a linear dependence on $\tau$ of the phase of both $\alpha_1$ and $\alpha_2$ for $\tau$ large with

$$\alpha_1(\tau) = \alpha_2^*(\tau) = \sqrt{\frac{1}{\operatorname{Re}\tilde{\chi}}}\, e^{i\omega\tau} \quad \text{for } \tau >> 1, \tag{5.15}$$

where $\omega := -\frac{\operatorname{Im}\tilde{\chi}}{\operatorname{Re}\tilde{\chi}}$. Of course, in light of the rotational invariance of all of the above equations, the asymptotic state (5.15) holds only up to a constant rotation. As an example, we computed the functionals $\chi_{11}$ and $\chi_{12}$ for $I = 20$. In this case the leading eigenvalues are $\lambda_{1,2} \approx 8.64 \pm 5.25\, i$, the amplitude is $\approx 0.92$ and $\omega \approx -1.81$. Unlike the case $I < I_c$, the transition from (N) to (P) is always stable.

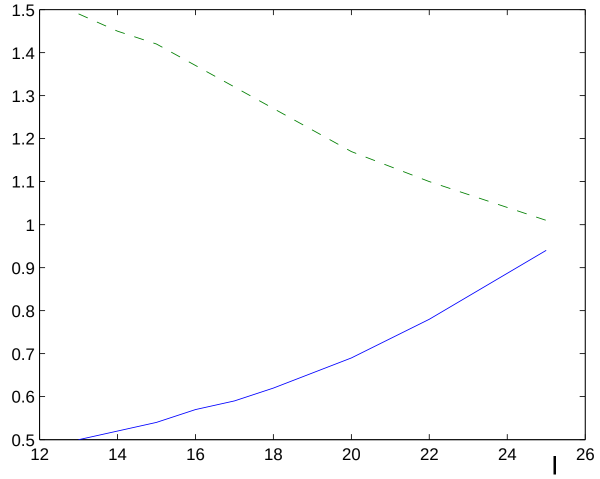


FIGURE 6. The dashed curve represents the graph of the parameter $\mathrm{Re}\,\hat{\chi}$ as a function of applied current $I$ and the solid curve represents the graph of $\mathrm{Re}\,\tilde{\chi}$.

The case where either $\alpha_1(0) = 0$ or $\alpha_2(0) = 0$ is treated separately. In view of (5.6)-(5.7), if, for example, $\alpha_1(0) = 0$, then necessarily $\alpha_1(\tau) \equiv 0$. Hence, $\alpha_2$ evolves by

$$(\alpha_2)_\tau = \left(1 + \chi_{11}^* |\alpha_2|^2\right) \alpha_2.$$

From this and (5.11) it easily follows that

$$\alpha_2(\tau) \sim \frac{1}{\sqrt{-\mathrm{Re}\,\chi_{11}}} e^{i \frac{\mathrm{Im}\chi_{11}}{\mathrm{Re}\chi_{11}} \tau} \quad \text{for } \tau >> 1 \tag{5.16}$$

provided that $\mathrm{Re}\,\chi_{11} < 0$. Since $\chi_{11} = -\frac{1}{2}(\tilde{\chi} + \hat{\chi})$, and both $\tilde{\chi}$ and $\hat{\chi}$ have been shown numerically to be positive, we do indeed have this condition on $\chi_{11}$ met. Similarly, when $\alpha_2(0) = 0$, one finds that $\alpha_2(\tau) \equiv 0$ and

$$\alpha_1(\tau) \sim \frac{1}{\sqrt{-\mathrm{Re}\,\chi_{11}}} e^{-i \frac{\mathrm{Im}\chi_{11}}{\mathrm{Re}\chi_{11}} \tau} \quad \text{for } \tau >> 1 \tag{5.17}$$

provided again that $\mathrm{Re}\chi_{11} < 0$.

To summarize, in the generic case where neither $\alpha_1(0)$ nor $\alpha_2(0)$ vanish, the solution in the (P) state is to leading order

$$\psi(x,t) \sim \varepsilon^{1/2} A \left(e^{-i(\mathrm{Im}\,\lambda_1 + \omega\varepsilon)t} u_1(x) + e^{i(\mathrm{Im}\,\lambda_1 + \omega\varepsilon)t} u_2(x)\right). \tag{5.18}$$

Therefore the solution is time-periodic with period

$$p_\varepsilon \sim 2\pi/\left(\operatorname{Im}\lambda_1 + \omega\varepsilon\right) + o(\varepsilon). \tag{5.19}$$

In a subsequent article, we will examine in some detail how the asymptotic solution (5.18) extends deeper into the nonlinear regime of the (P) state.

## 6. Rigorous bifurcation theory

In this section we will establish a rigorous justification for the formally derived expansions and bifurcations of the previous two sections. As was done earlier, we will set the normal conductivity $\sigma$ equal to one and set the interval $[-L, L]$ to be $[-1, 1]$, thus focusing on the interplay between the externally forced current $I$ and the temperature-dependent parameter $\Gamma$. As was done earlier, we will treat the case of Dirichlet boundary conditions, though similar conclusions can be rigorously established for the case of Neumann boundary conditions. Though in our previously derived formal asymptotics, we only pursued the case of bifurcation off of the principal eigenvalue, at the end of this section we also treat bifurcation off of any eigenvalue.

### 6.1. **Spectral properties of linear operator.**

We begin by recalling some notation and collecting some facts about the linear operator $M$ given by (1.9).

**Lemma 6.1.** *The spectrum of $M$ consists only of point spectrum, denoted by $\{-\lambda_j\}$ with corresponding eigenfunctions $\{u_j\}$. If $(\lambda_j, u_j)$ is an eigenpair satisfying*

$$Mu_j = -\lambda_j u_j, \quad u_j(\pm 1) = 0,$$

*then*

$$\operatorname{Re}\lambda_j > 0, \quad \text{and} \quad |\operatorname{Im}\lambda_j| < I. \tag{6.1}$$

*Thus, in particular we may order the eigenvalues $\lambda_1, \lambda_2, \dots$ according to the size of their real part, with $0 < \operatorname{Re}\lambda_1 \le \operatorname{Re}\lambda_2 \le \dots$. The PT-symmetry of the operator is reflected in the fact that if $(\lambda_j, u_j)$ is an eigenpair then so is $(\lambda_j^*, u_j^\dagger)$ where $u_j^\dagger(x) := u_j^*(-x)$. Finally, for each* *positive integer $\ell$, there exists a positive value of the current $I$, which we denote by $I_\ell$, with $I_\ell < I_{\ell+1}$ such that*

$$\lambda_{2\ell-1},\ \lambda_{2\ell} \in \mathbb{R} \quad \textit{for} \quad I \le I_\ell \quad \textit{while} \quad \lambda_{2\ell-1} = \lambda_{2\ell}^* \notin \mathbb{R} \quad \textit{for} \quad I > I_\ell. \tag{6.2}$$

**Remark 6.2.** We note that in earlier parts of this paper, the critical value $I_1$ was denoted by $I_c$. In the next subsection, we revert to this notation to keep consistency with earlier sections.

*Proof.* The fact that the spectrum consists entirely of eigenvalues follows from standard spectral theory. To verify (6.1), multiply the equation $Mu = -\lambda u$ by $u^*$ and integrate to obtain

$$\operatorname{Re}\lambda = \frac{\int_{-1}^{1} |u_x|^2\, dx}{\int_{-1}^{1} |u|^2\, dx} \quad \text{and} \quad \operatorname{Im}\lambda = \frac{I\int_{-1}^{1} x\,|u|^2\, dx}{\int_{-1}^{1} |u|^2\, dx}.$$

The fact that if $(\lambda, u)$ is an eigenpair then so is $(\lambda^*, u^\dagger)$ follows by inspection. The assertion that the spectrum is real for small $I$ follows from Theorem 4.1 of [12] while the existence of a critical values $\{I_\ell\}$ beyond which pairs of real eigenvalues collide to form conjugate pairs is a result of [14, 15]. ☐

Next, for any fixed positive integer $\ell$, we introduce the operator

$$L_\ell u := Mu + (\operatorname{Re}\lambda_{2\ell-1})\, u. \tag{6.3}$$

Then $L_\ell$ has spectrum shifted from that of $M$ by $\operatorname{Re}\lambda_{2\ell-1}$ so that $L_\ell u_j = -\mu_j\, u_j$ where $\mu_j = \lambda_j - \operatorname{Re}\lambda_{2\ell-1}$. Based on the behavior of the spectrum of $M$ described in Lemma 6.1, we have the following scenario for $L_\ell$.

$$\text{For } 0 \le I < I_\ell : \tag{6.4}$$
$$\mu_{2\ell-1} = 0, \quad \operatorname{Re}\mu_j < 0 \text{ for } 1 \le j < 2\ell - 1, \quad \mu_j > 0 \text{ for } j > 2\ell - 1.$$

$$\text{For } I = I_\ell : \tag{6.5}$$
$$\mu_{2\ell-1} = \mu_{2\ell} = 0, \quad \operatorname{Re}\mu_j < 0 \text{ for } 1 \le j < 2\ell - 1, \quad \mu_j > 0 \text{ for } j > 2\ell.$$

$$\text{For } I > I_\ell : \tag{6.6}$$
$$\operatorname{Re}\mu_{2\ell-1} = \operatorname{Re}\mu_{2\ell} = 0, \quad \operatorname{Im}\mu_{2\ell-1} = -\operatorname{Im}\mu_{2\ell} \neq 0,$$
$$\operatorname{Re}\mu_j < 0 \text{ for } 1 \le j < 2\ell - 1, \quad \operatorname{Re}\mu_j > 0 \text{ for } j > 2\ell.$$

We also note that by PT-symmetry, if we normalize all eigenfunctions so that $u_\ell(0) = 1$, then we must have

$$u_{2\ell-1} = u_{2\ell-1}^\dagger \quad \text{when} \quad I \le I_\ell \quad \text{while} \quad u_{2\ell-1} = u_{2\ell}^\dagger \quad \text{when} \quad I > I_l. \tag{6.7}$$

For later use, we also introduce the spectral gap $\zeta_\ell$ given by

$$\zeta_\ell := \inf_j \{|\operatorname{Re}\mu_j| \,:\, \mu_j \text{ is an eigenvalue of } L_\ell \text{ with non-zero real part}\}. \tag{6.8}$$

In what follows it will be convenient to choose a basis for the eigenspace of $L_1$ that is PT-symmetric. This has already been taken care of when $I < I_1$. However, for $I > I_1$ we introduce the basis $v_1$ and $v_2$ given by

$$v_1 := u_1 + u_2 \quad \text{and} \quad v_2 := i(u_1 - u_2). \tag{6.9}$$

The PT-symmetry of this basis follows from (6.7). Note also that this basis satisfies the relations

$$L_1 v_1 = -\operatorname{Im}\lambda_1 v_2, \quad L_1 v_2 = \operatorname{Im}\lambda_1 v_1 \tag{6.10}$$

.

6.2. **Bifurcation from first eigenvalue.** We now develop the rigorous bifurcation theory associated with the stationary and periodic solution branches formally derived in earlier sections. To this end, we wish to reformulate the full nonlinear system (1.3)-(1.6) in such a way as to make it amenable to standard center manifold and bifurcation theory. Accordingly, we first solve for the electric potential $\phi$ in (1.5), thereby obtaining

$$\phi = -Ix + \frac{i}{2}\int_0^x (\psi\psi_x^* - \psi^*\psi_x)\, dx'. \tag{6.11}$$

In this section, we will focus on the stable bifurcations that occurs off of the first eigenvalue of the linear operator and so we henceforth fix the positive integer $\ell$ from the previous section to equal 1 and pick $\Gamma$ to be of the form $\Gamma = \operatorname{Re}\lambda_1 + \varepsilon$. Substituting (4.1) into (1.4), we obtain a single, nonlocal complex equation so that (1.4)-(1.5) can be rewritten as

$$\psi_t = L_1\psi + \mathcal{N}(\psi, \varepsilon) \tag{6.12}$$

where we have introduced $\mathcal{N}(y,\varepsilon) := \mathcal{N}[y] + \varepsilon y$ with

$$\mathcal{N}[y] := -|y|^2 y + \frac{1}{2}\, y \int_0^x (yy_x^* - y^*y_x)\, dx'. \tag{6.13}$$

We recall that as before, the system is augmented with Dirichlet boundary conditions $\psi(\pm1, t) = 0$ along with the normalization $\phi(0, t) = 0$ for $t \geq 0$, and initial conditions, say $\psi(x, 0) = \psi_0(x)$. We take $\varepsilon$ to be small and positive (unless otherwise specified).

We remark that since $L_1 y^\dagger = (L_1 y)^\dagger$ and $\mathcal{N}[y^\dagger] = (\mathcal{N}[y])^\dagger$, it follows easily that the flow (6.12) preserves PT-symmetry in the sense that if $\psi$ is a solution to (6.12), then so is $\psi^\dagger$. Hence, by uniqueness, we note that if the initial data $\psi_0$ is PT-symmetric, i.e. if $\psi_0 = \psi_0^\dagger$, then so is the resulting solution $\psi$.

For later use, we also record the estimate:

**Lemma 6.3.** *There exists a positive constant $C_0$ such that*

$$\|\mathcal{N}[y]\|_{H^1} \leq C_0 \|y\|_{H^1}^3 \quad \textit{for all } y \in H_0^1((-1,1));\mathbb{C}). \tag{6.14}$$

*Proof.* For $y \in H_0^1((-1,1))$ recall that both $\|y\|_{L^\infty}$ and $\|y\|_{H^1}$ are controlled by $\|y_x\|_{L^2}$. We begin by estimating the $H^1$ norm of the local part of $\mathcal{N}$. We find

$$\left\||y|^2 y\right\|_{L^2} \leq \|y\|_{L^6}^3 \leq \|y\|_{L^\infty}^2 \|y\|_{L^2} \leq C\|y\|_{H^1}^3.$$

Similarly,

$$\left\|\left(|y|^2 y\right)_x\right\|_{L^2} \le C\Big(\int_{-1}^{1} |y|^4 \, |y_x|^2 \; dx\Big)^{1/2} \le C \, \|y\|_{L^\infty}^2 \, \|y_x\|_{L^2} \le C \, \|y\|_{H^1}^3$$

Turning now to the nonlocal part of $\mathcal{N}$ we find

$$\left\| y \int_0^x (yy_x^* - y^* y_x)) \; dx' \right\|_{L^2} \le C\Big(\int_{-1}^{1} \big(|y| \int_{-1}^{1} |y| \, |y_x| \; dx'\big)^2 \, dx\Big)^{1/2}$$

$$\le C\Big(\int_{-1}^{1} |y| \, |y_x| \; dx\Big) \, \|y\|_{L^2} \le C \, \|y\|_{L^2}^2 \, \|y_x\|_{L^2} \le C \, \|y\|_{H^1}^3 .$$

Finally, we check that

$$\left\| \Big( y \int_0^x (yy_x^* - y^* y_x)) \; dx' \Big)_x \right\|_{L^2} \le$$

$$C\Big(\int_{-1}^{1} \big(|y_x| \int_{-1}^{1} |y| \, |y_x| \; dx'\big)^2 \, dx\Big)^{1/2} + C\Big(\int_{-1}^{1} |y|^4 \, |y_x|^2 \Big)^{1/2}$$

$$\le C \, \|y\|_{L^2} \, \|y_x\|_{L^2}^2 + C \, \|y\|_{L^\infty}^2 \, \|y_x\|_{L^2} \le C \, \|y\|_{H^1}^3$$

□

Armed with a full understanding of the linearized operator and control on the nonlinear operator provided by Lemma 6.3, we can proceed to construct a center manifold for the flow (6.12). To this end, we will denote by $S_c$ the center subspace associated with $L_1$; that is, $S_c$ is the eigenspace associated with any eigenvalues of $L_1$ having zero real part. For $I \le I_c \; (= I_1)$, we have $S_c = \text{span}\,\{u_1\}$ while for $I > I_c$, $S_c = \text{span}\,\{v_1, v_2\}$ $(= \text{span}\,\{u_1, u_2\})$.

As $\varepsilon$ will play the role of a bifurcation parameter, we then augment (6.12) with the equation

$$\varepsilon_t = 0. \tag{6.15}$$

The theorem below provides for the existence of a finite dimensional invariant manifold associated with the flow (6.12) for each fixed small $\varepsilon$ describing all orbits of sufficiently small norm. This, in effect, allows us to rigorize the formal bifurcation calculations of the previous two chapters by reducing the analysis of the nonlocal P.D.E. to a study of a local system of O.D.E.'s with accompanying rigorous error estimates.

**Theorem 6.4.** *For each value $I > 0$ and positive integer $k$, there is a $C^k$ local center manifold $\mathcal{M} \subset H^1((-1,1);\mathbb{C}) \times \mathbb{R}$ of* (6.12), (6.15) *tangent to the center subspace. The center manifold*

*$\mathcal{M}$ is expressible as a graph over the center subspace in the sense that there exists a $C^k$ map $\Phi : S_c \times \mathbb{R} \to H^1((-1,1);\mathbb{C})$ such that*

$$\mathcal{M} = \bigcup_{|\varepsilon|<\varepsilon_0} \left(\mathcal{M}_\varepsilon \times \{\varepsilon\}\right) \quad \text{where} \quad \mathcal{M}_\varepsilon := \{\Phi(u,\varepsilon) :\, u \in S_c,\ \|u\|_{H^1} < \delta_0,\ |\varepsilon| < \varepsilon_0\} \tag{6.16}$$

*for some sufficiently small positive constants $\delta_0$ and $\varepsilon_0$ depending in particular on $k$. The center manifold is invariant under complex rotation, i.e. $(\psi,\varepsilon) \in \mathcal{M} \implies (e^{i\theta}\psi,\varepsilon) \in \mathcal{M}$ for all $\theta \in \mathbb{R}$ and in fact*

$$e^{i\theta}\Phi(u,\varepsilon) = \Phi(e^{i\theta}u,\varepsilon) \quad \text{for all}\ \theta \in \mathbb{R}. \tag{6.17}$$

*The center manifold is also PT-symmetric, i.e. $(\psi,\varepsilon) \in \mathcal{M} \implies (\psi^\dagger,\varepsilon) \in \mathcal{M}$. If $u = u^\dagger$ then $\Phi(u,\varepsilon) = \Phi^\dagger(u,\varepsilon)$.*

*The discrepancy between the center manifold and the center subspace can be expressed through the estimate*

$$\|\Phi(u,\varepsilon) - u\|_{H^1} \le C_1 \left(\|u\|_{H^1}^3 + |\varepsilon|\, \|u\|_{H^1}\right) \tag{6.18}$$

*which holds for any pair $(u,\varepsilon)$ such that $u \in S_c$ with $\|u\|_{H^1} < \delta_0$ and $|\varepsilon| < \varepsilon_0$, where $C_1$ is a positive constant independent of $u$ and $\varepsilon$.*

*The center manifold is locally invariant for the flow (6.12) in the sense that if $|\varepsilon| < \varepsilon_0$ and the initial data $\psi_0$ lies on $\mathcal{M}_\varepsilon$, then so does the solution $\psi^\varepsilon$ to (6.12) so long as $\|\psi^\varepsilon(\cdot,t)\|_{H^1}$ stays sufficiently small. Hence, for such initial data, one can describe the resulting solution $\psi^\varepsilon(t) = \psi^\varepsilon(\cdot,t)$ through either one or two maps $\beta_j^\varepsilon : [0,\infty) \to \mathbb{C}$ via $\psi^\varepsilon(t) = \Phi(\beta_1^\varepsilon(t)u_1,\varepsilon)$ when $I < I_c$ or $\psi^\varepsilon(t) = \Phi(\beta_1^\varepsilon(t)v_1, \beta_2^\varepsilon(t)v_2,\varepsilon)$ when $I > I_c$). Finally, $\mathcal{M}$ contains all nearby bounded solutions of (6.12) in $H^1$, and in particular, it contains any nearby steady-state or time-periodic solutions.*

*Proof.* We follow a standard center manifold construction, along the lines for example, of [3]. To outline this approach, we first note that in light of conditions (6.1), the spectrum of the operator $-L_1$ lies within the set

$$\{\lambda \in \mathbb{C} :\, |\arg(\lambda + a)| < \frac{\pi}{4}\},$$

for some positive number $a = a(I)$. Hence, $-L_1$ is sectorial and we may assert the existence of an analytic semi-group $\{e^{L_1 t}\}_{t\geq 0}$, cf. [7], Theorem 1.3.4 or [19], section 2.2.3.

We will denote the ($L^2$) projection operators from $H^1((-1,1);\mathbb{C})$ onto the center and stable subspaces of $L_1$ by $\Pi_c$ and $\Pi_s$ respectively. Here by stable subspace we mean the span of all eigenvectors of $L_1$ whose corresonding eigenvalues have negative real aprts. We note that since the real part of all eigenvalues of $L_1$ are non-positive, $L_1$ has no unstable subspace.

A local center manifold is constructed by first constructing a global center manifold for a problem with a truncated nonlinearity through the introduction of a cut-off function $\rho \in C^\infty([0,\infty);[0,1])$ satisfying $\rho(s) \equiv 1$ for $0 \leq s \leq 1$ and $\rho(s) \equiv 0$ for $s \geq 2$. For any $\delta > 0$ we then let $\rho^\delta(s) := \rho(s/\delta)$. We use this cut-off to truncate the nonlinearity $\mathcal{N}(y,\varepsilon)$ by defining $\mathcal{N}^\delta(y,\varepsilon) := \rho^\delta(\|y\|_{H^1})\,\mathcal{N}(y,\varepsilon)$.

The graph map $\Phi : S_c \times \mathbb{R} \to H^1((-1,1);\mathbb{C})$ is then defined by the following procedure: For $u \in S_c$ and $\varepsilon$ fixed, we use a "variation of constants" approach, rephrasing the P.D.E. (6.12) (with the original nonlinearity replaced by $\mathcal{N}^\delta(y,\varepsilon)$) as an integral equation:

$$y(x,t) = \Gamma(u,\varepsilon,y) := e^{L_1 t}u + \int_0^t e^{L_1(t-\tau)}\Pi_c \mathcal{N}^\delta\left(y(x,\tau),\varepsilon\right)\,d\tau$$
$$+ \int_{-\infty}^t e^{L_1(t-\tau)}\Pi_s \mathcal{N}^\delta\left(y(x,\tau),\varepsilon\right)\,d\tau. \tag{6.19}$$

Then one argues that there exists a unique fixed point $y_{u,\varepsilon} = y_{u,\varepsilon}(x,t)$ to (6.19) in the space of functions that grow sufficiently slowly at $t = \pm\infty$ given by

$$Y_\eta := \{y \in C\left((-\infty,\infty); H^1((-1,1))\right) :\ \|y\|_\eta < \infty\}.$$

Here $\|y\|_\eta := \sup_{t\in\mathbb{R}} e^{-\eta|t|}\,\|y(\cdot,t)\|_{H^1((-1,1))}$ and $\eta$ is any fixed positive number less than the spectral gap $\zeta_1$, cf. (6.8). Once the existence of this fixed point is established, we define the map $\Phi$ by

$$\Phi(u,\varepsilon) = y_{u,\varepsilon}(\cdot,0).$$

We should remark that when $t$ is negative, we interpret $e^{L_1 t}\Pi_c$ in (6.19) to mean flow projected onto the finite dimensional center subspace; thus it reduces to a finite number of ordinary differential equations. To see this and to carry out the application of the contraction mapping principle to (6.19), one considers the inverse Laplace transform representations

$$e^{tL_1}\Pi_c := \int_{\Gamma_c} e^{\lambda t}(\lambda I - L_1)^{-1}\, d\lambda, \tag{6.20}$$

and

$$e^{tL_1}\Pi_s := \int_{\Gamma_s} e^{\lambda t}(\lambda I - L_1)^{-1}\, d\lambda, \tag{6.21}$$

where $\Gamma_c$ is a bounded contour enclosing the eigenvalues with zero real part and $\Gamma_s$ is a contour in the left half-plane enclosing the stable spectrum that tends asymptotically to infinity along the lines $a(1-s) \pm ais$ as $s \to \pm\infty$. Since the resolvent is bounded along these contours, it follows from these representations that for every $s_1 \in (0, \zeta_1)$ one has

$$\left\| e^{tL_1}\Pi_c \right\|_{H^1 \to H^1} \leq C, \quad \text{for all } t \in \mathbb{R}, \tag{6.22}$$

$$\left\| e^{tL_1}\Pi_s \right\|_{H^1 \to H^1} \leq Ce^{-(\zeta_1 - s_1)t} \quad \text{for all } t \geq 0. \tag{6.23}$$

Invoking these bounds, and by choosing the parameter $\delta$ in the cut-off of the nonlinearity sufficiently small, the existence of a fixed point to (6.19) follows from (6.14) by the contraction mapping principle, from which we find easily also Lipshitz regularity of $\Phi$. The asserted $C^k$ regularity of $\Phi$ may be established by a careful iterative argument as described in [3], using $C^(k+1)$ regularity of the truncated equations (in general, the center manifold inherits one degree less regularity than the underlying equations); we omit discussion of this delicate point. The local center manifold for the untruncated problem is then realized through (6.16) by choosing $\delta_0$ and $\varepsilon_0$ sufficiently small.

The rotational and PT invariance of $\mathcal{M}$ follow from the fact that the center subspace $S_c$ enjoys these invariances and the fact that for any $\theta_0 \in \mathbb{R}$ one has

$$\Gamma(e^{i\theta_0}u, \varepsilon, e^{i\theta_0}y) = e^{i\theta_0}\Gamma(u, \varepsilon, y), \quad \text{as well as} \quad \Gamma(u^\dagger, \varepsilon, y^\dagger) = (\Gamma(u, \varepsilon, y))^\dagger .$$

With regard to this last assertion, note in particular that $\mathcal{N}^\delta(y^\dagger, \varepsilon) = \left(\mathcal{N}^\delta(y, \varepsilon)\right)^\dagger$. Also if $u = u^\dagger$ and if $y_{u,\varepsilon} = \Gamma(u, \varepsilon, y_{u,\varepsilon})$ then necessarily

$$y^\dagger_{u,\varepsilon} = (\Gamma(u, \varepsilon, y_{u,\varepsilon}))^\dagger = \Gamma(u, \varepsilon, y^\dagger_{u,\varepsilon})$$

and so by the uniqueness of the fixed point, necessarily $y^{\dagger}_{u,\varepsilon} = y_{u,\varepsilon}$. Hence, in particular $y^{\dagger}_{u,\varepsilon}(\cdot,0) = y_{u,\varepsilon}(\cdot,0)$ and we have $\Phi(u,\varepsilon)^{\dagger} = \Phi(u,\varepsilon)$. Similarly, $e^{i\theta_0}\Phi(u,\varepsilon) = \Phi(e^{i\theta_0}u,\varepsilon)$ for all $\theta_0 \in \mathbb{R}$.

Finally, we turn to the verification of (6.18). This comes from an examination of the iteration procedure leading to the fixed point as follows. Picking $\delta_0$ sufficiently small, we may argue that, for instance,

$$\|\Gamma(u,\varepsilon,y_1) - \Gamma(u,\varepsilon,y_2)\|_\eta < \frac{1}{2}\|y_1 - y_2\|_\eta$$

for all $u \in S_c$, all sufficiently small $\varepsilon$ and all $y_1$, $y_2 \in Y_\eta$. Letting $y_u$ denote the solution to the linear problem, i.e. $y_u := e^{L_1 t}u$, it then easily follows that

$$\|y_u - y_{u,\varepsilon}\|_\eta = \|y_u - \Gamma(u,\varepsilon,y_{u,\varepsilon})\|_\eta \le \|y_u - \Gamma(u,\varepsilon,y_u)\|_\eta + \|\Gamma(u,\varepsilon,y_u) - \Gamma(u,\varepsilon,y_{u,\varepsilon})\|_\eta$$

and so

$$\|y_u - y_{u,\varepsilon}\|_\eta \le 2\|y_u - \Gamma(u,\varepsilon,y_u)\|_\eta .$$

Then we calculate

$$\begin{aligned} &\|\Phi(u,\varepsilon) - u\|_{H^1} = \|y_{u,\varepsilon}(\cdot,0) - y_u(\cdot,0)\|_{H^1} \\ &\le \sup_{t\in\mathbb{R}} \|y_{u,\varepsilon}(\cdot,t) - y_u(\cdot,t)\|_{H^1} e^{-\eta|t|} = \|y_{u,\varepsilon} - y_u\|_\eta \\ &\le 2\|y_u - \Gamma(u,\varepsilon,y_u)\|_\eta \le 2C\left(\|u\|^3_{H^1} + |\varepsilon|\,\|u\|_{H^1}\right), \end{aligned}$$

where in the last estimate we invoked (6.14), (6.19) and (6.20)–(6.21).

The final conclusion of Theorem 6.4, namely that the center manifold contains all $H^1$-nearby solutions of (6.12), follows from the construction of $\mathcal{M}$. More precisely, any such solution would necessarily have slow growth at $\pm\infty$, that is, it would have finite $\|\cdot\|_\eta$ norm. Hence, it would arise as the unique fixed point of $\Lambda$ (cf. (6.19)) with $\mathcal{N}^\delta = \mathcal{N}$.

□

We also will need a version of the standard result on exponential attraction to an orbit on the center manifold in the absence of any unstable manifold. Again the proof we sketch is an adaptation of a more general but somewhat weaker result in [3] that is valid in the presence of an unstable manifold.

**Theorem 6.5.** *For any positive integer $k$ and $r = r(k) > 0$ sufficiently small, there exists a $C^k$ map $P_\varepsilon$ from $B(0, r) \subset H^1$ to $\mathcal{M}_\varepsilon$, equal to the identity when restricted to $\mathcal{M}_\varepsilon$, such that, for all solutions $\psi^\varepsilon$ of* (6.12) *originating at time $t = 0$ within $B(0, r)$,*

$$\|\psi^\varepsilon(t) - \hat{\psi}^\varepsilon(t)\|_{H^1} \leq C_1 e^{-\eta t} d_{H^1}(\psi^\varepsilon(0), \mathcal{M}_\varepsilon), \tag{6.24}$$

*so long as $\psi^\varepsilon$ remains in $B(0, r)$, where $\hat{\psi}^\varepsilon \in \mathcal{M}_\varepsilon$ denotes the trajectory along $\mathcal{M}_\varepsilon$ originating at time $t = 0$ at $P_\varepsilon(\psi^\varepsilon(0))$ and $\eta > 0$ and $C_1$ are uniform constants. Here $d_{H^1}(\cdot, \cdot)$ denotes the distance in $H^1$.*

*Proof.* As noted in [16], this follows by the proof of the more general approximation property (v) of the Center Manifold Theorem stated in [3], restricted to the case that the underlying linearized operator ($L_1$ in this case) has no unstable manifold.

To say a bit more about our adaptation of the approach presented in [3], given a solution $\psi^\varepsilon = \psi^\varepsilon(x, t)$ to (6.12), one first extends $\psi^\varepsilon$ to a function $\bar{\psi}^\varepsilon$ defined for negative $t$-values via

$$\bar{\psi}^\varepsilon = \begin{cases} \psi^\varepsilon & \text{for } t \geq 0 \\ \psi^\varepsilon(0) & \text{for } t < 0. \end{cases}$$

Thus, $\bar{\psi}^\varepsilon$ represents a globally bounded solution to the equation ${\bar{\psi}^\varepsilon}_t = L_1 \bar{\psi}^\varepsilon + \mathcal{N}(\bar{\psi}^\varepsilon, \varepsilon) + \phi^\varepsilon$ where

$$\phi^\varepsilon = \phi^\varepsilon(x, t) := \begin{cases} 0 & \text{for } t > 0 \\ -L_1 \psi^\varepsilon(0) - \mathcal{N}(\psi^\varepsilon, \varepsilon) & \text{for } t < 0. \end{cases} \tag{6.25}$$

Then we fix any positive $\eta$ such that $\eta < \zeta_1$ (cf. (6.8)) and seek a function $z$ such that $\bar{\psi}^\varepsilon + z \in \mathcal{M}_\varepsilon$. We will find such a $z$ in the set

$$Z_\eta := \{z \in C\left((\infty, \infty); H^1((-1, 1))\right) : \ |z|_\eta < \infty\}$$

where $|z|_\eta := \sup_{t \in \mathbb{R}} e^{\eta t} \|z(\cdot, t)\|_{H^1((-1,1))}$ and then define the projection $P_\varepsilon$ onto $\mathcal{M}_\varepsilon$ via $P_\varepsilon(\psi^\varepsilon(0)) := \psi^\varepsilon(0) + z(0)$. Thus, the trajectory on the center manifold satisfying (6.24) will be $\hat{\psi}^\varepsilon(t) := \psi^\varepsilon(t) + z(t)$. The function $z$ is produced as follows: plugging $\bar{\psi}^\varepsilon + z$ into the integral equation (6.19) leads one to seek $z$ as a fixed point of the mapping $\Lambda : Z_\eta \to Z_\eta$ defined by

$$\Lambda(z) := -\int_t^\infty e^{L_1(t-\tau)}\Pi_c \left[\mathcal{N}^\delta\left(\bar{\psi}^\varepsilon(x,\tau)+z(x,\tau),\varepsilon\right)-\mathcal{N}^\delta\left(\bar{\psi}^\varepsilon(x,\tau),\varepsilon\right)\right]\,d\tau +$$
$$\int_t^\infty e^{L_1(t-\tau)}\Pi_c\,\phi^\varepsilon(x,\tau)\,d\tau + \int_{-\infty}^t e^{L_1(t-\tau)}\Pi_s\left[\mathcal{N}^\delta\left(\bar{\psi}^\varepsilon(x,\tau)+z(x,\tau),\varepsilon\right)-\mathcal{N}^\delta\left(\bar{\psi}^\varepsilon(x,\tau),\varepsilon\right)\right]$$
$$-\int_{-\infty}^t e^{L_1(t-\tau)}\Pi_s\,\phi^\varepsilon(x,\tau)\,d\tau. \tag{6.26}$$

Again the existence of a (unique such) fixed point $z \in Z_\eta$ follows readily from the contraction mapping principle since one can check that $|\Lambda(z_1)-\Lambda(z_2)|_\eta \le \theta|z_1-z_2|_\eta$ for some $\theta \in (0,1)$. The fixed point $z$, and thus the map $P_\varepsilon$ that it determines, is Lipshitz in $\psi^\varepsilon(0)$ by construction. With further effort, it may be shown to be $C^k$ for any $k$, by a procedure similar to that used to show smoothness of the center manifold [3] in the analogous fixed-point construction of Proposition 6.4, using $C^k$ regularity of both the center manifold and the truncated equations. (In general, $P_\varepsilon$ inherits the regularity of the center manifold.) Note that the center manifold solution $\hat{\psi}^\varepsilon(t)$ so constructed satisfies the truncated equations (6.19) and not (6.12), since the righthand side of (6.26) involves the truncated nonlinearity $\mathcal{N}^\delta$ in place of $\mathcal{N}$. However, this makes no difference since the equations agree on the ball $B(0,r)$ under consideration, for $r>0$ sufficiently small.

Hence, we have

$$|z|_\eta \le |z-\Lambda(0)|_\eta + |\Lambda(0)|_\eta = |\Lambda(z)-\Lambda(0)|_\eta + |\Lambda(0)|_\eta \le \theta|z|_\eta + |\Lambda(0)|_\eta,$$

and so we conclude that

$$|z|_\eta \le C|\Lambda(0)|_\eta. \tag{6.27}$$

One then observes from (6.25) and (6.26) that for $t \ge 0$,

$$\Lambda(0)(t) = e^{L_1 t}\Pi_s F(\psi^\varepsilon(0)), \tag{6.28}$$

where $F : H^1 \to H^1$ is given by

$$F(v) := \int_{-\infty}^0 e^{-L_1\tau}\Pi_s\left(L_1 v + N^\delta(v,\varepsilon)\right)\,d\tau.$$

Note that $F$ is evidently bounded and Lipshitz. We next claim that if $\psi(0) \in \mathcal{M}_\varepsilon$, then $F(\psi^\varepsilon(0)) = 0$. To see this, note that in this case the invariance property of the center manifold implies that $\psi^\varepsilon \in \mathcal{M}_\varepsilon$ for $t \neq 0$ as well. Hence the unique fixed point of (6.26) must be $z \equiv 0$ and so in particular $\Lambda(0)(t) = 0$ for $t > 0$, which establishes the claim.

Finally, fixing any element $\psi^\varepsilon(0) \in H^1 \setminus \mathcal{M}_\varepsilon$ with sufficiently small $H^1$-norm, one uses this last observation to obtain

$$\|F(\psi^\varepsilon(0))\|_{H^1} = \inf_{\psi_1 \in \mathcal{M}\varepsilon} \|F(\psi^\varepsilon(0)) - F(\psi_1)\|_{H^1} \leq C d_{H^1}(\psi^\varepsilon(0), \mathcal{M}_\varepsilon). \tag{6.29}$$

The bound (6.24) now follows by combining (6.27), (6.28) and (6.29) and using the bound (6.23), since $z = \hat{\psi} - \psi$. □

**Remark 6.6.** Using the Implicit Function Theorem and the fact that $P_\varepsilon$ is the identity on $\mathcal{M}^\varepsilon$, we find that $H^1$ is foliated on a small neighborhood of $\mathcal{M}^\varepsilon$ by transverse smooth manifolds $P_\varepsilon^{-1}(w)$ through each $w \in \mathcal{M}^\varepsilon$, depending in a smooth fashion on the value of $w$. In particular, for $\overline{C} > 0$ sufficiently large and $a > 0$ sufficiently small, the $H^1$-ball $B(0, a)$ is foliated by leaves $P_\varepsilon^{-1}(w)$ for $w \in \mathcal{M}^\varepsilon \cap B(0, \overline{C}a)$, carried one to the other under the flow of the underlying ODE, uniquely specified by the property that each solution initiating in $P_\varepsilon^{-1}(w_0)$ approaches the solution on the center manifold with initial data $w_0$ at uniform exponential rate $\sim 1 >> \varepsilon$. As a consequence, a $C^k$ stable manifold $\mathcal{N}_s \subset \mathcal{M}^\varepsilon \cap B(0, a)$ of an orbit or manifold of orbits within the center manifold $\mathcal{M}^\varepsilon$ extends to a $C^k$ stable manifold $\tilde{\mathcal{N}_s} := \cup_{w \in \mathcal{N}} P_\varepsilon^{-1}(w) \cap B(0, a)$ in $B(0, a)$ of the same codimension in $B(0, a)$ as the codimension of $\mathcal{N}$ in $\mathcal{M}^\varepsilon$. That is, not only is asymptotic stability in $B(0, a)$ determined completely by asymptotic stability within the center manifold, but also conditional stability as measured by codimension of the stable manifold.

We now apply the previous result on existence of a center manifold to assert the existence of bifurcating stationary and periodic states for equation (6.12).

We begin with the case of stationary states bifurcating from the normal state. We refer to Section 4 for the definition (4.9) of the parameter $\chi_{11}$ which was found numerically to be real for $I \leq I_c$, positive for $I_k < I < I_c$ and negative for $0 < I < I_k$ where $I_k \approx 10.93$ and $I_c \approx 12.31$.

**Proposition 6.7.** *For $I$ fixed in the interval $(0, I_k)$, equation (6.12) exhibits a stable supercritical pitchfork bifurcation of stationary states $\{e^{i\theta_0}\psi_e(\cdot,\varepsilon) : \ \theta_0 \in [0, 2\pi)\}$ branching from the normal state for all sufficiently small and positive values of $\varepsilon$. These equilibria satisfy the bound*

$$\left\| \psi_e - \Big(\frac{1}{\sqrt{-\chi_{11}}}\Big)\, \varepsilon^{1/2}\, u_1 \right\|_{H^1} < C\varepsilon^{3/2} \tag{6.30}$$

*as predicted formally in Section 4*

*For $I$ fixed in the interval $(I_k, I_c)$, the equation exhibits an unstable subcritical pitchfork bifurcation of stationary states $e^{i\theta_0}\tilde{\psi}_e(\cdot,\varepsilon)$, $\theta_0 \in [0, 2\pi)$, branching from the normal state for all sufficiently small and negative values of $\varepsilon$. These equilibria satisfy the bound*

$$\left\| \tilde{\psi}_e - \Big(\frac{1}{\sqrt{\chi_{11}}}\Big)\, |\varepsilon|^{1/2}\, u_1 \right\|_{H^1} < C|\varepsilon|^{3/2} \tag{6.31}$$

*as predicted formally in Section 4.*

*Proof.* Since we work here in the setting where $I \leq I_c$, the center subspace is spanned by the single eigenfunction $u_1$. Hence, we will express any point on the center manifold $\mathcal{M}$ as $(\Phi(\beta,\varepsilon),\varepsilon)$ where $\beta \in \mathbb{C}$ corresponds to the coefficient of the point $\beta u_1$ on the center subspace. We begin with the case where $I$ is fixed to lie in the interval $(0, I_k)$. Then for any small value $\beta_0 \in \mathbb{C}$ we let $\psi^\varepsilon$ denote the solution to (6.12) satisfying Dirichlet boundary conditions and initial condition $\psi^\varepsilon(\cdot, 0) = \Phi(\beta_0,\varepsilon)$. For all small, positive $t$, we know from Theorem 6.4 that $(\psi^\varepsilon(\cdot,t),\varepsilon) \in \mathcal{M}$ and so there exists a smooth function, which we denote by $\beta^\varepsilon = \beta^\varepsilon(t)$, such that

$$\psi^\varepsilon(x,t) = \Phi(\beta^\varepsilon(t),\varepsilon). \tag{6.32}$$

Recalling the definition $\beta_1 := \int_{-1}^1 u_1^2\,dx$ we now apply the projection $\Pi_c$ to every term in the equation

$$\psi^\varepsilon_t = L_1\psi^\varepsilon + \mathcal{N}(\psi^\varepsilon,\varepsilon)$$

satisfied by $\psi^\varepsilon$ and then integrate against $u_1$. We find that

$$\int_{-1}^1 \Pi_c\big(\psi^\varepsilon_t\big)u_1\,dx = \frac{\partial}{\partial t}\int_{-1}^1 \Pi_c\big(\psi^\varepsilon\big)u_1\,dx = \frac{\partial}{\partial t}\int_{-1}^1 \Pi_c\big(\Phi(\beta^\varepsilon,\varepsilon)u_1\,dx\big) = \beta_1\beta^\varepsilon_t.$$

We also have

$$\int_{-1}^{1} \Pi_c\big(L_1\psi^\varepsilon\big)u_1\,dx = \beta_1 \int_{-1}^{1} u_1\,L_1\psi^\varepsilon\,dx = \beta_1 \int_{-1}^{1} L_1u_1\,\psi^\varepsilon\,dx = 0$$

and $\int_{-1}^{1} \Pi_c\big(\varepsilon\psi^\varepsilon\big)u_1\,dx = \varepsilon\beta_1\beta^\varepsilon$. Consequently, we obtain for $\beta^\varepsilon$ the O.D.E.

$$\begin{aligned} \beta_t^\varepsilon &= \varepsilon\beta^\varepsilon + \frac{1}{\beta_1}\int_{-1}^{1} \Pi_c\big(\mathcal{N}[\beta^\varepsilon u_1]\big)u_1\,dx + e(\beta^\varepsilon,\varepsilon) \\ &= \varepsilon\beta^\varepsilon + \chi_{11}\,|\beta^\varepsilon|^2\,\beta^\varepsilon + e(\beta^\varepsilon,\varepsilon) \end{aligned} \tag{6.33}$$

where we recall the calculation of $\int_{-1}^{1} \Pi_c\big(\mathcal{N}[\beta^\varepsilon u_1]\big)u_1\,dx$ carried out in Section 4, and we have introduced

$$e(\beta^\varepsilon,\varepsilon) := \frac{1}{\beta_1}\Pi_c\big(\mathcal{N}[\Phi(\beta^\varepsilon,\varepsilon)]\big) - \Pi_c\big(\mathcal{N}[\beta^\varepsilon u_1]\big). \tag{6.34}$$

Due to (6.17), we know that

$$e(e^{i\theta_0}\beta^\varepsilon,\varepsilon) = e^{i\theta_0}e(\beta^\varepsilon,\varepsilon) \quad \text{for any } \theta_0 \in \mathbb{R} \tag{6.35}$$

and through elementary use of the triangle and Cauchy-Schwartz inequalities applied to the nonlinearity $\mathcal{N}$, along with (6.18), we estimate

$$\begin{aligned} &|e(\beta^\varepsilon,\varepsilon)| \le C\,\|\mathcal{N}[\Phi(\beta^\varepsilon,\varepsilon)] - \mathcal{N}[\beta^\varepsilon u_1]\|_{L^\infty} \\ &\le C\left(\|\Phi(\beta^\varepsilon,\varepsilon)\|_{H^1}^2 + \|\beta^\varepsilon u_1\|_{H^1}^2\right)\left(\|\Phi(\beta^\varepsilon,\varepsilon) - \beta^\varepsilon u_1\|_{H^1}\right) \\ &= \mathcal{O}\left(\varepsilon(\beta^\varepsilon)^3 + (\beta^\varepsilon)^5\right). \end{aligned} \tag{6.36}$$

Returning to (6.33), consider first the case where $\beta^\varepsilon(0) = \beta_0 \in \mathbb{R}$. We first claim that the function $\beta^\varepsilon(t)$ must be real. To see this, we begin by noting that since $\beta_0$ is real, the quantity $\beta_0 u_1$ is PT-symmetric. Hence, in particular $\psi^\varepsilon(\cdot,t_1) = (\psi^\varepsilon(\cdot,t_1))^\dagger$ as well for any fixed $t_1 > 0$ since $\psi^\varepsilon$ satisfies a PT-symmetric initial condition $\Phi(\beta_0 u_1,\varepsilon)$. Now denote by $\psi^{\varepsilon,1}$ the unique solution in $Y_\eta$ to the equation

$$\psi^{\varepsilon,1} = \Gamma(\beta^\varepsilon(t_1)u_1,\varepsilon,\psi^{\varepsilon,1}), \tag{6.37}$$

where $\Gamma$ is defined by (6.19). By (6.32) we have $\psi^\varepsilon(\cdot,t_1) = \psi^{\varepsilon,1}(\cdot,0)$, and consequently, $\psi^{\varepsilon,1}(\cdot,0) = (\psi^{\varepsilon,1}(\cdot,0))^\dagger$. Evaluating (6.37) at $t=0$ and applying the † operation to both sides, we then conclude that

$$\psi^{\varepsilon,1}(\cdot,0) = \Gamma(\beta^\varepsilon(t_1)^* u_1, \varepsilon, \psi^{\varepsilon,1})(\cdot,0) \tag{6.38}$$

as well. Applying the projection $\Pi_c$ to both (6.37) evaluated at $t = 0$ and (6.38), we see that indeed $\beta^\varepsilon(t_1) = \beta^\varepsilon(t_1)^*$ as claimed.

An easy application of the implicit function theorem then reveals the existence of a smooth curve of zeros $\varepsilon = \varepsilon(\beta^\varepsilon)$ to the equation

$$\frac{\varepsilon\beta^\varepsilon + \chi_{11}(\beta^\varepsilon)^3 + e(\beta^\varepsilon, \varepsilon)}{\beta^\varepsilon} = 0$$

such that $\varepsilon = -\chi_{11}(\beta^\varepsilon)^2 + \mathcal{O}((\beta^\varepsilon)^4)$. Hence, there exist smooth curves of equilibria $\beta^\pm(\varepsilon)$ to (6.33) for all small, positive $\varepsilon$ with

$$\beta^\pm(\varepsilon) = \pm\frac{1}{\sqrt{-\chi_{11}}}\,\varepsilon^{1/2} + \mathcal{O}(\varepsilon^{3/2}). \tag{6.39}$$

Consequently, within the collection of points on the center manifold of the form $\Phi(\beta, \varepsilon)$ with $\beta$ real, the functions $\psi_e^\pm := \Phi(\beta^\pm(\varepsilon), \varepsilon)$ represent a supercritical pitchfork bifurcation of equilibria from the normal state. The bound (6.30) follows immediately from (6.18). In light of the rotation invariance of the problem, it immediately follows that there is in fact a circle of equilibria $e^{i\theta_0}\psi_e$, $\theta_0 \in [0, 2\pi)$ where we have written simply $\psi_e$ for $\psi_e^+$.

Regarding stability of these equilibria, it is clear from (6.33) and the estimate (6.36) that given any initial data on the center manifold of the form $\Phi(\beta_0 u_1, \varepsilon)$ with $\beta_0$ real, positive and say bounded by $C\sqrt{\varepsilon}$, the solution to (6.33) will converge to $\beta^+(\varepsilon)$ and so the solution to (6.12) will converge to $\psi_e$. Then since in light of (6.35), (6.33) is clearly rotationally invariant, it follows that for complex initial data on the center manifold, i.e. initial data of the form $\Phi(\beta_0 u_1, \varepsilon)$ where $\beta_0 = |\beta_0|\, e^{i\theta_0}$ for some non-zero phase $\theta_0$, necessarily the solution will converge to $e^{i\theta_0}\Phi(\beta^+(\varepsilon), \varepsilon)$. Thus, one concludes that the circle of equilibrium states $\{e^{i\theta_0}\psi_e : \theta_0 \in [0, 2\pi)\}$ is asymptotically stable on the center manifold.

Finally, suppose that we start with initial conditions for (6.12) that are close to the circle of equilibria but that do not lie on the center manifold. That is, suppose we have

$$\left\|\psi^\varepsilon(0) - e^{i\theta_0}\psi_e\right\|_{H^1} < r_1 \quad \text{for some } \theta_0 \in [0, 2\pi) \tag{6.40}$$

but that $\psi^\varepsilon(0) \notin \mathcal{M}_\varepsilon$. Without loss of generality, we ignore this rotation for the remainder of the argument and take $\theta_0 = 0$. We will argue that for $r_1$ and $\varepsilon$ sufficiently small, again the trajectory $\psi^\varepsilon(t)$ is exponentially attracted to $\psi_e$. We take in particular, $r_1 < \frac{1}{4C_1} r$ where $C_1 \geq 1$ and $r$ are the constants appearing in Theorem 6.5. Since $\|\psi_e\|_{H^1} \leq \frac{2}{\sqrt{-\chi_{11}}}\sqrt{\varepsilon}$ we can assert that

$$\|\psi^\varepsilon(0)\|_{H^1} < r/2 \tag{6.41}$$

by choosing $\varepsilon$ sufficiently small and appealing to (6.40). Denoting by $\hat{\psi}^\varepsilon$ the trajectory on $\mathcal{M}_\varepsilon$, it then follows from (6.24) and (6.40) that for as long as $\psi(t)$ obeys the bound $\|\psi^\varepsilon(t)\|_{H^1} < r$, one has the estimate

$$\left\|\psi^\varepsilon(t) - \hat{\psi}^\varepsilon(t)\right\|_{H^1} \leq C_1 r_1 e^{-\eta t} < \frac{r}{4} e^{-\eta t}. \tag{6.42}$$

Then the triangle inequality implies that

$$\left\|\hat{\psi}^\varepsilon(0) - \psi_e\right\|_{H^1} \leq (1 + C_1) r_1. \tag{6.43}$$

Now $\hat{\psi}^\varepsilon(t) \in \mathcal{M}_\varepsilon$ is necessarily given by $\hat{\psi}^\varepsilon(t) = \Phi(\beta^\varepsilon(t), \varepsilon)$ where $\beta^\varepsilon$ is governed by (6.33). As we already noted, up to a rotation which we again ignore, the equilibrium value $\beta^+(\varepsilon)$ that is stable under this flow, both in the sense of (exponential) asymptotic approach, $\beta^\varepsilon(t) \to \beta^+(\varepsilon)$ and in the sense that $\beta^\varepsilon(t)$ will stay close to this equilibrium for all time. Choosing $r_1$ still smaller if necessary, we may appeal to (6.43) to conclude that $|\beta^\varepsilon(0) - \beta^+(\varepsilon)|$ is small and then the Lipschitz property of the map $\Phi$ allows us to assert that, for instance,

$$\left\|\hat{\psi}^\varepsilon(t) - \psi_e\right\|_{H^1} < r/4 \quad \text{for all } t \geq 0. \tag{6.44}$$

It follows from (6.41), (6.42) and (6.44) that in fact $\|\psi^\varepsilon(t)\|_{H^1} < r$ for all $t \geq 0$ and so (6.42) is valid for all time. Combining (6.42) with the exponential approach of $\hat{\psi}^\varepsilon$ to $\psi_e$ along the center manifold, we obtain the asymptotic stability of all trajectories $\psi^\varepsilon(t)$ satisfying (6.40).

The case where $I_k < I < I_c$ is handled similarly. Recall that in this case, the parameter $\chi_{11}$ takes a positive value. Working then with $\varepsilon$ small and negative, and again starting with initial data of the form $\Phi(\beta_0, \varepsilon)$ with $\beta_0$ real, we find that (6.33) is now replaced by

$$\beta^\varepsilon_t = \varepsilon\beta^\varepsilon + \chi_{11}(\beta^\varepsilon)^3 + e(\beta^\varepsilon, \varepsilon). \tag{6.45}$$

Another application of the implicit function theorem reveals that (6.45) possesses a pair of unstable equilibria $\tilde{\beta}^{\pm}(\varepsilon)$ for all small negative $\varepsilon$-values with $\tilde{\beta}^{\pm}(\varepsilon) = \pm\frac{1}{\sqrt{\chi_{11}}}\,|\varepsilon|^{1/2} + \mathcal{O}(|\varepsilon|^{3/2})$. An examination of (6.45) shows that the corresponding equilibria $\tilde{\psi}_e^{\pm} := \Phi(\tilde{\beta}^{\pm}(\varepsilon), \varepsilon)$ bifurcating subcritically from the normal state are unstable as well since even nearby PT-symmetric points on the center manifold, that is points of the form $\beta u_1$ for $\beta$ real and near $\beta^{\pm}(\varepsilon)$, flow away from them. Writing simply $\tilde{\psi}_e$ for $\tilde{\psi}_e^{+}$, the same is of course true for any of the equilibria on the circle $\{e^{i\theta_0}\tilde{\psi}_e :\ \theta_0 \in [0, 2\pi)\}$.

□

We turn now to the case where the applied current $I$ satisfies $I > I_c$ and a bifurcation to a periodic state occurs. We recall that in this parameter regime the constants $\chi_{11}$ and $\chi_{12}$ are not real.

**Proposition 6.8.** *Fix the applied current $I$ in the interval $(I_c, \infty)$. Then provided that the constants $\hat{\chi}$ and $\tilde{\chi}$ given by and* (5.13) *and* (5.14) *respectively both have positive real parts, the equation* (6.12) *exhibits a stable Hopf bifurcation to a periodic state $\psi_p = \psi_p(x, t, \varepsilon)$ branching from the normal state for all sufficiently small and positive values of $\varepsilon$. This bifurcating solution, $\psi_p$ obeys the estimate*

$$\left\|\psi_p - e^{i\theta_0}\left(\beta_{1,p}^{\varepsilon}(t)v_1 + \beta_{2,p}^{\varepsilon}(t)v_2\right)\right\|_{H^1} < C\varepsilon^{3/2} \tag{6.46}$$

*for some $\theta_0 \in [0, 2\pi)$, where the real functions $\beta_{1,p}^{\varepsilon}$ and $\beta_{2,p}^{\varepsilon}$ take the form*

$$\begin{aligned}(\beta_{1,p}^{\varepsilon}(t), \beta_{2,p}^{\varepsilon}(t)) &= \left(\frac{\sqrt{\varepsilon}}{\sqrt{\operatorname{Re}\tilde{\chi}}} + \mathcal{O}(\varepsilon^{3/2})\right)\\ &\times\left(\cos\left[\left(\operatorname{Im}\lambda_1 + \frac{\operatorname{Im}\tilde{\chi}}{\operatorname{Re}\tilde{\chi}}\varepsilon + \mathcal{O}(\varepsilon^{3/2})\right)t\right], \sin\left[\left(\operatorname{Im}\lambda_1 + \frac{\operatorname{Im}\tilde{\chi}}{\operatorname{Re}\tilde{\chi}}\varepsilon + \mathcal{O}(\varepsilon^{3/2})\right)t\right]\right).\end{aligned} \tag{6.47}$$

**Remark 6.9.** Recalling the relationship between the functions $v_1$ and $v_2$ given by (6.9), one checks that the coefficients $\beta_{1,p}^{\varepsilon}$ and $\beta_{2,p}^{\varepsilon}$ introduced above are related to the coefficients $\alpha_1$ and $\alpha_2$ introduced in (5.3) via the formulas

$$\alpha_1(\varepsilon t) = \frac{1}{\sqrt{\varepsilon}}\,e^{i\operatorname{Im}\lambda_1 t}\left(\beta_{1,p}^{\varepsilon}(t) + i\beta_{2,p}^{\varepsilon}(t)\right), \quad \alpha_2(\varepsilon t) = \frac{1}{\sqrt{\varepsilon}}\,e^{-i\operatorname{Im}\lambda_1 t}\left(\beta_{1,p}^{\varepsilon}(t) - i\beta_{2,p}^{\varepsilon}(t)\right)$$

**Remark 6.10.** We recall from the previous section that numerically, we indeed find that $\operatorname{Re}\tilde{\chi} > 0$ and $\operatorname{Re}\hat{\chi} > 0$ for $I > I_c$. See Figure 6.

*Proof.* Invoking Theorem 6.4, given any two complex numbers $\beta_1^0$ and $\beta_2^0$ of sufficiently small modulus, let $\psi^\varepsilon$ denote the solution to (6.12) subject to Dirichlet boundary conditions and initial conditions given by $\Phi(\beta_1^0, \beta_2^0, \varepsilon)$. Then we may describe $\psi^\varepsilon$ via $\Phi$ at all future times as $\psi^\varepsilon = \Phi(\beta_1^\varepsilon(t), \beta_2^\varepsilon(t), \varepsilon)$ for complex-valued functions $\beta_1^\varepsilon(t)$ and $\beta_2^\varepsilon(t)$.

We then project (6.12) onto the center subspace and use (6.10) to obtain

$$\begin{aligned}
&(\beta_1^\varepsilon)' v_1 + (\beta_2^\varepsilon)' v_2 = \\
&-\operatorname{Im}\lambda_1 \beta_2^\varepsilon v_1 + \operatorname{Im}\lambda_1 \beta_1^\varepsilon v_2 + \varepsilon \beta_1^\varepsilon v_1 + \varepsilon \beta_2^\varepsilon v_2 \\
&+\left(\int_{-1}^1 v_1 \mathcal{N}\left[\beta_1^\varepsilon v_1 + \beta_2^\varepsilon v_2\right]\, dx\right) v_1 + \left(\int_{-1}^1 v_2 \mathcal{N}\left[\beta_1^\varepsilon v_1 + \beta_2^\varepsilon v_2\right]\, dx\right) v_2 \\
&+\left(\int_{-1}^1 v_1 \left(\mathcal{N}[\Phi(\beta_1^\varepsilon, \beta_2^\varepsilon, \varepsilon)] - \mathcal{N}[\beta_1^\varepsilon v_1 + \beta_2^\varepsilon v_2]\right) dx\right) v_1 \\
&+\left(\int_{-1}^1 v_2 \left(\mathcal{N}[\Phi(\beta_1^\varepsilon, \beta_2^\varepsilon, \varepsilon)] - \mathcal{N}[\beta_1^\varepsilon v_1 + \beta_2^\varepsilon v_2]\right) dx\right) v_2.
\end{aligned}$$

Integrating this equation first against $v_1$ and then against $v_2$, we use the resulting two by two linear system in ${\beta_1^\varepsilon}'$ and ${\beta_2^\varepsilon}'$ to find

$$\begin{aligned}
\begin{pmatrix} \beta_1^\varepsilon \\ \beta_2^\varepsilon \end{pmatrix}' &= \begin{pmatrix} \varepsilon & -\operatorname{Im}\lambda_1 \\ \operatorname{Im}\lambda_1 & \varepsilon \end{pmatrix} \begin{pmatrix} \beta_1^\varepsilon \\ \beta_2^\varepsilon \end{pmatrix} + \begin{pmatrix} \int_{-1}^1 v_1 \mathcal{N}\left(\beta_1^\varepsilon v_1 + \beta_2^\varepsilon v_2\right)\, dx \\ \int_{-1}^1 v_2 \mathcal{N}\left(\beta_1^\varepsilon v_1 + \beta_2^\varepsilon v_2\right)\, dx \end{pmatrix} \\
&+ \begin{pmatrix} \int_{-1}^1 v_1 \left(\mathcal{N}[\Phi(\beta_1^\varepsilon, \beta_2^\varepsilon, \varepsilon)] - \mathcal{N}[\beta_1^\varepsilon v_1 + \beta_2^\varepsilon v_2]\right) dx \\ \int_{-1}^1 v_2 \left(\mathcal{N}[\Phi(\beta_1^\varepsilon, \beta_2^\varepsilon, \varepsilon)] - \mathcal{N}[\beta_1^\varepsilon v_1 + \beta_2^\varepsilon v_2]\right) dx \end{pmatrix}.
\end{aligned}$$

Appealing to the center manifold estimate (6.18) and carrying out a lengthy calculation similar to that of Section 5, we finally arrive at a system of the form

$$\begin{aligned}
\begin{pmatrix} \beta_1^\varepsilon \\ \beta_2^\varepsilon \end{pmatrix}' &= \begin{pmatrix} \varepsilon & -\operatorname{Im}\lambda_1 \\ \operatorname{Im}\lambda_1 & \varepsilon \end{pmatrix} \begin{pmatrix} \beta_1^\varepsilon \\ \beta_2^\varepsilon \end{pmatrix} + \begin{pmatrix} -[\operatorname{Re}\tilde{\chi}(R^\varepsilon)^2 + i\operatorname{Im}\hat{\chi}\gamma^\varepsilon]\beta_1^\varepsilon + [\operatorname{Im}\tilde{\chi}(R^\varepsilon)^2 - i\operatorname{Re}\hat{\chi}\gamma^\varepsilon]\beta_2^\varepsilon \\ -[\operatorname{Im}\tilde{\chi}(R^\varepsilon)^2 - i\operatorname{Re}\hat{\chi}\gamma^\varepsilon]\beta_1^\varepsilon - [\operatorname{Re}\tilde{\chi}(R^\varepsilon)^2 + i\operatorname{Im}\hat{\chi}\gamma^\varepsilon]\beta_2^\varepsilon \end{pmatrix} \\
&+\mathcal{O}\left(\varepsilon (R^\varepsilon)^3 + (R^\varepsilon)^5\right). \qquad (6.48)
\end{aligned}$$

In the system above we have introduced the notation

$$R^\varepsilon := \sqrt{|\beta_1^\varepsilon|^2 + |\beta_1^\varepsilon|^2} \quad \text{and} \quad \gamma^\varepsilon := i\left((\beta_1^\varepsilon)^*\beta_2^\varepsilon - \beta_1^\varepsilon(\beta_2^\varepsilon)^*\right) = |\beta_1^\varepsilon|\,|\beta_2^\varepsilon|\sin(\theta_1^\varepsilon - \theta_2^\varepsilon), \tag{6.49}$$

where $\beta_j^\varepsilon = \left|\beta_j^\varepsilon\right| e^{i\theta_j^\varepsilon}$.

We now apply the standard method for proving the existence of a periodic solution to (6.48) via a Hopf bifurcation and to compute rigorously the amplitude and period of the oscillations. To this end, we consider first the case where the initial values $\beta_1^0$ and $\beta_2^0$ are real. Then the resulting PT-symmetric initial data $\Phi(\beta_1^0, \beta_2^0, \varepsilon)$ for (6.12) will lead to the PT-symmetry of the solution at all future times. Consequently, for all $t > 0$, the projection of $\psi^\varepsilon$ onto the center subspace must take the form $\beta_1^\varepsilon(t)v_1 + \beta_2^\varepsilon(t)v_2$ where $\beta_1^\varepsilon$ and $\beta_2^\varepsilon$ are real. This leads to a significant simplification of (6.48) in that $\gamma^\varepsilon \equiv 0$.

Converting to polar coordinates, $R^\varepsilon$ and $\theta^\varepsilon := \tan^{-1}\left(\beta_2^\varepsilon/\beta_1^\varepsilon\right)$, we derive the system

$$(R^\varepsilon)' = \varepsilon R^\varepsilon - \operatorname{Re}\tilde{\chi}(R^\varepsilon)^3 + \mathcal{O}\left(\varepsilon(R^\varepsilon)^4 + (R^\varepsilon)^6\right), \tag{6.50}$$

$$(\theta^\varepsilon)' = \operatorname{Im}\lambda_1 - \operatorname{Im}\tilde{\chi}(R^\varepsilon)^2 + \mathcal{O}\left(\varepsilon(R^\varepsilon)^2 + (R^\varepsilon)^4\right). \tag{6.51}$$

Starting with any initial condition with small amplitude $a := R^\varepsilon(0)$, it is easy to argue that $\theta^\varepsilon$ is a monotone increasing function of $t$, thus justifying the description of $R^\varepsilon$ as $R^\varepsilon(a, \theta^\varepsilon)$ via the scalar O.D.E.

$$\frac{dR^\varepsilon}{d\theta^\varepsilon} = \frac{\varepsilon}{\operatorname{Im}\lambda_1}R^\varepsilon - \frac{\operatorname{Re}\tilde{\chi}}{\operatorname{Im}\lambda_1}(R^\varepsilon)^3 + g(\varepsilon, R^\varepsilon, \theta^\varepsilon) \tag{6.52}$$

where

$$g(\varepsilon, R^\varepsilon, \theta^\varepsilon) = \mathcal{O}\left(\varepsilon(R^\varepsilon)^3 + (R^\varepsilon)^5\right).$$

A periodic orbit for (6.48) corresponds to a value of the amplitude $a$ such that $R^\varepsilon(a, 2\pi) = a$. Thus, we seek a fixed point of the Poincaré return map $\Pi$ given by

(6.53)
$$\Pi(\varepsilon, a) := ae^{2\pi\varepsilon/\operatorname{Im}\lambda_1} + e^{2\pi\varepsilon/\operatorname{Im}\lambda_1}\int_0^{2\pi} e^{-(\varepsilon/\operatorname{Im}\lambda_1)\theta}\left(\frac{-\operatorname{Re}\tilde{\chi}}{\operatorname{Im}\lambda_1}R^\varepsilon(a,\theta)^3 + g(\varepsilon, R^\varepsilon(a,\theta), \theta)\right) d\theta.$$

Using (6.52) one readily checks that

$$R^\varepsilon(a, \theta) = a + \mathcal{O}(\varepsilon a + a^3) \quad \text{and so} \quad R^\varepsilon(a,\theta)^3 = a^3 + \mathcal{O}(\varepsilon a^3 + a^5). \tag{6.54}$$

Therefore, expanding $\Pi$ for small $\varepsilon$ we see that

$$\Pi(\varepsilon, a) = a + \frac{2\pi\varepsilon}{\operatorname{Im}\lambda_1}a - \frac{2\pi \operatorname{Re}\tilde{\chi}}{\operatorname{Im}\lambda_1}a^3 + \mathcal{O}(a\varepsilon^2 + \varepsilon a^3 + a^5)$$

Via the implicit function theorem one derives a curve of zeros $\varepsilon(a)$ for the expression

$$\frac{\Pi(\varepsilon, a) - a}{a}.$$

Hence, we obtain a curve of fixed points of $\Pi$ with

$$\varepsilon(a) = (\operatorname{Re}\tilde{\chi})a^2 + \mathcal{O}(a^3),$$

or inverting this relationship,

$$a = \frac{1}{\sqrt{\operatorname{Re}\tilde{\chi}}}\,\varepsilon^{1/2} + \mathcal{O}(\varepsilon). \tag{6.55}$$

Denoting the resulting periodic solution to (6.50)-(6.51) by $(R_p^\varepsilon(t), \theta_p^\varepsilon(t))$ and letting $(\beta_{1,p}^\varepsilon, \beta_{2,p}^\varepsilon) = (R_p^\varepsilon \cos(\theta_p^\varepsilon), R_p^\varepsilon \sin(\theta_p^\varepsilon))$ denote the corresponding periodic solution to (6.48), the asymptotic estimates (6.46) and (6.47) then readily follow from (6.51), (6.54) and (6.55).

We also note that the asymptotic stability of this periodic orbit, and hence the asymptotic stability of the periodic solutions to (6.12) among nearby PT-symmetric competitors on the center manifold, is a consequence of the condition

$$\frac{\partial \Pi}{\partial a}(\varepsilon(a), a) \sim 1 - \frac{4\pi \operatorname{Re}\tilde{\chi}}{\operatorname{Im}\lambda_1}a^2 < 1,$$

cf. [6], Thm. 12.13. In light of the rotational invariance of the system (6.48), this means not only that real initial data $(\beta_1^\varepsilon(0), \beta_2^\varepsilon(0))$ lying sufficiently close to the periodic orbit will be asymptotically drawn by the flow into the orbit but also that any rotation, say $e^{i\theta_0}(\beta_1^\varepsilon(0), \beta_2^\varepsilon(0))$ of such initial data will be drawn to the corresponding rotation of this orbit as well.

We next wish to argue that these periodic orbits are stable within the class of all flows starting nearby on the center manifold. For this purpose we return to (6.48); that is, we consider the situation where the initial data for $\beta_1^\varepsilon$ and $\beta_2^\varepsilon$ are not necessarily real. Note that this corresponds to initial data that is then not assumed to be PT-symmetric. We claim that if we start with complex initial data $(\beta_1^\varepsilon(0), \beta_2^\varepsilon(0))$ sufficiently close to any of the orbits given by (6.46)-(6.47), then again the flow will draw the resulting solution into one of these periodic orbits.

To establish this claim it is useful to derive a system of O.D.E.'s for the quantities $(R^\varepsilon)^2$ and $\gamma^\varepsilon$. Differentiating the defining formulas in (6.49) and using the system satisfied by $\beta_1^\varepsilon$ and $\beta_2^\varepsilon$ given in (6.48), a routine calculation leads us to:

$$(6.56)\qquad \begin{pmatrix} (R^\varepsilon)^2 \\ \gamma^\varepsilon \end{pmatrix}' = \begin{pmatrix} 2\left[\left(\varepsilon - \mathrm{Re}\,\tilde\chi\,(R^\varepsilon)^2\right)(R^\varepsilon)^2 - \mathrm{Re}\,\hat\chi(\gamma^\varepsilon)^2\right] \\ 2\left[\varepsilon - \mathrm{Re}\,(\tilde\chi+\hat\chi)(R^\varepsilon)^2\right]\gamma^\varepsilon \end{pmatrix} \quad + \quad \mathcal{O}(\varepsilon(R^\varepsilon)^4 + (R^\varepsilon)^6).$$

Note that $\beta_j^\varepsilon$ are parallel if and only if $\gamma^\varepsilon = 0$. Recalling that PT-symmetric solutions correspond to $\beta_j^\varepsilon$ real, we thus see that $\gamma^\varepsilon = 0$ corresponds to the invariant three-dimensional manifold of PT-symmetric solutions and their complex rotations that we have already considered, with the single remaining dimension parametrized conveniently by $\gamma^\varepsilon$.

If we momentarily ignore the error term, then we can readily identify an equilibrium point for (6.56) at $(\varepsilon/\mathrm{Re}\,\tilde\chi, 0)$. Linearization about this critical point immediately yields linear stability and in fact a standard phase plane analysis yields nonlinear asymptotic stability. To treat the full system (6.56) (that is, including the error term), we must linearize instead about the nearby periodic solution $((R_p^\epsilon)^2, 0)$, corresponding to the (real) periodic orbit $(\beta_{1,p}^\varepsilon, \beta_{2,p}^\varepsilon)$ constructed above, yielding

$$(\gamma^\epsilon)' = 2\left[\varepsilon - \mathrm{Re}\,(\tilde\chi+\hat\chi)(R_p^\varepsilon)^2 + \mathcal{O}(\varepsilon(R_p^\varepsilon)^2 + (R_p^\varepsilon)^4)\right]\gamma^\varepsilon.$$

Since the error term is clearly lower order, the condition $(R_p^\varepsilon)^2 \approx \varepsilon/\mathrm{Re}\,\tilde\chi$ again implies exponential linearized stability in the $\gamma^\epsilon$ direction. As exponential orbital stability has already been verified in the remaining directions, we may conclude exponential linearized and nonlinear orbital stability of the family of periodic orbits within the full four-dimensional center manifold and not only within the invariant three-dimensional $\gamma = 0$ manifold.

It remains to argue that the periodic orbits attract nearby initial data for (6.12) off of the center manifold. The argument for this fact follows exactly as did the corresponding point in the proof of Proposition 6.7 regarding stability of stationary solutions through an appeal to Theorem 6.5. □

Proposition 6.8 describes the bifurcating stable periodic solutions of main physical interest. With a little further effort, we may also rigorously obtain the existence of the two unstable periodic branches derived formally at the end of Section 5 and obtain essentially a complete description of the full four-dimensional bifurcation.

**Proposition 6.11.** *Under the assumptions of Proposition 6.8, the equation* (6.12) *also exhibits an unstable bifurcation for small* $\varepsilon > 0$ *to periodic states* $\psi_{p,\pm} = \psi_{p,\pm}(x,t,\varepsilon)$ *of form*

$$\left\|\psi_{p,\pm} - e^{i\theta_0}\left(\beta^{\varepsilon}_{1,\pm}(t)v_1 + \beta^{\varepsilon}_{2,\pm}(t)v_2\right)\right\|_{H^1} < C\varepsilon^{3/2}, \tag{6.57}$$

*where, recalling that* $\chi_{11} = -\frac{1}{2}(\tilde{\chi} + \hat{\chi})$ *so that* $-\mathrm{Re}\,\chi_{11} > 0$,

$$\beta^{\varepsilon,\pm}_1(t) = \left(\frac{\sqrt{\varepsilon}}{2\sqrt{-\mathrm{Re}\,\chi_{11}}} + \mathcal{O}(\varepsilon^{3/2})\right) e^{\pm i\left((\mathrm{Im}\,\lambda_1 + \varepsilon\frac{\mathrm{Im}\,\chi_{11}}{\mathrm{Re}\,\chi_{11}})t + \mathcal{O}(\varepsilon^{3/2})\right)} \quad \textit{and} \quad \beta^{\varepsilon,\pm}_2 = \pm i\beta^{\varepsilon,\pm}_1. \tag{6.58}$$

*Furthermore, there exists a positive value* $a$ *independent of* $\varepsilon$ *such that for all small* $\varepsilon > 0$, *these two states, along with the stable periodic state* $\psi_p$ *constructed in Theorem 6.8 and the normal state, represent the only persistent states in* $B(0,a) := \{\psi :\ \|\psi\|_{H^1} < a\}$. *Indeed, the phase portrait within* $B(0,a)$ *consists of two repelling codimension two* $C^\infty$ *stable manifolds* $\tilde{\mathcal{N}}_{p,\pm}$ *of the unstable periodic solutions* $\psi_{p,\pm}$, *a repelling codimension four* $C^\infty$ *stable manifold* $\tilde{\mathcal{N}}_0$ *of the normal state* $\psi = 0$, *and an attracting codimension one* $C^\infty$ *stable manifold* $\tilde{\mathcal{N}}_{PT}$ *of the invariant manifold* $\mathcal{N}_{PT}$ *of PT-symmetric solutions and their rotations lying within* $\mathcal{M}^\varepsilon \cap B(0,\overline{C}a)$, *cf. Remark 6.6. The latter contains both the unstable normal equilibrium and the stable periodic solutions, with all other solutions flowing from repelling submanifolds* $\tilde{\mathcal{N}}_{p,\pm}$ *and* $\tilde{\mathcal{N}}_0$ *to the attracting submanifold* $\tilde{\mathcal{N}}_{PT}$ *and thereafter to the stable periodic solutions.*

*In particular, all solutions originating in* $B(0,a)$ *converge either to the stable periodic, an unstable periodic, or the normal state. For generic initial data* $\psi_0 \in B(0,a)$, *the corresponding solution converges to the stable periodic state, the only exceptional data lying on the codimension two and four submanifolds* $\tilde{\mathcal{N}}_{p,\pm}$ *and* $\tilde{\mathcal{N}}_0$, *respectively.*

**Remark 6.12.** The description (6.57)-(6.58) of the unstable periodic solutions given in $\beta$-coordinates may be recognized as profile (5.16)-(5.17) given in the $\alpha$-coordinates of Section 5.

*Proof.* Our approach here is generally to establish the claims of the theorem first for the system of differential equations (6.48) or (6.56) ignoring higher order error terms and then to broaden the claims to the full equations including the perturbations. Hence, we first note that in light of the smoothness of these higher order terms and the polynomial error bounds, the perturbations are seen to be small in the $C^1$ topology in a small neighborhood of the origin, i.e. for $\beta_1^\varepsilon$, $\beta_2^\varepsilon$ small or equivalently, for $R^\varepsilon$ small.

We begin our analysis by turning to the system (6.56). For convenience, let us denote the quantity $(R^\varepsilon)^2$ by $A^\varepsilon$ and so view this system as one for $\gamma^\varepsilon$ given by (6.49) and $A^\varepsilon := |\beta_1^\varepsilon|^2 + |\beta_2^\varepsilon|^2$. We list some simple observations:

• Elementary phase plane analysis of this system reveals that there exists a small positive number $r$, independent of $\varepsilon$, such that the triangle

$$T := \{(\gamma^\varepsilon, A^\varepsilon) : \ 0 \le |\gamma^\varepsilon| \le A^\varepsilon \le r\}$$

is invariant for the flow (6.56) for all $\varepsilon$ sufficiently small. That the flow cannot exit the two bottom sides of this triangle is trivial in light of the algebra

$$|\gamma^\varepsilon| \le 2\,|\beta_1^\varepsilon|\,|\beta_2^\varepsilon| \le |\beta_1^\varepsilon|^2 + |\beta_2^\varepsilon|^2 = A^\varepsilon.$$

That the flow cannot exit the top where $A^\varepsilon = r$ follows immediately from the fact that

$$(A^\varepsilon)' \le 2\Big(\varepsilon - \operatorname{Re}\tilde{\chi}(A^\varepsilon) + C\varepsilon(A^\varepsilon) + C(A^\varepsilon)^2\Big)A^\varepsilon < 0 \tag{6.59}$$

provided that $\varepsilon << A^\varepsilon$ and $A^\varepsilon = r$ is small. Here $C$ is a positive constant coming from the error bounds in (6.56).

• A further consequence of (6.59) for the full system (6.56) and its "parent" system (6.48) is that if one starts with data lying in the triangle $T$, then the corresponding orbit will exponentially approach a sub-triangle in which $A^\varepsilon = \mathcal{O}(\varepsilon)$. The rate of approach may be obtained by direct computation from the differential inequality $(A^\varepsilon)' \le -C_1(A^\varepsilon)^2$, $C_1 > 0$.

• If one ignores the error terms in this system, then it is easy to check that there are exactly four critical points located at

$$(\gamma^\varepsilon, A^\varepsilon) = (0, \frac{\varepsilon}{\operatorname{Re}\tilde{\chi}}),\ (0,0) \quad \text{and} \quad \frac{\varepsilon}{2\,|\operatorname{Re}\chi_{11}|}(\pm 1, 1)$$

and the Jacobian at each of these points is nonzero (see the next item below).

• Linearizing (6.56) about these four critical points, one finds: (i) The critical point near $(0, \frac{\varepsilon}{\operatorname{Re}\tilde{\chi}})$ is stable, with two negative eigenvalues, as has been noted already in the proof of Theorem 6.8. This point corresponds to the stable periodic orbit arising from a Hopf bifurcation. (ii) The normal state $(0,0)$ is unstable with a multiplicity two positive eigenvalue. (iii) $\frac{\varepsilon}{2|\mathrm{Re}\,\chi_{11}|}(\pm 1, 1)$ are both saddles with one positive and one negative eigenvalue. The negative eigenvalue corresponds to the eigenvector $(\pm 1, 1)$.

• Not only do $(\pm 1, 1)$ correspond to the stable eigendirections of the linearization about $\frac{\varepsilon}{2|\mathrm{Re}\,\chi_{11}|}(\pm 1, 1)$ respectively, but in fact, the bottom sides of the triangle $T$, i.e. $\gamma^\varepsilon = \pm A^\varepsilon$ are exactly the stable manifolds of these unstable critical points for the system (6.56) ignoring error terms.

• Introducing $s$ as the ratio $\gamma^\varepsilon / A^\varepsilon$, we use (6.56) to calculate that

$$s' = -2\mathrm{Re}\,\hat{\chi} A^\varepsilon s(1 - s^2) + \mathcal{O}(\varepsilon A^\varepsilon + (A^\varepsilon)^2). \tag{6.60}$$

As a consequence, if one ignores error terms, then one sees the flow repels away from the triangle's lower sides $s = \pm 1$ towards the manifold of $PT$-symmetric solutions and their rotations, $s = 0$, i.e. towards $\gamma^\varepsilon = 0$. We should note that since the flow (6.12) preserves PT-symmetry, it follows that $s = 0$ must in fact be a critical point of (6.60) even with inclusion of error terms.

Having noted these simple properties of (6.56), we next establish the existence of the unstable periodic orbits for (6.48) and hence for (6.12). As we shall see, these correspond precisely to the two unstable critical points near $\frac{\varepsilon}{2|\mathrm{Re}\,\chi_{11}|}(\pm 1, 1)$ found above in the $\gamma^\varepsilon A^\varepsilon$ plane. Let us first note that back in Section 5, we identified exact periodic solutions on the invariant manifolds $\alpha_1 \equiv 0$ and $\alpha_2 \equiv 0$ for the equations (5.6)-(5.7) obtained by neglecting higher-order error terms. In light of the relations (6.9) this corresponds to exact solutions (again neglecting error terms) to (6.48) found along the manifold $\beta_2^\varepsilon = \pm i\beta_1^\varepsilon$. This, in turn, corresponds to solutions of (6.56) for which $\gamma^\varepsilon = \pm A^\varepsilon$.

The explicit formulas for these two unstable periodic solutions can be obtained by directly solving (6.48) under the constraint $\beta_2^\varepsilon = \pm i\beta_1^\varepsilon$ and are given in (6.58). Note in particular that the two equations in (6.48) for $(\beta_1^\varepsilon)'$ and $(\beta_2^\varepsilon)'$ reduce to the same equation under this constraint. Their saddle-type instability follows from the observations listed above and in particular from (6.60).

In order to argue that these saddle-type periodic solutions persist with the inclusion of error terms in (6.48), we define $\omega_1^\varepsilon := (\beta_1^\varepsilon + i\beta_2^\varepsilon)$ and $\omega_2^\varepsilon := (\beta_1^\varepsilon - i\beta_2^\varepsilon)$ so that $\omega_1^\varepsilon$ and $\omega_2^\varepsilon$ are closely related to the $\alpha_j$ variables of Section 5 via the formulas $\omega_j^\varepsilon = \sqrt{\varepsilon} e^{(-1)^j \mathrm{Im}\,\lambda_1 t}\alpha_j$ for $j = 1, 2$. We will describe how to find a periodic solution to the full system (6.48) nearby the solution corresponding to $\beta_1^\varepsilon = i\beta_2^\varepsilon$, (i.e. $\omega_2^\varepsilon = 0$) for the truncated system. The same analysis can be repeated to find the other unstable solution to the full system as well.

To this end, it is perhaps best to introduce a Poincaré map as follows: Consider (6.48) without error terms and start with initial data $\omega_1^\varepsilon(0)$, $\omega_2^\varepsilon(0)$ such that $\mathrm{Im}\,\omega_1^\varepsilon(0) = 0$ while $\mathrm{Re}\,\omega_1^\varepsilon(0)$ is near $\sqrt{\varepsilon}/\sqrt{-\mathrm{Re}\,\chi_{11}}$ and $|\omega_2^\varepsilon(0)|$ is small. We then measure the values of $\mathrm{Re}\,\omega_1^\varepsilon$, $\mathrm{Re}\,\omega_2^\varepsilon$ and $\mathrm{Im}\,\omega_2^\varepsilon$ at the next time, say $T = T(\omega_1^\varepsilon(0), \omega_2^\varepsilon(0), \varepsilon)$, at which the trajectory $(\omega_1^\varepsilon(t), \omega_2^\varepsilon(t))$ crosses the 3-plane $\{(\omega_1, \omega_2) \in \mathbb{C} \times \mathbb{C} : \mathrm{Im}\,\omega_1 = 0\}$. Let us denote this map

$$(\mathrm{Re}\,\omega_1^\varepsilon(0), \mathrm{Re}\,\omega_2^\varepsilon(0), \mathrm{Im}\,\omega_2^\varepsilon(0)) \to (\mathrm{Re}\,\omega_1^\varepsilon(T), \mathrm{Re}\,\omega_2^\varepsilon(T), \mathrm{Im}\,\omega_2^\varepsilon(T)) \tag{6.61}$$

by $F^\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3$. Note that $(\frac{\sqrt{\varepsilon}}{\sqrt{-\mathrm{Re}\,\chi_{11}}}, 0, 0)$ is a fixed point of this map corresponding to the exact unstable periodic solution of the truncated system satisfying $\beta_1^\varepsilon = i\beta_2^\varepsilon$.

By converting (6.48) into a system for $\omega_j^\varepsilon$, $j = 1, 2$ one can carry out a tedious but straightforward calculation to determine that when the Jacobian matrix of $F^\varepsilon$ is evaluated at the fixed point $(\frac{\sqrt{\varepsilon}}{\sqrt{-\mathrm{Re}\,\chi_{11}}}, 0, 0)$, it takes the form

$$DF^\varepsilon = \begin{pmatrix} \mu^\varepsilon & 0 & 0 \\ 0 & e^{a^\varepsilon}\cos b^\varepsilon & -e^{a^\varepsilon}\sin b^\varepsilon \\ 0 & e^{a^\varepsilon}\sin b^\varepsilon & e^{a^\varepsilon}\cos b^\varepsilon \end{pmatrix}. \tag{6.62}$$

Here

$$\mu^\varepsilon = e^{-2\varepsilon p^\varepsilon}, \quad a^\varepsilon = \varepsilon\left(\frac{\mathrm{Re}\,\hat{\chi}}{-\mathrm{Re}\,\chi_{11}}\right)p^\varepsilon, \quad b^\varepsilon = \left(\mathrm{Im}\,\lambda_1 - \varepsilon\frac{\mathrm{Im}\,\chi_{12}}{\mathrm{Re}\,\chi_{11}}\right)p^\varepsilon,$$

and

$$p^\varepsilon = \frac{2\pi}{\mathrm{Im}\,\lambda_1 + \frac{\mathrm{Im}\,\chi_{11}}{\mathrm{Re}\,\chi_{11}}\varepsilon},$$

which we recognize as the period of the unstable periodic solution with error terms ignored.

Note that there is one eigenvalue of $DF^\varepsilon$ of modulus less than one, namely $\mu^\varepsilon$, giving one stable direction to the map, while the other two unstable eigenvalues $e^{a^\varepsilon \pm ib^\varepsilon}$ have modulus greater than one since $a^\varepsilon > 0$. Thus, one again sees the saddle-type instability of this periodic solution from the perspective of this Poincaré map.

To argue that this picture persists under perturbation (i.e. under inclusion of the error terms in (6.48)), we observe that the three eigenvalues of $D(F^\varepsilon - I)$ take the form

$$(6.63) \qquad -\frac{4\pi}{\operatorname{Im}\lambda_1}\varepsilon + \mathcal{O}(\varepsilon^2), \quad \frac{2\pi}{(\operatorname{Im}\lambda_1)\,(-\operatorname{Re}\chi_{11})}\Big(\operatorname{Re}\hat{\chi} \pm i\operatorname{Im}\tilde{\chi}\Big)\varepsilon + \mathcal{O}(\varepsilon^2).$$

Once we incorporate the error terms from (6.48) in our analysis, we use the fact that within the $\mathcal{O}(\sqrt{\varepsilon})$ ball where we are working, these $\mathcal{O}(\varepsilon|\beta|^3+|\beta|^5)$ error terms are necessarily $\mathcal{O}(\varepsilon^{5/2})$ with a contribution to the entries of the original Jacobian matrix $DF^\varepsilon$ of at most $\mathcal{O}(\varepsilon|\beta|^2+|\beta|^4) = \mathcal{O}(\varepsilon^2)$. Hence, again subtracting the identity to form the displacement map of the full system including error terms, (6.63) guarantees that the differential of this map is still nonsingular. Necessarily it then must still map a neighborhood of the original critical point onto the origin, that is, there must still exist a fixed point of the map corresponding to $F$ perturbed by higher order terms. What is more, the Jacobian matrix of this perturbed map evaluated at the fixed point must still possess one eigenvalue of modulus less than one and two of modulus greater than one so the saddle-type instability persists under perturbation for these periodic orbits.

It remains to establish the asserted global behavior of solutions originating in the $H^1$-ball $B(0,a)$. We begin by describing the behavior in $\mathcal{M}^\varepsilon \cap B(0,\overline{C}a)$, where $\overline{C} > 0$ is the geometric constant of Remark 6.6, to be used later. As observed previously, the manifold $\mathcal{N}_{PT} \subset \mathcal{M}^\varepsilon$ of PT-symmetric solutions and their rotations remains invariant for the full as well as the unperturbed equations, and is locally attracting by (6.60). Likewise, the normal state ($\beta_1^\varepsilon = \beta_2^\varepsilon = 0$) is an equilibrium of the full system that is locally repelling. Since the unstable periodic solutions correspond to hyperbolic, saddle-type equilibria of the associated return map, with a single stable eigenvalue and a pair of unstable eigenvalues, we find that they possess, for fixed $\varepsilon$, a $C^k$ stable manifold of dimension one of the return map. Under time-evolution, this induces stable manifolds $\mathcal{N}_{p,\pm}$ of dimension two within the center manifold, as claimed.

The structure and attracting or repelling properties of $\mathcal{N}_{p,\pm}$ are difficult to determine outside an $\varepsilon$-neighborhood of the periodic solution, due to the fast, order one, angular flow relative to the order $\varepsilon$ flow measured by the Poincaré return map. Thus, it would appear that a determination of the global structure in $\mathcal{M}^\varepsilon \cap B(0, Ca)$ would require more complicated averaging arguments outside the scope of the present analysis. However, we may finesse this point using more elementary tools together with the special structure of our equations.

We first observe by (6.60) that for $C > 0$ sufficiently large, the neighborhoods $\mathcal{K}_\pm$ of the triangle $T$ given by

$$\mathcal{K}_\pm := \{(\gamma^\varepsilon, A^\varepsilon) \in T: \ s := \gamma^e/A^\varepsilon \text{ satisfies } |s \pm 1| \leq C(\varepsilon + A^\varepsilon)\} \tag{6.64}$$

are invariant under backward flow of the ODE on the center manifold. Hence, by estimate (6.57)-(6.58) showing that $||s|-1| \leq C\varepsilon$ along the unstable periodics, the stable manifolds $\mathcal{N}_{p,\pm}$ of the unstable periodics are confined to $\mathcal{K}_\pm$. In particular, they lie no more than $C(\varepsilon + A^\varepsilon)$ from the sides of the triangle corresponding in $\gamma^\varepsilon$-$A^\varepsilon$ coordinates to the stable manifolds of the unstable periodic solutions of the unperturbed equations. Moreover, from (6.56), we see that for $(\gamma^e, A^\varepsilon) \in \mathcal{K}_\pm$ one has

$$(A^\varepsilon)' = 2(\varepsilon - \mathrm{Re}\,(\tilde{\chi} + \hat{\chi})A^\varepsilon)A^\varepsilon + \mathcal{O}(\varepsilon + A^\varepsilon)(A^\varepsilon)^2.$$

Thus, for $(\gamma^\varepsilon, A^\varepsilon) \in \mathcal{K}_\pm$ outside an $\varepsilon^2$-neighborhood of the critical point $\frac{\varepsilon}{\mathrm{Re}\,(\tilde{\chi}+\hat{\chi})}(\pm 1, 1)$ of the unperturbed $\gamma^\varepsilon$-$A^\varepsilon$ system, $A^\varepsilon$ is strictly decreasing for $A^\varepsilon > \frac{\varepsilon}{\mathrm{Re}\,(\tilde{\chi}+\hat{\chi})}$ and strictly decreasing for $A^\varepsilon < \frac{\varepsilon}{\mathrm{Re}\,(\tilde{\chi}+\hat{\chi})}$. Hence in these regions, $\mathcal{N}_{p,\pm}$ are graphs over their unperturbed counterparts and $\mathcal{O}(\varepsilon + A^\varepsilon)$ close to them.

It remains to treat the excluded $\varepsilon^2$ neighborhood in $\gamma^\varepsilon$-$A^\varepsilon$ coordinates of the critical point $\frac{\varepsilon}{\mathrm{Re}\,(\tilde{\chi}+\hat{\chi})}(\pm 1, 1)$ of the unperturbed $\gamma^\varepsilon$-$A^\varepsilon$ system. By a straightforward computation, we find that this neighborhood is contained in the image of an $\varepsilon$-neighborhood in $\beta^\varepsilon$-coordinates of the orbit of the order $\varepsilon^{1/2}$ amplitude unstable periodic solution (6.58) of the unperturbed $\beta^\varepsilon$-equations.

Indeed, a brief examination in $\beta^\varepsilon$-coordinates shows that the $\gamma^\varepsilon$-$A^\varepsilon$ estimate is overly conservative (a result of the singularity in certain directions of the coordinate change between the two coordinate systems). Changing to the more convenient $\omega$-coordinates, for which $|\omega_1^\varepsilon| = \frac{\sqrt{\varepsilon}}{\sqrt{-\mathrm{Re}\,\chi_{11}}}$, $\omega_2^\varepsilon = 0$ corresponds to the unstable periodic orbit, and $\omega_2^\varepsilon = 0$ its stable manifold, we find compute that

$$|\omega_1^\varepsilon|_t = (\varepsilon + (\operatorname{Re}\chi_{11}|\omega_1^\varepsilon|^2 + \operatorname{Re}\chi_{12}|\omega_2^\varepsilon|^2)|\omega_1^\varepsilon| + O(\varepsilon|\omega^\varepsilon|^3 + |\omega^\varepsilon|^5),$$
$$|\omega_2^\varepsilon|_t = (\varepsilon + (\operatorname{Re}\chi_{11}|\omega_2^\varepsilon|^2 + \operatorname{Re}\chi_{12}|\omega_1^\varepsilon|^2)|\omega_2^\varepsilon| + O(\varepsilon|\omega^\varepsilon|^3 + |\omega^\varepsilon|^5)$$

(cf. (5.10)–(5.11)). Using this system, we can check that as long as the deviations $\tilde{\omega}_1^\varepsilon$ and $\tilde{\omega}_2^\varepsilon$ from the unstable periodic solution to the unperturbed system satisfy

$$\varepsilon^{3/2} << |\tilde{\omega}_1^\varepsilon| << \varepsilon^{1/2} \quad \text{and} \quad \varepsilon^{3/2} << |\tilde{\omega}_2^\varepsilon| << \varepsilon^{1/2},$$

the stable manifolds $\mathcal{N}_{p,\pm}$ are graphs over their unperturbed counterparts ($\omega_2^\varepsilon \equiv 0$ in this case) and all other solutions are repelled toward $\mathcal{N}_{PT}$ since $\frac{d}{dt}|\tilde{\omega}_1^\varepsilon|$ will be negative in this regime while $\frac{d}{dt}|\tilde{\omega}_2^\varepsilon|$ will be positive.

On the other hand, by our previous estimates on the orders of perturbation terms and their Jacobians in the Poincaré return map $F^\varepsilon$ given by (6.61), the Taylor expansion of this map at the unstable fixed point corresponding to the unstable periodic, denoted here by $p_*^\varepsilon$, is

$$\tilde{F}^\varepsilon(z) := F^\varepsilon(p_*^\varepsilon + z) - p_*^\varepsilon = dF^\varepsilon(p_*^\varepsilon)z + N^\varepsilon(z,z) + \Theta^\varepsilon(z).$$

Here the quadratic order Taylor remainder term $N^\varepsilon$ is order

$$N^\varepsilon = \mathcal{O}(|p_*^\varepsilon||z|^2), \quad N_z^\varepsilon = \mathcal{O}(|z|^2 + |p_*^\varepsilon||z|) \tag{6.65}$$

and the term $\Theta^\varepsilon$ which incorporates the perturbation term is order

$$\Theta^\varepsilon = \mathcal{O}(\varepsilon|p_*^\varepsilon|^3 + |p_*^\varepsilon|^5), \quad \Theta_z^\varepsilon = \mathcal{O}(\varepsilon|p_*^\varepsilon|^2 + |p_*^\varepsilon|^4) \tag{6.66}$$

in an $\varepsilon$-neighborhood of $p_*^\varepsilon$.

Reviewing the standard invariant manifold constructions by fixed point/contraction arguments (see, e.g., [3, 6]), we find that they yield existence and closeness in angle of these manifolds to corresponding invariant subspaces of $dF^\varepsilon$ on a ball about $p_*^\varepsilon$ for which the Lipshitz norm of the (total) nonlinear term $N^\varepsilon + \Theta^\varepsilon$ is sufficiently small compared to the spectral gap of $d\tilde{F}^\varepsilon(0) = dF^\varepsilon(p_*^\varepsilon)$ so long as the norm of $\log d\tilde{F}^\varepsilon = \log dF^\varepsilon$ is no larger than some specified multiple of the spectral gap, as it is here, cf. (6.62). Here the spectral gap is defined as the minimum distance between one and the modulus of eigenvalues that are not modulus one. By (6.63), the spectral gap is greater than $\eta\varepsilon$ for some positive $\eta$, while by (6.65)–(6.66) and the fact that $|p_*^\varepsilon| \sim \varepsilon^{1/2}$, we have $|N_z^\varepsilon| + |\Theta_z^\varepsilon| \leq C(|z|^2 + \varepsilon^{1/2}|z| + \varepsilon^2) = o(\varepsilon)$ as desired for $|z| << \varepsilon^{1/2}$. Thus, we obtain a detailed "microscopic" description of the behavior of the Poincaré return map on a ball about $p_*^\varepsilon$ of radius $\eta\varepsilon^{1/2}$, for $\eta > 0$ and sufficiently small. This translates to a detailed description of $\mathcal{N}_{p,\pm}$ and asymptotic behavior on an $\eta\varepsilon^{1/2}$-neighborhood of the unstable periodic orbit, far more than what was needed (the excluded $\varepsilon^{3/2}$-neighborhood in $\omega^\varepsilon$ coordinates or for that matter the excluded $\varepsilon^2$ neighborhood in $\gamma^\varepsilon, A^\varepsilon$ coordinates) to complete the argument.

Indeed, though we do not need it, the faster decay in $\varepsilon$ of perturbation terms $\Theta$, $\Theta_z$ yields uniform convergence of perturbed to unperturbed flow and stable manifold in "blown-up" coordinates $\tilde{\omega} := \omega/\varepsilon^{1/2}$, by the same argument.

Thus, solutions originating outside $\mathcal{K}_\pm$ remain outside, with $s$ by (6.60) strictly decreasing at an exponential rate. Moreover, if ever solutions leave $\mathcal{K}_\pm$, they are attracted at an exponential rate to the PT-symmetric manifold $\mathcal{N}_{PT}$ corresponding to $s = 0$. On the other hand, solutions remaining in $\mathcal{K}_\pm$ sufficiently long must eventually enter the small neighborhood $B_\varepsilon \cap \mathcal{K}_\pm$, after which solutions not on $\mathcal{N}_{p,\pm}$ must (by our microscopic description carried out in $\omega$-coordinates) eventually leave. Piecing together this information, we find that all solutions originating in $\mathcal{M}^\varepsilon \cap B(0, Ca)$ and not on the stable manifolds $\mathcal{N}_{p,\pm}$ of the unstable periodics time-asymptotically approach the attracting PT-symmetric manifold $\mathcal{N}_{PT}$, as claimed. This completes the description of asymptotic behavior within the center manifold.

By Remark 6.6, the $C^\infty$ stable manifolds $\mathcal{N}_{p,\pm}$ within the center manifold, of codimension two in $\mathcal{M}^\varepsilon \cap B(0, a)$, extend to $C^\infty$ stable manifolds

$$\tilde{N}_{p,\pm} := \cup_{w \in \mathcal{N}_{p,\pm}} P_\varepsilon^{-1}(w) \cap B(0, \overline{C}a)$$

of codimension two in $B(0, Ca)$. Consequently, for fixed $\varepsilon > 0$, $(\tilde{\mathcal{N}}_{p,+} \cup \tilde{\mathcal{N}}_{p,-}) \cap B(0, a)$ is exactly the set of data in $B(0, a)$ whose solutions converge asymptotically to an unstable branch. Likewise, there is a $C^\infty$ stable manifold $\tilde{\mathcal{N}}_0 := P_\varepsilon^{-1}(0)$ of the normal equilibrium, of codimension four, containing all solutions originating in $B(0, a)$ and converging to the normal state. Finally, $\mathcal{N}_{PT}$ has a $C^\infty$ stable manifold $\tilde{\mathcal{N}}_{PT} := \cup_{w \in \mathcal{N}_{PT}} P_\varepsilon^{-1}(w)$ of codimension one. By Proposition 6.5, together with our description of asymptotic behavior on the center manifold, we find that all solutions originating in $B(0, a)$ outside the sets $\tilde{\mathcal{N}}_{p,\pm}$ and $\tilde{\mathcal{N}}_0$ are attracted to $\mathcal{N}_{PT}$ (hence to the larger manifold $\tilde{\mathcal{N}}_{PT}$ of solutions converging to $\mathcal{N}_{PT}$) and ultimately to the stable periodic solutions or their rotations. This completes the description of asymptotic behavior and the proof. □

**Remark 6.13.** An implication of Proposition 6.11 is that if one chooses complex initial data $(\beta_1^\varepsilon(0), \beta_2^\varepsilon(0))$ away from the special two-dimensional manifolds containing the unstable branch, then there is a kind of phase-locking phenomenon whereby the flow (6.48) pushes the solution towards a pair $(\beta_1^\varepsilon(t), \beta_2^\varepsilon(t))$ which is a complex rotation of a real pair, that is, $\gamma^\varepsilon$ tends to zero and the corresponding solution to (6.12) approaches a PT-symmetric profile. On the other hand, it is not true that all solutions exhibit this phase-locking, as evidenced by the existence of the unstable periodic solutions.

**Remark 6.14.** As noted parenthetically in the proof, a side-consequence of our analysis is to rigorously validate the formal asymptotics of Section 5 by verifying convergence of perturbed to unperturbed phase portrait on a ball of size $\varepsilon^{1/2}$ about the normal state, in "blown-up" coordinates $\tilde{\omega} := \omega/\varepsilon^{1/2}$ (equivalently, $\tilde{\beta} := \beta/\varepsilon^{1/2}$) equivalent to the $\alpha$-coordinates of the earlier section.

6.3. **Bifurcation off of higher eigenvalues.** We conclude this section by remarking on the bifurcation situation off of higher eigenvalues. Recalling the notation (6.3), we fix any integer $\ell > 1$, and set the parameter $\Gamma$ in (1.4) equal to $\operatorname{Re} \lambda_{2\ell-1} + \varepsilon$. Then (1.4)-(1.5) can be rewritten as

$$\psi_t = L_\ell[\psi] + \mathcal{N}(\psi, \varepsilon) \tag{6.67}$$

All of the analysis of the preceding subsection, including the center manifold construction, applies to produce the existence of stationary and periodic solutions to this system for $I < I_\ell$ and $I > I_\ell$ respectively, cf. (6.2). The difference here is that for $\ell > 1$, there will always be a nonempty unstable subspace corresponding to $\lambda_j$ with $1 \leq j < 2\ell - 1$. Hence, these bifurcating solutions will always be unstable. As far as the center manifold construction is concerned, the primary change is that the fixed point argument must now be applied to the integral equation

$$y(x,t) = \Gamma(u,\varepsilon,y) := e^{L_1 t}u + \int_0^t e^{L_1(t-\tau)}\Pi_c \mathcal{N}^\delta\left(y(x,\tau),\varepsilon\right)\, d\tau$$
$$+ \int_{-\infty}^t e^{L_1(t-\tau)}\Pi_s \mathcal{N}^\delta\left(y(x,\tau),\varepsilon\right)\, d\tau - \int_t^\infty e^{L_1(t-\tau)}\Pi_u \mathcal{N}^\delta\left(y(x,\tau),\varepsilon\right)\, d\tau,$$

rather than (6.19). Here $\Pi_u$ denotes the projection onto the unstable subspace and through its realization as a contour integral

$$e^{tL_1}\Pi_u := \int_{\Gamma_u} e^{\lambda t}(\lambda I - L_1)^{-1}\, d\lambda, \tag{6.68}$$

(where $\Gamma_u$ is any bounded contour enclosing the finite number of eigenvalues with positive real part) one establishes the necessary bound

$$\left\| e^{tL_1}\Pi_u \right\|_{H^1\to H^1} \le Ce^{(\zeta_1 - s)t} \quad \text{for all } t \le 0$$

to augment (6.22)-(6.23).

## 7. Phase slip centers

The physics literature associates periodic solutions with the existence of phase slip centers (PSCs), that is, zeros of the order parameter. Indeed, an immediate conclusion of the formal calculation of Section 5, rigorously confirmed in Section 6, is that the leading order term in the expansion for the periodic solution $\psi_p$ to (6.12) along the bifurcation branch in the regime $I > I_c$ has a periodic array of zeros at $x = 0$. More precisely, referring back to (6.46)-(6.47), we see that the approximate solution $\beta_1(t)v_1(x) + \beta_2(t)v_2(x)$, when evaluated at $x = 0$, has zeros whenever the expression

$$\cos[\omega^\varepsilon t] := \cos\left[\left(\operatorname{Im}\lambda_1 + \frac{\operatorname{Im}\tilde{\chi}}{\operatorname{Re}\tilde{\chi}}\varepsilon + \mathcal{O}(\varepsilon^{3/2})\right)t\right]$$

vanishes, since $v_2(0) = i(u_1(0) - u_2(0)) = 0$. In other words, there are PSC's located periodically at $(x,t) = (0, T_k^\varepsilon)$, $k = 0,1,2,\dots$ where $T_k^\varepsilon \approx \frac{1}{\omega^\varepsilon}(\pi/2 + k\pi)$. Since the actual solution $\psi_p$ is uniformly close to $\beta_1 v_1 + \beta_2 v_2$, it follows that the two functions have the same Brouwer degree in a neighborhood of the points $\{(0, T_k^\varepsilon)\}$. Hence, we will rigorously conclude that $\psi_p$ possesses an array of zeros close to the points $\{(0, T_k^\varepsilon)\}$ once we check that the degree of

$\beta_1 v_1 + \beta_2 v_2$ is nonzero at these points. But a direct calculation of the Jacobian determinant at these points indeed reveals that

$$\text{Jac}\,(\beta_1 v_1 + \beta_2 v_2)\,(0, T_k^\varepsilon) \sim -\frac{8\pi\varepsilon}{\kappa_r \text{Im}\,\lambda_1 + \kappa_i \varepsilon} \text{Re}\, u_1'(0).$$

A straight-forward numerical calculation of the first eigenfunction for the operator $M$ (cf. (1.9)), for which there are rigorous error bounds, shows that $\text{Re}\, u_1'(0) \neq 0$, with values ranging monotonically from about $-0.2234$ for $I = 12.5$ down to about $-0.3578$ for $I = 20$. Hence, we obtain a rigorous confirmation of a periodic array of PSC's for the periodic solution.

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