Local Minimizers with Vortices to the Ginzburg-Landau System in 3-d

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Abstract

We construct local minimizers to the Ginzburg-Landau energy in certain 3-d domains based on the asymptotic connection between the energy and the total length of vortices using the theory of weak Jacobians. Whenever there exists a collection of locally minimal line segments spanning the domain, we can find local minimizers with arbitrarily assigned degrees with respect to each segment.

1 Introduction

We pursue here the construction and asymptotic characterization of local minimizers to the Ginzburg-Landau energy

$$\tilde{E}_\varepsilon(u) = \int_\Omega \left\{ \frac{|\nabla u|^2}{2} + \frac{(1-|u|^2)^2}{4\varepsilon^2} \right\} dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded, open set and $u \in H^1(\Omega;\mathbb{C})$. For small ε, one expects that any such minimizers will be largely characterized by their zero sets or vortices. There has been extensive study of $\tilde{E}_\varepsilon$ in this vein by many authors, some of whom we refer to below, but what distinguishes the present investigation from most others is that we seek local minimizers possessing vortices while adhering to the natural homogeneous Neumann boundary conditions rather than Dirichlet. In particular then, the local minimizers we construct represent nontrivial solutions to the problem

$$\Delta u = \frac{1}{\varepsilon^2}(|u|^2-1)u \text{ in } \Omega, \tag{1.2}$$

$$\nabla u \cdot \nu = 0 \text{ on } \partial\Omega, \tag{1.3}$$

where ν represents the outer unit normal to $\partial\Omega$.

As alluded to above, the situation where a nontrivial Dirichlet condition replaces (1.3) presents a quite different picture with regard to the

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existence of stable vortex solutions. For example, when $n = 2$, S^1−valued data of non-zero degree prescribed on $\partial\Omega$ necessarily leads to vortices for the *global* minimizer of $\tilde{E}_\varepsilon$ whereas, of course, the global minimizer in our setting is simply any constant unit vector. Thus, in the Dirichlet case, the goal is generally not existence of local minimizers but rather a description of the asymptotic geometry and location of the vortices of a sequence of global minimizers as $\varepsilon \to 0$. We refer to [3] for the resolution of this question in two dimensions, where the vortices are generically points, and to the more recent works [26], [23], [28] and [5] for its consideration in three and higher dimensions where vortices are generically of co-dimension two.

The energy $\tilde{E}_\varepsilon$ arises in certain models for super-fluids but also stands as a kind of "warm-up problem" for the full Ginzburg-Landau energy of superconductivity given by

$$G_\varepsilon(u, A) = \int_\Omega \left(\frac{1}{2}|(\nabla - iA)u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2\right) dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla \times A - H_{ap}|^2 \, dx \tag{1.4}$$

for $n = 2$ or 3 (cf. [10]). Here $A : \mathbb{R}^n \to \mathbb{R}^n$ represents the vector potential of the induced magnetic field and $H_{ap} : \mathbb{R}^n \to \mathbb{R}^n$ is an applied magnetic field. As regards vortex solutions, one can make an analogy between topologically nontrivial Dirichlet data for $\tilde{E}_\varepsilon$ and a sufficiently large applied field for G_ε. Both conditions are known to force the presence of vortices in even the global energy minimizer (see e.g. [30] for the analysis of G_ε in this setting). For the energy G_ε, the setting analogous to our present study, however, would be the case of zero applied field, where nontrivial solutions correspond physically to so-called "permanent currents" (cf. [7]). Over the past ten years, the development of tools yielding results in the study of $\tilde{E}_\varepsilon$ has often lead to tools and results applicable to G_ε and the same is true in our setting. In a subsequent article we will present existence results for local minimizers to G_ε possessing vortices when $H_{ap} \equiv 0$ ([11]).

Before describing in more detail our methods and results we wish to mention a number of existence and non-existence results in the Neumann setting that highlight the subtle role played by the topology, geometry and dimension of the domain Ω in these problems. For Ω in any dimension, the results of [15] demonstrate the non-existence of non-constant stable critical points to $\tilde{E}_\varepsilon$ when Ω is convex. Here "stable" refers to the hypothesis of non-negative second variation. This non-existence result has since been extended for $n = 2$ to the case of G_ε with $H_{ap} \equiv 0$ in [19]. We suspect non-existence of stable vortex solutions in two dimensions for nonconvex, simply-connected domains in the plane as well, but to our knowledge this question is open.

On the other hand, if one takes $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ to be non-simply-connected, then non-constant local minimizers have been found for both $\tilde{E}_\varepsilon$ and G_ε (again with $H_{ap} \equiv 0$) when ε is sufficiently small (see [16], [18], [20] and [27]). For example, if one takes Ω to be a topological torus in $\mathbb{R}^3$, then for each integer m, one can construct local minimizers of $\tilde{E}_\varepsilon$ whose restriction to loops encircling the hole of the torus have winding number m. At the heart of these arguments is the fact that for topologically non-trivial domains, the space $W^{1,2}(\Omega; S^1)$ can be decomposed according to 1-homotopy type ([31]). However, another property of these solutions is that they have no vortices; indeed they are uniformly close to 1 in modulus.

Closely related to our work are the results in [6], [17] and [21] on existence of local minimizers of $\tilde{E}_\varepsilon$ and G_ε (with $H_{ap} \equiv 0$) when $\Omega \subset \mathbb{R}^3$ is a simply-connected perturbation of a non-simply-connected domain. As such, these results serve to demonstrate that there is no topological obstruction to producing non-constant stable critical points. They also serve as the first examples of stable vortices in the Neumann setting, since a simple maximum principle argument applied to (1.2)-(1.3) demonstrates that non-constant critical points in simply-connected domains must vanish. However, these existence results in 'almost' non-simply-connected domains beg the question of whether there exists a more natural *geometric* criterion on Ω guaranteeing existence of stable critical points. Furthermore, these results carry no information on the geometry or location of the vortices.

To address this question, we bring to bear the recent work of Jerrard and Soner [12] on weak Jacobians and the Ginzburg-Landau energy. (See also [2].) A more precise description of the theory, along with the necessary background from geometric measure theory, is provided in Section 2 of this paper. Roughly speaking though, given a sequence $\{v_\varepsilon\} \subset H^1(\Omega; \mathbb{C})$, these authors view the weak Jacobians of the v_ε as generalized curves and argue that if the quantity

$$E_\varepsilon(v_\varepsilon) \equiv \frac{\tilde{E}_\varepsilon(v_\varepsilon)}{|\ln \varepsilon|}$$

is bounded, then the Jacobians are compact in a suitable topology–the dual of the Hölder space, $C_c^{0,\alpha}$, for any $\alpha \in (0, 1]$. Hence, up to a subsequence, they converge to a countable union of limiting curves in Ω. Loosely speaking, one can think of these limiting curves as the asymptotic location of the sets $\{v_\varepsilon(x) = 0\}$. Along with this, it is shown that the sequence of energies $E_\varepsilon(v_\varepsilon)$ is bounded below asymptotically by a constant multiplying the total length of the limiting curve or curves. (See Theorem 2.1 for a precise statement.) We will exploit this connection between the energy E_ε and length of vortices to produce local minimizers. The approach is very much in the spirit of the one found in [22] for the scalar version of $\tilde{E}_\varepsilon$, where one takes advantage of the connection established in [24] between the energy $\varepsilon\tilde{E}_\varepsilon$ and the surface area of the co-dimension-1 zero sets of appropriate bounded sequences $\{v_\varepsilon\} \subset H^1(\Omega;\mathbb{R}^1)$.

Given this asymptotic connection between the Ginzburg-Landau energy and the length of vortices, we construct local minimizers of $\tilde{E}_\varepsilon$ in any domain $\Omega \subset \mathbb{R}^3$ possessing a finite collection of line segments $l_1, l_2, \ldots, l_N \subset \Omega$, $(N \geq 1)$ with endpoints on $\partial\Omega$ such that each l_j locally minimizes length. See assumptions (3.1)–(3.3) and (4.1) for a precise statement of the hypotheses on Ω. We accomplish this so that in a certain topology, the line segments serve as the asymptotic location of the vortices for the constructed sequence. What is more, given any collection of integers $m_1, m_2, \ldots, m_N$, we can carry out this construction so as to build local minimizers which, roughly speaking, wind m_j times around l_j. See Theorem 4.2 for a precise statement of the existence result and Theorem 5.4 for a precise statement on the location of the vortices.

Parallel to our efforts has been the investigation by the authors of [8]. Under an assumption that a single line segment $l \subset \Omega$ with endpoints on $\partial\Omega$ represents a non-degenerate critical point of length, they construct a sequence of critical points of $\tilde{E}_\varepsilon$. In particular then, they also obtain a local minimizer of the energy when l is a local minimizer of length. They also obtain a precise asymptotic description of these degree-one vortex solutions including the description of the vortex itself as a C^1 curve. Perhaps the most striking aspect of a comparison between the two articles is that the techniques are totally different. The approach in [8] is based upon a very carefully constructed approximate sequence of critical points followed by an infinite-dimensional reduction argument whereas our approach, as we mentioned earlier, is based upon the theory of weak Jacobians.

To carry out our program rigorously, we invoke the theory of currents from geometric measure theory. The necessary elements of the theory are reviewed in Section 2, along with a description of the needed results from [12]. In Section 3 we construct sequences of approximate local minimizers and in Section 4 we present our main existence result. Then in Section 5 we present a result strengthening the topology in which the vortices are "close" to the line segments.

2 Preliminaries

Throughout this article, $\Omega \subset \mathbb{R}^3$ will denote a bounded open set with Lipschitz continuous boundary. We begin by briefly introducing certain concepts and notation from the theory of currents. We refer the reader to [9] or [29] for more details. Throughout, we use $H^{(n)}$ to denote n-dimensional Hausdorff measure.

2.1 Currents

For integers $0 \le k \le n$, the space of Grassman k-covectors is denoted by $\wedge^k(\mathbb{R}^n)$ endowed with the usual Euclidean norm $|\cdot|$. A differential k-form ϕ on Ω is a mapping $\phi\colon \Omega \to \wedge^k(\mathbb{R}^n)$ with norm $|\phi| := \sup_{x\in\Omega} |\phi(x)|$. Relative to the usual basis in $\wedge^k(\mathbb{R}^n)$, $\varphi(x)$ has coordinate functions $\varphi(x) = \{\varphi^1(x), \ldots .\varphi^{n^*}(x)\}$ where $n^* = \binom{n}{k}$. The space of C^∞ k-forms compactly supported within Ω are denoted by $\mathcal{D}^k(\Omega)$. Its topology implies that $\phi_k \to 0$ in $\mathcal{D}^k(\Omega)$ if and only if there is a fixed compact set $K \subset \Omega$ with $\operatorname{spt}\phi_k \subset K$ and $\phi_k \to 0$ uniformly along with all derivatives of any order. A $k-$current in Ω is a continuous linear functional on the space $\mathcal{D}^k(\Omega)$ and the space of such k-currents is denoted by $\mathcal{D}_k(\Omega)$. We recall that the boundary of an k-current T, denoted by ∂T, is the $(k-1)$-current defined by the relation

$$\partial T(\phi) = T(d\phi) \text{ for all } \phi \in \mathcal{D}^{k-1}(\Omega),$$

where $d\phi$ represents the k-form obtained by exterior differentiation of ϕ. In particular, we note that a k-current T has zero boundary relative to the set Ω if $T(d\phi) = 0$ for all $\phi \in \mathcal{D}^{k-1}(\Omega)$.

For $T \in \mathcal{D}_k(\Omega)$, we denote the mass of T in Ω by

$$M_\Omega(T) \equiv \sup_{\{\phi\in D^k(\Omega):\ |\phi|\le 1\}} |T(\phi)|\,. \tag{2.1}$$

Throughout this paper, the dependence of the mass on the domain will be clear and in such instances we will suppress this dependence in the notation and write simply $M(T)$ rather than $M_\Omega(T)$.

Another norm on k-currents that will play a crucial role in what follows is the dual to the $C^{0,1}$-norm, given by

$$\|T\|^*_{0,1} \equiv \sup |T(\phi)|\,. \tag{2.2}$$

where the supremum is taken over all $\phi \in D^k(\Omega)$ such that

$$|\phi| \le 1,\ \sum_{j=1}^{n^*} \left|\nabla\phi^j\right| \le 1.$$

At times, we will use this norm in various subdomains $\Omega' \subset \Omega$ and though we do not explicitly express the dependence of the domain in our notation $\|T\|^*_{0,1}$, this dependence will be clear from the fact that $T \in \mathcal{D}_1(\Omega')$.

If an element $T \in \mathcal{D}_k(\Omega)$ satisfies the condition $M_W(T) < \infty$ for all $W \subset\subset \Omega$ then one can associate with T a Radon measure μ_T on Ω, the total variation measure associated with T. It is characterized by

$$\mu_T(W) = \sup_{\{\phi \in D^k(\Omega):\ |\phi| \leq 1,\, spt\,\phi \subset W\}} T(\phi) \tag{2.3}$$

for any open $W \subset \Omega$. In particular, one has

$$\mu_T(\Omega) = M_\Omega(T) \tag{2.4}$$

(cf. [29], pg. 134).

Most prominent in our approach will be the class $\mathcal{R}_k(\Omega)$ of rectifiable, integer multiplicity k-currents. Especially crucial will be elements of $\mathcal{R}_1(\Omega)$, a geometric measure theoretic generalization of a Lipschitz curve. To describe this class, let us first recall that a set $\Gamma \subset \mathbb{R}^n$ is said to be 1-rectifiable if

$$\Gamma = \Gamma_0 \cup \left(\cup_{j=1}^{\infty} f_j(\gamma_j) \right)$$

where $H^{(1)}(\Gamma_0) = 0$, $\gamma_j \subset \mathbb{R}^1$ and $f_j : \gamma_j \to \mathbb{R}^n$ are Lipschitz functions. A 1-current T on Ω is said to be rectifiable and integer multiplicity if its action on a 1-form $\phi \in \mathcal{D}^1(\Omega)$ is given by

$$T(\phi) = \int_\Gamma \langle \phi(x), \tau(x) \rangle\, m(x)\, dH^{(1)}(x) \tag{2.5}$$

where Γ is a 1-rectifiable set, τ is a unit vector orienting the approximate tangent space $T_x\Gamma$ and m is an $H^{(1)}$-measurable, positive integer-valued function referred to as the multiplicity of the current. The notation $\langle \cdot, \cdot \rangle$ above refers to the dual pairing of a vector and co-vector. At other times in this article we shall use this notation to denote the usual inner product between two forms. We trust that this will cause no confusion.

Useful to us will be the characterization of any $T \in \mathcal{R}_1(\Omega)$ with $\partial T = 0$ relative to Ω as

$$T = \sum_{i \geq 1} T_i \quad \text{with} \quad M(T) = \sum_{i \geq 1} M(T_i), \tag{2.6}$$

where each T_i is multiplicity one and indecomposable in the sense that for each i there is no 1-current R such that

$$R \neq 0,\ T_i - R \neq 0 \text{ and}$$
$$M(T_i) + M(\partial T_i) = M(R) + M(\partial R) + M(T_i - R) + M(\partial(T_i - R))$$

(cf. [9], 4.2.25). More importantly, in this characterization there are one-to-one Lipschitz functions $f_i : [0, M(T_i)] \to \mathbb{R}^3$ with Lipschitz constant less than or equal to 1 such that $(f_i)_\#([0, M(T_i)]) = T_i$. Here $(f_i)_\#(\cdot)$ refers to the push forward. Finally, we note that if $\partial T = 0$, that is, if T is a cycle relative to Ω, then for each i one has either $f_i(0) = f_i(M(T_i))$ (a closed curve within Ω) or $f_i(0) \in \partial\Omega$ and $f_i(M(T_i)) \in \partial\Omega$ (an arc with both endpoints on $\partial\Omega$).

Finally, we wish to recall the notion of the slice of a 1-current T by a Lipschitz function $f : \Omega \to \mathbb{R}^1$ (cf. [29], p.155). If T is described by a 1-rectifiable set Γ, a tangent vector $\xi : \Gamma \to \mathbb{R}^3$ and a multiplicity $m : \Gamma \to \mathbb{Z}$, then the slice of T by f at a value $t \in \mathbb{R}^1$, denoted by $\langle T, f, t\rangle$, is a 0-current defined for H^1-a.e. $t \in \mathbb{R}^1$. Letting $\nabla^\Gamma f$ denote the gradient of f relative to Γ and $\Gamma_+ \equiv \{x \in \Gamma : \left|\nabla^\Gamma f(x)\right| > 0\}$, the slice is supported on the countable set of points $f^{-1}(t) \cap \Gamma_+$, carries orientation ± 1 according to the relation $\xi(x) = \pm \frac{\nabla^\Gamma f(x)}{|\nabla^\Gamma f(x)|}$, and carries multiplicity $m \llcorner \Gamma_+$. Of most significance to our purposes is the inequality

$$\int_{-\infty}^{\infty} M(\langle T, f, t\rangle)\, dt \leq \sup_{x\in\Gamma} \left|\nabla^\Gamma f(x)\right| M(T). \tag{2.7}$$

2.2 Weak Jacobians as 1-Currents

Our approach is based upon the notion of weak Jacobian, as developed in [13]. For a smooth map $u : \mathbb{R}^3 \to \mathbb{C}$, the classical Jacobian is a 3-vector whose components consist of the three 2×2 minors of the 2×3 matrix ∇u (associating $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way). However, through integration by parts, it has been observed that one can define a weak Jacobian over appropriate classes of functions that are not so smooth. Depending on one's intentions, one can alternatively view this weak Jacobian of maps from $\mathbb{R}^3 \to \mathbb{C}$ as a distribution, a vector-valued Radon measure, a current or a differential form. For our purposes, we choose to view it as a 1-current associated with functions $u \in W^{1,1}(\Omega; S^1)$ or $u \in W^{1,2}(\Omega;\mathbb{C})$. To this end, we define the 1-form $j(u)$ via the formula

$$j(u) = \frac{1}{2i} \sum_{k=1}^{3} (\overline{u} u_{x_k} - u \overline{u}_{x_k}) dx_k. \tag{2.8}$$

where $\bar{\cdot}$ denotes complex conjugation. Note that $|j(u)| \in L^1(\Omega)$ for these classes of functions u. Then we define the 1-current $J(u)$ by the relation

$$J(u)(\phi) = \frac{1}{2} \int_\Omega \langle j(u), \star d\phi\rangle\, dx \tag{2.9}$$

for any $\phi \in \mathcal{D}^1(\Omega)$, where $\star$ denotes the Hodge-star operator which for any $\phi \in \mathcal{D}^1(\Omega)$ given by $\sum_{k=1}^3 \phi^{(k)} dx_k$ converts the 2-form

$$d\phi = (\phi^{(2)}_{x_1} - \phi^{(1)}_{x_2})dx_1 \wedge dx_2 + (\phi^{(1)}_{x_3} - \phi^{(3)}_{x_1})dx_1 \wedge dx_3 + (\phi^{(3)}_{x_2} - \phi^{(2)}_{x_3})dx_2 \wedge dx_3$$

into the 1-form

$$\star d\phi = (\phi^{(3)}_{x_2} - \phi^{(2)}_{x_3})dx_1 + (\phi^{(1)}_{x_3} - \phi^{(3)}_{x_1})dx_2 + (\phi^{(2)}_{x_1} - \phi^{(1)}_{x_2})dx_3.$$

Two simple but important consequences of (2.9) are that for any $u \in W^{1,2}(\Omega;\mathbb{C}) \cup W^{1,1}(\Omega;S^1)$ one has

$$\partial J(u) = 0 \quad (\text{since } d^2\phi = 0) \tag{2.10}$$

and for any $u, v \in W^{1,2}(\Omega;C) \cup W^{1,1}(\Omega;S^1)$ one has

$$\|J(u) - J(v)\|^*_{0,1} \le 6\,\|j(u) - j(v)\|_{L^1(\Omega)}\,. \tag{2.11}$$

For certain functions u in $W^{1,1}(\Omega,S^1)$, it was shown in [13] that the weak Jacobian $J(u)$ can be identified with a 1-rectifiable current where $\frac{1}{\pi}J(u)$ is integer multiplicity. For the reader unfamiliar with this subject we now present an example of such a function which will in fact play a crucial role in the next section. To this end, consider a given bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ such that the set

$$\Gamma = \{(x_1,x_2,x_3) \in \Omega : x_1 = x_2 = 0\}$$

consists of a line segment with endpoints $(0,0,0)$ and $(0,0,L)$ for some $L > 0$. Let us introduce a cylindrical coordinate system (r,θ,z) where (r,θ) represent polar coordinates in the x_1x_2-plane and z points in the positive x_3 direction. Given any integer m, we proceed to compute the 1-current $J(u_m)$ associated with the $W^{1,1}(\Omega,S^1)$ function $u_m = e^{im\theta}$.

From (2.8) we calculate that $j(u_m) = m\sum_{k=1}^3 \theta_{x_k} dx_k$. Then we find from (2.9) that

$$\begin{aligned} J(u_m)(\phi) &= \frac{m}{2}\int_\Omega \langle \sum_{k=1}^3 \theta_{x_k} dx_k, \star d\phi\rangle\, dx \\ &= \lim_{\delta\to 0} \frac{m}{2}\int_{\Omega\setminus C_\delta} \langle \sum_{k=1}^3 \theta_{x_k} dx_k, \star d\phi\rangle\, dx, \end{aligned}$$

for any $\phi \in \mathcal{D}^1(\Omega)$, where we have let C_δ denote the infinite vertical cylinder $C_\delta = \{(x_1,x_2,x_3) : x_1^2 + x_2^2 < \delta^2\}$. Since $\nabla \times (\nabla\theta) = 0$, an integration by parts reveals that

$$\begin{aligned} J(u_m)(\phi) &= \lim_{\delta\to 0} \tfrac{m}{2} \textstyle\int_{\partial C_\delta \cap \Omega} \langle (\nabla\theta \times \nu), \phi\rangle\, dH^2(x) \\ &= \lim_{\delta\to 0} \tfrac{m}{2} \textstyle\int_{\partial C_\delta \cap \Omega} \langle \tfrac{1}{\delta} e_3, \phi\rangle\, dH^2(x) \\ &= m\pi \textstyle\int_0^L \langle e_3, \phi(0,0,z)\rangle\, dz, \end{aligned}$$

where ν above denotes the (inner) unit normal to ∂C_δ and e_3 denotes the unit vector pointing in the positive x_3 direction. Comparing this result with (2.5), we see that $\frac{1}{\pi} J(u_m) \in \mathcal{R}_1(\Omega)$ corresponds to the multiplicity m, 1-rectifiable current supported on the line segment joining $(0,0,0)$ to $(0,0,L)$ with orienting tangent vector e_3. We note, in particular, that

$$M(J(u_m)) = \pi\,|m|\,L. \tag{2.12}$$

In fact, replacing Ω above by a sequence of open cylinders contracting to Γ, one readily checks that the calculation above also implies

$$\mu_{J(u_m)}(\Gamma) = \pi\,|m|\,L, \tag{2.13}$$

where $\mu_{J(u_m)}$ is the total variation measure associated with the 1-current $J(u_m)$ (cf. (2.3)).

2.3 The Compactness and Lower-Semi-Continuity of Jerrard and Soner

Our existence result in the next section is an application of the work of Jerrard and Soner in [12] on the compactness properties and Γ-limit of the Ginzburg-Landau energy

$$E_\varepsilon \equiv \frac{1}{|\ln\varepsilon|}\tilde{E}_\varepsilon = \frac{1}{|\ln\varepsilon|}\int_\Omega \left\{ \frac{|\nabla u|^2}{2} + \frac{(1-|u|^2)^2}{4\varepsilon^2} \right\} dx.$$

(Similar results have also more recently appeared in [2].) Below we state their result on compactness and lower-semi-continuity of the weak Jacobians associated with sequences of competitors having bounded energy. We have stated it in terms of 1-currents in $\mathbb{R}^3$ since this will be most convenient for our purposes, but a version of the theorem holds more generally in any dimension.

THEOREM 2.1 *([12]) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary. Suppose that $\{u_\varepsilon\}_{\varepsilon\in(0,1]} \subset W^{1,2}(\Omega;\mathbb{C})$ satisfies the uniform bound $E_\varepsilon(u_\varepsilon) \le C$ for some $C > 0$. Then there is a sequence $\varepsilon_k \to 0$ and a rectifiable 1-current $\bar{J}$ such that $\partial\bar{J} = 0$ relative to Ω, $\frac{1}{\pi}\bar{J}$ is integer multiplicity and*

$$\lim_{k\to\infty} \left\|J(u_{\varepsilon_k}) - \bar{J}\right\|_{0,1}^* = 0, \tag{2.14}$$

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \ge M(\bar{J}). \tag{2.15}$$

Finally, denote by μ_ε the sequence of Radon measures characterized by the property

$$\mu_\varepsilon(W) = \frac{1}{|\ln\varepsilon|}\int_W \left\{ \frac{|\nabla u_\varepsilon|^2}{2} + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2} \right\} dx,$$

for any $W \subset \Omega$. Then for any Radon measure μ arising as a weak limit of a subsequence of $\{\mu_{\varepsilon_k}\}$, one has

$$\mu_{\bar{J}} << \mu \quad \text{and} \quad \frac{d\mu_{\bar{J}}}{d\mu} \leq 1 \ \mu \ a.e. \tag{2.16}$$

Regarding (2.16), recall that in our notation, $\mu_{\bar{J}}$ refers to the total variation measure associated with the 1-current $\bar{J}$ (cf. (2.3)).

We should note here that the inequality (2.15) only constitutes half of the statement of Γ-convergence for the sequence $\{J(u_\varepsilon)\}$. The other part would be a statement asserting that for any $\bar{J} \in \mathcal{R}_1(\Omega)$, there exists a sequence of $W^{1,2}(\Omega;\mathbb{C})$ functions for which the associated Jacobians and the energies converge to $\bar{J}$ and $M(\bar{J})$ respectively. Such a sequence can indeed be constructed, as has recently been announced in [2] (see also [1]), but we do not explore it here since it will not be needed in what follows. In fact, we shall only need a special case of this construction, namely the situation where $\bar{J}$ is a finite union of line segments with arbitrary integer multiplicities. This is treated in the next section.

3 Construction of an Approximate Sequence

In this section we construct a sequence of functions having vortices of prescribed degree concentrating along a given set of line segments. The asymptotic value of the energy of this sequence will be proportional to the total length of the line segments, thus providing for us what will be a crucial example of equality in the inequality (2.15) of Theorem 2.1.

Constructions in this vein can be found in numerous references on Ginzburg-Landau theory such as ([3]),[12],[14],[28]) to list a few. However, as these are either in 2-dimensions or do not deal with arbitrary multiplicities, they do not quite fit our context. We should hasten to add, though, that our construction represents a simpler special case of a very general recent result in [2]. We present the construction below both for the sake of giving a self-contained treatment and because the special case we require here involving 1-currents supported only on unions of line segments allows us to make the construction quite explicit as compared with the general case treated in [2].

For the reader interested in seeing how the approach works for one line vortex of degree one, we mention that one could skip to Section 4 after observing that the formula (3.10) serves as a suitable construction.

Our construction will be valid for bounded, open, Lipschitz domains Ω meeting the following criteria. For some positive integer N we assume there exist lines $l_1, l_2, \ldots, l_N$ and a positive number R such that the collection of infinite solid cylinders $\{\mathcal{C}_{R,j}\}_{j=1}^N$ with axis l_j and radius R satisfy the following conditions:

$$\mathcal{C}_{R,j} \cap \Omega \text{ has only one component,} \tag{3.1}$$

$$\mathcal{C}_{R,j} \cap \mathcal{C}_{R,k} \cap \Omega = \emptyset \text{ for all } j \neq k, \tag{3.2}$$

and in a coordinate system where the x_3-axis coincides with l_j one has

$$\begin{aligned} &\mathcal{C}_{R,j} \cap \Omega = \\ &\{(x_1, x_2, x_3) : x_1^2 + x_2^2 < R^2,\, z_1^j(x_1, x_2) < x_3 < L_j + z_2^j(x_1, x_2)\}, \end{aligned} \tag{3.3}$$

for Lipschitz functions z_1^j and z_2^j, where we have introduced the notation $L_j = H^{(1)}(l_j \cap \Omega)$. Condition (3.3) should be viewed as saying that l_j meets $\partial\Omega$ transversely.

Now for each j, we denote by (r_j, θ_j) polar coordinates measured in any plane orthogonal to the line l_j with $r_j = 0$ corresponding to the intersection of the plane with l_j. Then it follows that for any integer m, the function $e^{im\theta_j}$ lies in $W^{1,1}_{\text{loc}}(\mathbb{R}^3; S^1)$. Also, based on (3.1), (3.3) and the same calculation that we used to derive (2.13), we conclude that

$$M(J(e^{im\theta_j})) = \pi\, |m|\, L_j. \tag{3.4}$$

For our construction we will also need to recall the properties of the monotone solution $\rho : [0, \infty) \to [0, 1)$ to the problem

$$\begin{gathered} \rho'' + \frac{1}{r}\rho' - \frac{1}{r^2}\rho = (\rho^2 - 1)\rho \ \text{ for } \ 0 < r < \infty, \\ \rho(0) = 0, \quad \rho(\infty) = 1. \end{gathered}$$

(cf. e.g. [3]). This function ρ represents the modulus of the canonical degree-one vortex solution to (1.2). It is well-known that ρ and ρ' satisfy the asymptotic conditions

$$\rho(r) = 1 - \frac{1}{r^2} + \mathcal{O}(\frac{1}{r^3}) \quad \text{and} \tag{3.5}$$

$$\rho'(r) = \frac{2}{r^3} + \mathcal{O}(\frac{1}{r^4}) \quad \text{for } r >> 1. \tag{3.6}$$

We are now ready to state and prove:

PROPOSITION 3.1 *Assume $\Omega \subset \mathbb{R}^3$ is a bounded, open Lipschitz domain satisfying conditions (3.1)–(3.3). Let $\alpha = (m_1, m_2, \ldots, m_N)$ be any element of $\mathbb{Z}^N$ and define $u_\alpha \in W^{1,1}(\Omega; S^1)$ by*

$$u_\alpha(x) = \prod_{j=1}^N e^{im_j\theta_j}. \tag{3.7}$$

Then there exists a sequence $\{v_\varepsilon\} \in W^{1,2}(\Omega;\mathbb{C})$ *such that*

$$\lim_{\varepsilon\to 0} E_\varepsilon(v_\varepsilon) = M(J(u_\alpha)) = \pi \sum_{j=1}^N |m_j|\, L_j. \tag{3.8}$$

$$\text{and } \|J(v_\varepsilon) - J(u_\alpha)\|_{0,1}^* \to 0 \text{ as } \varepsilon \to 0. \tag{3.9}$$

Remark 3.2 This proposition provides an affirmative answer for a special case of a much more general question, namely, whether given an arbitrary cycle $T \in \mathcal{R}_1(\Omega)$, one can find a sequence $\{v_\varepsilon\} \subset W^{1,2}(\Omega;\mathbb{C})$ such that the sequence of energies $E_\varepsilon(v_\varepsilon)$ approaches the mass of T and such that the associated weak Jacobians approach T in $(C^{0,1})^*$. In this generality, the question is much more subtle and is addressed in [2]. There it is shown that such an approximating sequence does exist provided that T bounds some 2-current in Ω. (See also [1] for related questions.) Our condition (3.1) is sufficient to guarantee this bounding property.

To see why some assumption along these lines is necessary, consider the following example: take $\Omega = B_1 \setminus \bar{B}_{1/2}$ where B_r denotes the open ball of radius r in $\mathbb{R}^3$. Then take T to be the multiplicity-1 oriented line segment joining the point $(0,0,1/2)$ to $(0,0,1)$. If one considers any circle in Ω looping around the segment, such a circle can be smoothly deformed to a point without passing through the segment, thus precluding the existence of a smooth map with zero set converging to the segment and with non-zero winding number when restricted to this circle.

Remark 3.3 Our constructed sequence $\{v_\varepsilon\}$ will also satisfy $v_\varepsilon \to u_\alpha$ in $W^{1,1}(\Omega;\mathbb{C})$ though we will not need this fact.

Proof. The fact that $M(J(u_\alpha)) = \pi \sum_{j=1}^N |m_j|\, L_j$ as is claimed in (3.8) follows from (3.4).

The plan of the proof is to first establish (3.8)–(3.9) for the case $N = 1$ (one cylinder) and then to discuss arbitrary $N > 1$. For $N = 1$, we will drop the subscript j and write simply l for l_1, m for m_1, etc.

To verify the first equality in (3.8), we begin with the simplest possible case: besides taking $N = 1$, we take $\alpha = m = 1$ (degree-one vortex). Without loss of generality, we may assume the line segment $l \cap \Omega$ has endpoints $(0,0,0)$ and $(0,0,L)$. Introducing cylindrical coordinates (r,θ,z) and corresponding unit vectors $(\hat{e}_r, \hat{e}_\theta, \hat{k})$, we now define the approximating sequence by choosing the canonical degree-one vortex solution to the Ginzburg-Landau equation (1.2):

$$v_\varepsilon(r,\theta,z) = \rho\left(\frac{r}{\varepsilon}\right) e^{i\theta}. \tag{3.10}$$

It is easy to check that $v_\varepsilon \in W^{1,2}(\Omega;\mathbb{C})$. Our immediate goal is to establish (3.8) which in this case takes the form

$$\lim_{\varepsilon\to 0}\frac{1}{|\ln(\varepsilon)|}\left(\int_\Omega \frac{1}{2}|\nabla v_\varepsilon|^2+\frac{1}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2\,dx\right)=\pi L. \tag{3.11}$$

To this end we note first that from property (3.5) one finds

$$\frac{1}{4\varepsilon^2}\int_\Omega (1-|v_\varepsilon|^2)^2\,dx\le C, \tag{3.12}$$

where C is independent of ε. To estimate the other integral we write

$$\nabla v_\varepsilon=(\frac{1}{\varepsilon}\rho'(\frac{r}{\varepsilon})\hat{e}_r+i\frac{1}{r}\rho(\frac{r}{\varepsilon})\hat{e}_\theta)e^{i\theta}.$$

This gives us

$$\int_\Omega |\nabla v_\varepsilon|^2\,dx=\frac{1}{\varepsilon^2}\int_\Omega |\rho'(\frac{r}{\varepsilon})|^2\,dx+\int_\Omega \rho^2(\frac{r}{\varepsilon})\frac{1}{r^2}\,dx=I_1+I_2.$$

Appealing this time to (3.6) we find that $I_1=\mathcal{O}(1)$. To analyze I_2 we introduce the notation

$$\Omega_\eta=\{x\in\Omega \text{ with } r<\eta\}$$

for any $\eta>0$ and then split I_2 into three integrals as

$$I_2=\int_{\Omega_{\varepsilon M}}\rho^2\left(\frac{r}{\varepsilon}\right)\frac{1}{r^2}\,dx+\int_{\Omega_R\setminus\Omega_{\varepsilon M}}\rho^2\left(\frac{r}{\varepsilon}\right)\frac{1}{r^2}\,dx+\int_{\Omega\setminus\Omega_R}\rho^2\left(\frac{r}{\varepsilon}\right)\frac{1}{r^2}\,dx.$$

Here $M>0$ is chosen so that (3.5) and (3.6) are valid for $r>M$. One readily verifies that the first and third term in this sum are $\mathcal{O}(1)$. The second term can be computed rather explicitly using (3.3) and (3.5):

$$\begin{aligned}\int_{\Omega_R\setminus\Omega_{\varepsilon M}}\rho^2\left(\frac{r}{\varepsilon}\right)\frac{1}{r^2}\,dx&=\int_{\Omega_R\setminus\Omega_{\varepsilon M}}\frac{1}{r^2}(1-\frac{\varepsilon^2}{r^2}+\mathcal{O}(\frac{\varepsilon^3}{r^3}))^2\,dx\\&=\int_{\Omega_R\setminus\Omega_{\varepsilon M}}\frac{1}{r^2}\,dx+o(|\ln(\varepsilon)|)\\&=\int_{\varepsilon M}^{R}\int_0^{2\pi}\int_{z_1(r,\theta)}^{L+z_2(r,\theta)}\frac{1}{r^2}r\,dz\,d\theta\,dr+o(|\ln(\varepsilon)|).\end{aligned}$$

Since z_1 and z_2 are Lipschitz continuous and $z_1(0,0)=z_2(0,0)=0$, we find

$$\begin{aligned}\int_{\Omega_R\setminus\Omega_{\varepsilon M}}\rho^2\left(\frac{r}{\varepsilon}\right)\frac{1}{r^2}\,dx&=2\pi L\ln(\frac{R}{\varepsilon M})\\&\quad+\int_{\varepsilon M}^{R}\int_0^{2\pi}\frac{z_2(r,\theta)-z_1(r,\theta)}{r}\,d\theta\,dr+o(|\ln(\varepsilon)|)\\&=2\pi L\ln(\frac{1}{\varepsilon})+o(|\ln(\varepsilon)|),\end{aligned}$$

which yields (3.11). To establish (3.8) when $N=1$ and $\alpha=m>1$, we set $\theta_m=\frac{2\pi}{m}$ and $a_{k,\varepsilon}=\frac{1}{|\ln(\varepsilon)|}e^{ik\theta_m}$ for $k=1,...,m$. For ease of notation we will suppress the ε-dependence and write simply (r_k,θ_k,z) for cylindrical coordinates with origin at $(a_{k,\varepsilon},0)$ where again the z-direction corresponds to the axis l. We caution the reader that, in particular, throughout the argument pertaining to the case $N=1$, the subscript k in θ_k does not refer to the different lines l_1,l_2, etc. (since for $N=1$ there is only one such line).

Now define the approximating sequence by

$$v_\varepsilon(x)=\prod_{k=1}^m v_{k,\varepsilon}(x) \quad \text{with} \tag{3.13}$$

$$v_{k,\varepsilon}(x)=\rho\left(\frac{r_k}{\varepsilon}\right)e^{i\theta_k}. \tag{3.14}$$

One again readily verifies that (3.12) holds so we now compute

$$\nabla v_\varepsilon(x)=\sum_{k=1}^m \rho_1\cdot...\cdot\hat{\rho}_k\cdot\dots\cdot\rho_m\cdot\dots\cdot(\nabla\rho_k(x)+i\rho_k\nabla\theta_k)e^{i\Psi}. \tag{3.15}$$

where we have denoted $\rho_k(x)=\rho(\frac{r_k}{\varepsilon})$, $\Psi=\sum_{k=1}^m\theta_k$ and the notation $\hat{\rho}_k$ means that this term is not present in the product. Then we calculate

$$\begin{aligned}|\nabla v_\varepsilon(x)|^2=\sum_{k,l=1}^m &\rho_1\cdot\dots\cdot\hat{\rho}_k\cdot\dots\cdot\rho_m\cdot\rho_1\cdot\dots\cdot\hat{\rho}_l\cdot\dots\cdot\rho_m\cdot\\ &(\nabla\rho_k(x)+i\rho_k\nabla\theta_k)\cdot(\nabla\rho_l(x)-i\rho_l\nabla\theta_l).\end{aligned} \tag{3.16}$$

From (3.5) and (3.6) it is easy to check that the integrals of the first three terms in the inner product above will be $\mathcal{O}(1)$ and so will not contribute to the asymptotic value of the energy.

To estimate the last term we proceed as follows. Choose $\eta_1>0$ to be a constant independent of ε such that $\operatorname{dist}(a_{k,\varepsilon},a_{l,\varepsilon})>\frac{2\eta_1}{|\ln\varepsilon|}$. (Recall that $a_{k,\varepsilon}=\frac{1}{|\ln\varepsilon|}e^{\frac{2\pi ki}{m}}$.) Then let $\eta_\varepsilon\equiv\frac{\eta_1}{|\ln\varepsilon|}$. For any $\eta>0$ we denote by Ω_η^k the set

$$\Omega_\eta^k=\{x\in\Omega: r_k\le\eta\}.$$

Now we shall estimate the last term in the inner product of (3.16). For $k\neq l$ we find

$$\begin{aligned}\left|\int_\Omega(\rho_1\cdot\dots\cdot\rho_m)^2\nabla\theta_k\cdot\nabla\theta_l\,dx\right|&\le\int_\Omega\frac{1}{r_k}\frac{1}{r_l}\,dx\\ \le\int_{\Omega\setminus(\Omega_{\eta_\varepsilon}^k\cup\Omega_{\eta_\varepsilon}^l)}\frac{1}{r_k}\frac{1}{r_l}\,dx&+\int_{\Omega_{\eta_\varepsilon}^k}\frac{1}{r_k}\frac{1}{r_l}\,dx+\int_{\Omega_{\eta_\varepsilon}^l}\frac{1}{r_k}\frac{1}{r_l}\,dx.\end{aligned} \tag{3.17}$$

After noting that $\Omega_{\eta_\varepsilon}^k\cap\Omega_{\eta_\varepsilon}^l=\emptyset$ for our choice of η_1, we observe that necessarily $r_k\ge\eta_\varepsilon$ for $x\in\Omega_{\eta_\varepsilon}^l$. Consequently,

$$\int_{\Omega_{\eta_\varepsilon}^k}\frac{1}{r_k}\frac{1}{r_l}\,dx+\int_{\Omega_{\eta_\varepsilon}^l}\frac{1}{r_k}\frac{1}{r_l}\,dx\le\frac{|\ln(\varepsilon)|}{\eta_1}\int_{\Omega_{\eta_\varepsilon}^k}\frac{1}{r_k}\,dx+\frac{|\ln(\varepsilon)|}{\eta_1}\int_{\Omega_{\eta_\varepsilon}^l}\frac{1}{r_l}\,dx=\mathcal{O}(1).$$

For the remaining term in (3.17) we can estimate

$$\int_{\Omega\setminus(\Omega^k_{\eta_\varepsilon}\cup\Omega^l_{\eta_\varepsilon})}\frac{1}{r_k}\frac{1}{r_l}\,dx \le \frac{1}{2}\int_{\Omega\setminus(\Omega^k_{\eta_\varepsilon}\cup\Omega^l_{\eta_\varepsilon})}\frac{1}{r_k^2}+\frac{1}{r_l^2}\,dx \le C\int_{\frac{\eta_1}{|\ln(\varepsilon)|}}^{\mathrm{diam}\,(\Omega)}\frac{1}{r}\,dr$$

where diam(Ω) denotes the diameter of Ω. Hence,

$$\frac{1}{|\ln(\varepsilon)|}\left|\int_\Omega(\rho_1\cdot\dots\cdot\rho_m)^2\nabla\theta_k\cdot\nabla\theta_l\,dx\right|\to 0$$

as $\varepsilon\to 0$, when $k\ne l$.

When $k=l$ we split the integral from the last term in (3.16) as follows:

$$\begin{aligned}\int_\Omega(\rho_1\cdot\dots\cdot\rho_m)^2|\nabla\theta_k|^2\,dx &= \int_\Omega(\rho_k)^2|\nabla\theta_k|^2\,dx\\ &-\int_\Omega(1-(\rho_1\cdot\dots\cdot\hat\rho_k\cdot\dots\cdot\rho_m)^2)(\rho_k)^2|\nabla\theta_k|^2\,dx.\end{aligned}\tag{3.18}$$

The first integral on the right can be handled in a manner similar to the $m=1$ case to yield an asymptotic value of $\pi\,|\ln\varepsilon|$, while an appeal to (3.5) reveals that the second term is $o(|\ln\varepsilon|)$. After summing over k, we conclude that

$$\lim_{\epsilon\to 0}\frac{1}{|\ln(\epsilon)|}\int_\Omega|\nabla v_\epsilon(x)|^2 = m\pi L.\tag{3.19}$$

This establishes (3.8) for the case of one cylinder and any multiplicity. (One treats the case where $m<0$ just as one does $m>0$.)

We now turn our focus to proving (3.9) in the case $N=1$. Here we will immediately handle the case of arbitrary $m\in\mathbb{Z}$ rather than starting with $m=1$. In view of (2.11), it will be sufficient to establish

$$j(v_\varepsilon)\to j(u_m)\text{ in }L^1(\Omega),\tag{3.20}$$

where we are letting u_m denote the function $e^{im\theta}$.

At this point it will be convenient to introduce a new function $v_\varepsilon^*\equiv e^{i\Psi}$ where again $\Psi=\sum_{k=1}^m\theta_k$. Then using definition (2.8) along with (3.15) we find

$$\begin{aligned}j(v_\varepsilon) &= (\rho_1\cdot\ldots\cdot\rho_m)^2\sum_{k=1}^m\nabla\theta_k,\\ j(v_\varepsilon^*) &= \sum_{k=1}^m\nabla\theta_k,\\ \text{and }j(u_m) &= m\nabla\theta.\end{aligned}$$

Then we can estimate

$$\int_\Omega|j(v_\varepsilon)-j(u_m)|\,dx \le \int_\Omega\Big\{|j(v_\varepsilon)-j(v_\varepsilon^*)|+|j(v_\varepsilon^*)-j(u_m)|\Big\}\,dx.\tag{3.21}$$

We note that

$$|j(v_\varepsilon) - j(v_\varepsilon^*)| \le \sum_{k=1}^m (1 - (\rho_1 \cdot \ldots \cdot \rho_m)^2)\, |\nabla \theta_k|$$

and from Hölder's inequality we conclude that

$$\int_\Omega |j(v_\varepsilon) - j(v_\varepsilon^*)|\ dx \le \left(\int_\Omega (1 - (\rho_1 \cdot \ldots \cdot \rho_m)^2)^p\, dx\right)^{\frac{1}{p}} \sum_{k=1}^m \left(\int_\Omega |\nabla \theta_k|^q\ dx\right)^{\frac{1}{q}}$$

for $1 < q < 2$. Since $|\nabla \theta_k| = \frac{1}{r_k}$, the sum of integrals on the right is uniformly bounded and so (3.5) and the dominated convergence theorem give

$$\int_\Omega |j(v_\varepsilon) - j(v_\varepsilon^*)|\ dx \to 0 \tag{3.22}$$

as $\varepsilon \to 0$. To handle the term $\int_\Omega |j(v_\varepsilon^*) - j(u_m)|\ dx$ in (3.21) we have

$$\begin{aligned} \int_\Omega |j(v_\varepsilon^*) - j(u_m)|\ dx &\le \sum_{k=1}^m \int_\Omega |\nabla \theta_k - \nabla \theta|\ dx \\ &= \sum_{k=1}^m \int_\Omega \left|\frac{\hat{e}_{\theta_k}}{r_k} - \frac{\hat{e}_\theta}{r}\right|\ dx \\ &\le \sum_{k=1}^m \int_\Omega \left(\left|\frac{\hat{e}_{\theta_k}}{r_k} - \frac{\hat{e}_{\theta_k}}{r}\right| + \left|\frac{\hat{e}_{\theta_k}}{r} - \frac{\hat{e}_\theta}{r}\right|\right)\ dx \\ &= \sum_{k=1}^m \left(\int_\Omega \left|\frac{1}{r_k} - \frac{1}{r}\right|\ dx + \int_\Omega \frac{1}{r}\,|\hat{e}_{\theta_k} - \hat{e}_\theta|\ dx\right). \end{aligned} \tag{3.23}$$

Of these two integrands, the second is dominated by an integrable function, namely $\frac{2}{r}$, and goes to zero pointwise so the second integral approaches zero by applying the dominated convergence theorem. To handle the first integral, fix any k and let $\tilde{C} \supset \Omega$ be any sufficiently large finite cylinder possessing an axis parallel to the x_3-axis that passes through the midpoint between 0 and $a_{k,\varepsilon}$. Then denote $\tilde{C}^+$ (resp. $\tilde{C}^-$) $= \{x \in \tilde{C} : r > r_k$ (resp. $r < r_k$)$\}$ and we compute

$$\int_\Omega \left|\frac{1}{r_k} - \frac{1}{r}\right|\ dx \le \int_{\tilde{C}} \left|\frac{1}{r_k} - \frac{1}{r}\right|\ dx = \int_{\tilde{C}^+} \left(\frac{1}{r_k} - \frac{1}{r}\right)\ dx + \int_{\tilde{C}^-} \left(\frac{1}{r} - \frac{1}{r_k}\right)\ dx. \tag{3.24}$$

Since

$$\int_{\tilde{C}^+} \left(\frac{1}{r_k} - \frac{1}{r}\right)\ dx = \int_{\tilde{C}^-} \left(\frac{1}{r} - \frac{1}{r_k}\right)\ dx = \int_{\tilde{C}^+} \frac{1}{r_k}\, dx - \int_{\tilde{C}^+} \frac{1}{r}\, dx, \tag{3.25}$$

and both of the two integrals in this last difference converge to the same number, we see that

$$\int_\Omega \left|\frac{1}{r_k} - \frac{1}{r}\right|\ dx \to 0, \tag{3.26}$$

so that $\int_\Omega |j(v_\varepsilon^*) - j(u_m)|\, dx \to 0$. Applying this to (3.23) and recalling (3.22), we arrive through (3.21) at the desired conclusion, (3.20). Then (3.9) follows. This completes the verification of (3.8)-(3.9) for the case $N = 1$.

For the case $N > 1$ where we assume the existence of N cylinders in Ω satisfying the conditions (3.1)–(3.3), we can take as a construction the product of N constructions as described above. That is, if we define v_ε^j as the construction given by (3.13) corresponding to the j^{th} segment l_j, then we can choose our sequence $\{v_\varepsilon\}$ to be

$$v_\varepsilon = \prod_{j=1}^{N} v_\varepsilon^j. \tag{3.27}$$

In light of the assumptions (3.1) and (3.2) on the N cylinders and the properties (3.5) and (3.6) of the function ρ, it is easy to check that definition (3.27) makes sense as a $W^{1,2}(\Omega;\mathbb{C})$ function and that any interaction terms make a lower order contribution to the total energy. Assertions (3.8)–(3.9) follow.

4 Existence of a Local Minimizer

In this section, we will prove the existence of a family of local minimizers to the energy E_ε, provided the line segments $l_j \cap \Omega$ introduced at the outset of the previous section represent local minimizers of length. To state this assumption more precisely, for each $j \in \{1, 2, \ldots, N\}$ let C_{R,L_j} denote the open solid cylinder of radius R and height L_j with axis consisting of $l_j \cap \Omega$. Let $a_j, b_j \in \partial\Omega$ denote the endpoints of l_j. Then we assume that for each j we have:

$$C_{R,L_j} \subset \Omega \text{ and } \bar{C}_{R,L_j} \cap \partial\Omega = \{a_j, b_j\} \tag{4.1}$$

REMARK 4.1 Condition (4.1) will in particular be satisfied by any domain for which there exist two points p_1 and p_2 on $\partial\Omega$ such that the line segment $\overline{p_1p_2}$ satisfies

$$\overline{p_1p_2} \setminus \{p_1, p_2\} \subset \Omega,\ \overline{p_1p_2} \perp \partial\Omega \text{ at } p_1 \text{ and } p_2$$

and such that $\partial\Omega$ is locally strictly convex in a neighborhood of p_1 and p_2.

Our main result is the following:

THEOREM 4.2 *Assume an open, bounded, Lipschitz domain $\Omega \subset \mathbb{R}^3$ satisfies (3.1)–(3.3) and (4.1) for some positive integer N. Let $\alpha =$*

$(m_1, m_2, \ldots, m_N)$ be any element of $\mathbb{Z}^N$. Then there exists an $\varepsilon_0 > 0$ and a sequence $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0} \subset W^{1,2}(\Omega;\mathbb{C})$ such that u_ε is a local minimizer in $W^{1,2}(\Omega;\mathbb{C})$ of E_ε and such that

$$\|J(u_\varepsilon) - J(u_\alpha)\|^*_{0,1} \to 0 \text{ as } \varepsilon \to 0, \tag{4.2}$$

where u_α is given by (3.7).

REMARK 4.3 A careful look at the proof to follow will reveal that one could in fact relax the second assumption in (4.1) a bit. As long as the top of the cylinder C_{R,L_j} only meets $\partial\Omega$ at b_j, the bottom of the cylinder can have any intersection with $\partial\Omega$ that includes a_j. Of course, the reverse of this situation is also allowable.

REMARK 4.4 Assumption (3.1) can be replaced in the theorem by the assumption that the integer multiplicity 1-current supported on the union of line segments bounds a 2-current. See Remark 3.2.

A crucial step in our approach is the contention that the union of oriented line segments joining the points a_j and b_j with multiplicities m_j, viewed as a 1-current, is a local minimizer of mass in the $(C^{0,1})^*$-topology among appropriate competitors in $\mathcal{R}_1(\Omega)$. To state this precisely, for any $\alpha = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}^N$ we denote by T_j the above-mentioned 1-current supported on $l_j \cap \Omega$ and let $T_\alpha = \sum_{j=1}^N T_j$.

THEOREM 4.5 *Assume a bounded, open domain Ω satisfies (4.1) for all $j \in \{1, 2, \ldots, N\}$ where N is any positive integer. For any $\alpha \in \mathbb{Z}^N$, let $T_\alpha \in \mathcal{R}_1(\Omega)$ be defined as above. Then there exists a positive number $\delta = \delta(\alpha, L_1, \ldots, L_N, R)$ such that for all $T \in \mathcal{R}_1(\Omega)$ with $\partial T = 0$ relative to Ω one has*

$$0 < \|T - T_\alpha\|^*_{0,1} \le \delta \implies M(T) > M(T_\alpha). \tag{4.3}$$

We will first present a proof of our main result, Theorem 4.2, under the assumption that Theorem 4.5 is valid. The proof of Theorem 4.5 is then presented afterwards.

PROOF OF THEOREM 4.2: We divide this proof into three steps. We begin by introducing the set

$$F_{\delta'} = \{u \in W^{1,2}(\Omega, \mathbb{C}) : \|J(u) - J(u_\alpha)\|^*_{0,1} \le \delta'\}$$

where $\delta' = \delta/\pi$ and $\delta > 0$ is the number provided by Theorem 4.5. Note that in light of (3.9), there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the functions v_ε provided by Lemma 3.1 lie in $F_{\delta'}$, so in particular, $F_{\delta'}$ is nonempty.

Step 1. We first claim that $F_{\delta'}$ is weakly closed in $W^{1,2}(\Omega;\mathbb{C})$. To show this, suppose $\{v_k\} \subset F_{\delta'}$ is a sequence such that $v_k \rightharpoonup v \in W^{1,2}(\Omega,\mathbb{C})$ for some $v \in W^{1,2}(\Omega;\mathbb{C})$. Since $v_k \to v$ strongly in L^2 and $\nabla v_k \rightharpoonup \nabla v$ in L^2, it is clear in view of (2.8) that $j(v_k) \rightharpoonup j(v)$ in L^1. By Mazur's lemma, for each k we can find a finite sequence of nonnegative numbers $\{\alpha_l^k\}$, $l = 1, \ldots, N(k)$ such that $\sum_{l=1}^{N(k)} \alpha_l^k = 1$ and

$$\sum_{l=1}^{N(k)} \alpha_l^k j(v_l) \to j(v) \text{ strongly in } L^1 \text{ as } k \to \infty. \tag{4.4}$$

Using the estimate (2.11), we find that

$$\begin{aligned}
&\|J(v) - J(u_\alpha)\|_{0,1}^* \\
&\leq \left\| J(v) - \textstyle\sum_{l=1}^{N(k)} \alpha_l^k J(v_l) \right\|_{0,1}^* + \left\| \sum_{l=1}^{N(k)} \alpha_l^k J(v_l) - J(u_\alpha) \right\|_{0,1}^* \\
&\leq 6 \left\| j(v) - \textstyle\sum_{l=1}^{N(k)} \alpha_l^k j(v_l) \right\|_{L^1(\Omega)} + \sum_{l=1}^{N(k)} \alpha_l^k \|J(v_l) - J(u_\alpha)\|_{0,1}^* \\
&\leq 6 \left\| j(v) - \textstyle\sum_{l=1}^{N(k)} \alpha_l^k j(v_l) \right\|_{L^1(\Omega)} + \delta',
\end{aligned}$$

and then letting $k \to \infty$ we see through (4.4) that $v \in F_\delta'$.

Step 2. Consider the set

$$O_\delta' = \{u \in W^{1,2}(\Omega;\mathbb{C}) : \|J(u) - J(u_\alpha)\|_{0,1}^* < \delta'\}.$$

We claim that $O_{\delta'}$ is open in $W^{1,2}(\Omega;\mathbb{C})$. To see this, fix any $u \in O_{\delta'}$ and consider any $v \in W^{1,2}(\Omega;\mathbb{C})$. We estimate $\|J(v) - J(u_\alpha)\|_{0,1}^*$ as follows:

$$\begin{aligned}
\|J(v) - J(u_\alpha)\|_{0,1}^* &\leq \|J(u) - J(u_\alpha)\|_{0,1}^* + \|J(v) - J(u)\|_{0,1}^* \\
&\leq \|J(u) - J(u_\alpha)\|_{0,1}^* + 6 \int_\Omega |j(u) - j(v)| \, dx \\
&\leq \|J(u) - J(u_\alpha)\|_{0,1}^* + 6 \int_\Omega |\bar{u}\nabla u - \bar{v}\nabla v| \\
&\leq \|J(u) - J(u_\alpha)\|_{0,1}^* + 6 \int_\Omega \{|\nabla u| \, |u - v| + |v| \, |\nabla u - \nabla v|\} \, dx \\
&\leq \|J(u) - J(u_\alpha)\|_{0,1}^* + 6(\|u\|_{W^{1,2}} + \|v\|_{W^{1,2}}) \, \|u - v\|_{W^{1,2}} .
\end{aligned}$$

If we now choose $\mu > 0$ so that

$$6 \ (2 \|u\|_{W^{1,2}} + \mu) \ \mu < \delta' - \|J(u) - J(u_\alpha)\|_{0,1}^*$$

we see that $\|v - u\|_{W^{1,2}} < \mu$ implies $v \in O_\delta$, so $O_{\delta'}$ is open.

Step 3. We apply the direct method to the variational problem

$$\inf_{v \in F_\delta} E_\varepsilon(v). \tag{4.5}$$

For ε sufficiently small and fixed, a minimizing sequence will be uniformly bounded in $W^{1,2}(\Omega;\mathbb{C})$ and by Step 1, a weakly convergent subsequence will converge to a limit, say u_ε, lying in $F_{\delta'}$. By the weak lower-semi-continuity of the $W^{1,2}$-norm, along with Fatou's lemma, it readily follows that u_ε solves (4.5). We claim that in fact

$$\|J(u_\varepsilon) - J(u_\alpha)\|_{0,1}^* < \delta' \tag{4.6}$$

for all ε sufficiently small. In light of Step 2, this would imply that u_ε is a local minimizer in $W^{1,2}$ of E_ε. To establish (4.6) we proceed by contradiction and suppose that for some sequence $\{\varepsilon_k\} \to 0$ we have

$$\|J(u_{\varepsilon_k}) - J(u_\alpha)\|_{0,1}^* = \delta. \tag{4.7}$$

Then, since $E_{\varepsilon_k}(u_{\varepsilon_k}) \leq E_{\varepsilon_k}(v_{\varepsilon_k})$, we find that $E_{\varepsilon_k}(u_{\varepsilon_k}) \leq C$. Applying Theorem 2.1, we conclude that for some subsequence (still denoted here by $\{\varepsilon_k\}$) we have $J(u_{\varepsilon_k}) \to \bar{J}$ in $(C^{0,1})^*$ for some $\bar{J}$ such that $\frac{1}{\pi}\bar{J} \in \mathcal{R}_1(\Omega)$ and $\partial\bar{J} = 0$ relative to Ω. By (4.7), it follows that

$$\|\bar{J} - J(u_\alpha)\|_{0,1}^* = \delta.$$

Then through an appeal to (2.15), (3.9) and (3.8) we find

$$M(\bar{J}) \leq \liminf E_{\varepsilon_k}(u_{\varepsilon_k}) \leq \liminf E_{\varepsilon_k}(v_{\varepsilon_k}) = M(J(u_\alpha)),$$

contradicting the statement of Theorem 4.5. (Recall that $M(J(u_\alpha)) = \pi M(T_\alpha)$.) Hence, $\{u_\varepsilon\}$ is a sequence of $W^{1,2}$ local minimizers of E_ε. Reviewing the argument, we see that the same contradiction would be reached under the hypothesis (4.7) with δ' replaced by any positive $\delta_1 < \delta'$. Consequently,

$$\|J(u_\varepsilon) - J(u_\alpha)\|_{0,1}^* \to 0 \text{ as } \varepsilon \to 0.$$

∎

Proof of Theorem 4.5: We present the proof for m positive, as the case $m < 0$ is virtually identical. Let $\hat{T} \in \mathcal{R}_1(\Omega)$ satisfy $\partial\hat{T} = 0$ relative to Ω and $\hat{T} \neq T_\alpha$. Recalling the assumption (4.1) that we make on Ω, let us first observe that for any $j \in \{1, 2, \ldots, N\}$, the restricted current $T \in \mathcal{R}_1(C_{R,L_j})$ given by $T \equiv \hat{T} \llcorner C_{R,L_j}$ necessarily satisfies $\partial T = 0$ relative to C_{R,L_j}. Also,

$$\|T - T_\alpha\|_{0,1}^* \leq \left\|\hat{T} - T_\alpha\right\|_{0,1}^*,$$

where the norm on the left is interpreted as a distance between two elements of $\mathcal{R}_1(C_{R,L_j})$ while the norm on the right is viewed as a distance between elements of $\mathcal{R}_1(\Omega)$.

In light of the above reasoning, we find that the argument is entirely local in that we can work with one cylinder at a time. For this reason, we consider any fixed j for the remainder of the proof and drop the subscript j. Hence, in particular, we write simply L for L_j, m for m_j and $C_{R,L}$ for C_{R,L_j} and we denote by T_* the oriented line segment joining a_j to b_j carrying multiplicity m. Then we pursue the inequality $M(\hat{T}) > M(T_*) = |m|\, L$. We shall only present the case $m > 0$. The argument is unchanged when $m < 0$.

As a final preliminary, recall from the discussion in Section 2, that we can write

$$T = \sum_{k=1}^{\infty} T_k$$

where each T_k is a multiplicity-one, rectifiable current, with support given by the image of a one-to-one Lipschitz map from $[0, M(T_k)]$ into $C_{R,L}$. Since $\partial T_k = 0$ in $C_{R,L}$, this image, denoted here by Γ_k, is either closed or has both endpoints lying on $\partial C_{R,L}$.

With the positive parameter δ controlling the closeness of $\hat{T}$ to T_* yet to be specified, we will now introduce a compactly supported 1-form ψ in order to gain a lower bound on the mass of the competitor T. Specifically, we define ψ by

$$\psi(x_1, x_2, x_3) = \rho(r) f(x_3) dx_3,$$

where $r = \sqrt{x_1^2 + x_2^2}$,

$$\rho(r) = \begin{cases} \delta^{1/4} - r & \text{for } r < \delta^{1/4}, \\ 0 & \text{for } r \geq \delta^{1/4}, \end{cases}$$

and

$$f(x_3) = \begin{cases} x_3 & \text{for } 0 \leq x_3 \leq \delta^{1/4}, \\ \delta^{1/4} & \text{for } \delta^{1/4} < x_3 < L - \delta^{1/4}, \\ L - x_3 & \text{for } L - \delta^{1/4} \leq x_3 \leq L. \end{cases}$$

Note that $\|\psi\|_{C^{0,1}(C_{R,L})} < 1$. Of course, since $\psi \notin C^\infty$ we do not have $\psi \in \mathcal{D}^1(C_{R,L})$. However, we can obviously approximate ψ in $C^{0,1}$ by a sequence in $\mathcal{D}^1(C_{R,L})$. As this is standard, we work directly with ψ below.

An easy integration yields the fact that

$$T_*(\psi) = m\delta^{1/2}(L - \delta^{1/4}). \tag{4.8}$$

Let us denote by τ_k the orienting unit tangent associated with T_k and also introduce the notation

$$\Gamma_k^+ = \{x \in \operatorname{supp} T_k : \tau_k \cdot e_3 \geq 0\},$$
$$\Gamma_k^- = \{x \in \operatorname{supp} T_k : \tau_k \cdot e_3 < 0\}.$$

Then the condition $\|T - T_*\|_{0,1}^* < \delta$ and (4.8) imply that

$$\begin{aligned}\delta^{1/2} \sum_{k=1}^{\infty} \int_{\Gamma_k^+ \cap \{0<r<\delta^{1/4}\}} e_3 \cdot \tau_k \, dH^{(1)}(x) &\geq \sum_{k=1}^{\infty} \int_{\Gamma_k^+ \cap \{0<r<\delta^{1/4}\}} \langle \psi, \tau_k \rangle \, dH^{(1)}(x) \\ &= \sum_{k=1}^{\infty} \int_{\Gamma_k^+} \langle \psi, \tau_k \rangle \, dH^{(1)}(x) \\ &\geq \sum_{k=1}^{\infty} T_k(\psi) = T(\psi) \\ &\geq m\delta^{1/2}(L - \delta^{1/4}) - \delta.\end{aligned}$$

It then follows that

$$\begin{aligned}\sum_{k=1}^{\infty} M\left(T_k \llcorner (\Gamma_k^+ \cap \{0 < r < \delta^{1/4}\})\right) &\geq \sum_{k=1}^{\infty} \int_{\Gamma_k^+ \cap \{0<r<\delta^{1/4}\}} e_3 \cdot \tau_k \, dH^{(1)}(x) \\ &\geq mL - (m+1)\delta^{1/4}. \qquad (4.9)\end{aligned}$$

We now complete the proof by distinguishing two cases:
Case 1. Let $g(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2}$ and suppose that for a.e. $r \in [R/2, R]$ one has

$$M(\langle T, g, r \rangle) \geq 1. \qquad (4.10)$$

We find using (2.7), (4.9) and (4.10) that for $\delta = \delta(m, R)$ sufficiently small one has

$$\begin{aligned}M(T) &\geq \left(\textstyle\sum_{k=1}^{\infty} M\left(T_k \llcorner (\Gamma_k^+ \cap \{0 < r < \delta^{1/4}\})\right) \right) + M(T \llcorner \{R/2 < r < R\}) \\ &\geq mL - (m+1)\delta^{1/4} + \tfrac{R}{2} > mL = M(T_*).\end{aligned}$$

Case 2. Now suppose

$$H^{(1)}\left(\{r \in [R/2, R] : M(\langle T, g, r \rangle) = 0\}\right) > 0. \qquad (4.11)$$

Define $\sigma \subset \mathbb{Z}^+$ by

$$\sigma = \left\{ k \in \mathbb{Z}^+ : M\left(T_k \llcorner (\Gamma_k^+ \cap \{0 < r < \delta^{1/4}\})\right) \right\} > 0.$$

For any $k \in \sigma$, it follows that Γ_k cannot be an arc with either endpoint lying on the lateral part of the cylinder $C_{R,L}$. That is, no endpoint of such a Γ_k can meet $\partial C_{R,L} \cap \{r = R\}$, for otherwise it would cross every cylinder $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 = r^2\}$ for $r \in [R/2, R]$, violating (4.11). Consequently, we can decompose σ into $\sigma = \sigma_1 \cup \sigma_2$ where

$$\sigma_1 = \{k \in \sigma : \Gamma_k \text{ is either a closed curve or an arc both of whose endpoints either lie on } \{x_3 = 0\} \text{ or on } \{x_3 = L\}\}$$

and

$$\sigma_2 = \{k \in \sigma : \Gamma_k \text{ is an arc having one endpoint on } \{x_3 = 0\} \text{ and the other on } \{x_3 = L\}\}.$$

Since for any $k \in \sigma_2$, the corresponding T_k satisfies $M(T_k) \geq L$, we see that if there exist at least m elements of σ_2 then necessarily,

$$M(T) \geq mL = M(T_*).$$

Then, from assumption (4.1), it follows that $M(\hat{T}) > mL$ unless $\hat{T} = T$ and consists of exactly m copies of the oriented line segment connecting $(0, 0, 0)$ to $(0, 0, L)$. But then $\hat{T} = T_*$, which we do not allow.

Therefore, we proceed under the assumption that σ_2 consists of m' elements where

$$0 \leq m' \leq m - 1. \tag{4.12}$$

Turning our attention for the moment to $k \in \sigma_1$, observe that any closed Γ_k will bound a surface $S_k \subset C_{R,L}$. Hence, by Stokes theorem we have

$$0 = \int_{S_k} (\nabla \times e_3) \cdot \nu_{S_k} \, dH^{(2)}(x) = \int_{\Gamma_k^+} e_3 \cdot \tau_k \, dH^{(1)}(x) + \int_{\Gamma_k^-} e_3 \cdot \tau_k \, dH^{(1)}(x). \tag{4.13}$$

A similar equation holds for any Γ_k having both endpoints on the top or both endpoints on the bottom of the cylinder since then Γ_k can be completed into a closed curve by including the line segment along the top or bottom joining the endpoints. Thus, (4.13) holds for all $k \in \sigma_1$.

Then we calculate

$$\begin{aligned} \sum_{k \in \sigma_1} M(T_k) &\geq \sum_{k \in \sigma_1} \left\{ \int_{\Gamma_k^+} e_3 \cdot \tau_k \, dH^{(1)}(x) - \int_{\Gamma_k^-} e_3 \cdot \tau_k \, dH^{(1)}(x) \right\} \\ &= 2 \sum_{k \in \sigma_1} \int_{\Gamma_k^+} e_3 \cdot \tau_k \, dH^{(1)}(x). \end{aligned} \tag{4.14}$$

Turning to any $k \in \sigma_2$, we see that

$$\int_{\Gamma_k} e_3 \cdot \tau_k \, dH^{(1)}(x) = \pm L,$$

where the sign depends on the orientation of T_k. Consequently, we have

$$\int_{\Gamma_k^+} e_3 \cdot \tau_k \, dH^{(1)}(x) + \int_{\Gamma_k^-} e_3 \cdot \tau_k \, dH^{(1)}(x) = \pm L.$$

This leads to the inequality

$$\begin{aligned} \sum_{k \in \sigma_2} M(T_k) &\geq \sum_{k \in \sigma_2} \left\{ \int_{\Gamma_k^+} e_3 \cdot \tau_k \, dH^{(1)}(x) - \int_{\Gamma_k^-} e_3 \cdot \tau_k \, dH^{(1)}(x) \right\} \\ &\geq \left(2 \sum_{k \in \sigma_2} \int_{\Gamma_k^+} e_3 \cdot \tau_k \, dH^{(1)}(x) \right) - m'L. \end{aligned} \tag{4.16}$$

Then an appeal to (4.9), (4.14) and (4.16) yields

$$\begin{aligned} \sum_{k \in \sigma} M(T_k) &\geq \left(2 \sum_{k \in \sigma} \int_{\Gamma_k^+} e_3 \cdot \tau_k \, dH^{(1)}(x) \right) - m'L \\ &\geq 2mL - m'L - 2(m+1)\delta^{1/4}. \end{aligned} \tag{4.17}$$

Recalling that $m' < m$, we conclude that

$$M(T) > mL$$

for $\delta = \delta(m, R, L)$ sufficiently small. ∎

5 Location of the zero set

Having obtained a sequence of local minimizers $\{u_\varepsilon\}$ in a domain Ω satisfying (4.1), we conclude with a stronger characterization of the location of the zero sets $\{u_\varepsilon = 0\}$ than that provided by (4.2).

We shall rely upon a version of the "η-compactness" property for solutions to the 3-d Ginzburg-Landau equation developed in [26], [23] and [4]. The property says roughly that there is a constant η independent of ε such that if the energy of a solution in a ball is not larger than $\eta\,|\ln \varepsilon|$, then in a smaller ball, the solution cannot vanish. A version of this property suitable for our purposes is given below:

PROPOSITION 5.1 *([5]) Let $R > 0$ be fixed. Then there exists a positive constant ε_0 and a positive constant η independent of R such that if U_ε is any solution to (1.2) in the ball $B(0, R)$ for any $\varepsilon \in (0, \varepsilon_0)$, the condition*

$$\frac{1}{R} \int_{B(0,R)} \left\{ \frac{|\nabla U_\varepsilon|^2}{2} + \frac{(1 - |U_\varepsilon|^2)^2}{4\varepsilon^2} \right\} dx \leq \eta \left| \ln \left(\frac{\varepsilon}{R} \right) \right|$$

implies that $|U_\varepsilon(0)| \geq \frac{1}{2}$.

With this result in hand, we now fix an integer m and let $\{u_\varepsilon\}$ denote the corresponding sequence of local minimizers guaranteed by Theorem 4.2 satisfying $J(u_\varepsilon) \to J(u_\alpha)$ in $(C_c^{0,1})^*$. Then we define

$$S_\varepsilon = \{x \in \Omega : |u_\varepsilon(x)| < \frac{1}{2}\} \tag{5.1}$$

and for any $\beta > 0$ we introduce the notation

$$\Omega_\beta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \beta\}.$$

We also introduce the notation

$$\Gamma \equiv \cup_{j=1}^N (l_j \cap \Omega)$$

(cf. assumptions (3.1)–(3.3) and (4.1)). Our result is:

THEOREM 5.2 *For any $\beta > 0$ and any $\delta_1 > 0$ there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ one has*

$$S_\varepsilon \cap \Omega_\beta \subset \{x \in \Omega_\beta : \operatorname{dist}(x, \Gamma) < \delta_1\} \tag{5.2}$$

REMARK 5.3 While (5.2) asserts that the zero set lies near the segments $\{l_j\}$ (at least away from $\partial\Omega$), in the case where $m_j \neq \pm 1$, one should not interpret Theorem 5.2 as saying that near $l_j \cap \Omega$, the vortex consists of a single curve. Indeed, in light of the structure required of good approximating sequences as in Proposition 3.1, one would expect that in this case, the vortices would consist of $|m_j|$ curves all lying close to l_j, with u_ε winding once around each.

REMARK 5.4 We suspect that a condition much stronger than (5.2) holds, namely that $\{u_\varepsilon = 0\}$ converges to Γ in the Hausdorff metric, even near $\partial\Omega$. We will pursue this point in a subsequent article.

PROOF: We begin by introducing the sequence of non-negative Radon measures $\{\mu_\varepsilon\}$ characterized by

$$\mu_\varepsilon(W) = \frac{1}{|\ln \varepsilon|} \int_W \{\frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1 - |u_\varepsilon|^2)^2\} \, dx \tag{5.3}$$

for any set $W \subset \Omega$. From the proof of Theorem 4.2 and (3.8), it follows that

$$\mu_\varepsilon(\Omega) = E_\varepsilon(u_\varepsilon) \leq E_\varepsilon(v_\varepsilon) \to \pi \sum_{j=1}^N |m_j| \, L_j, \tag{5.4}$$

where $\{v_\varepsilon\}$ is the sequence of approximate minimizers given in Proposition 3.1. Hence, for a subsequence $\{\varepsilon_k\} \to 0$ one has

$$\mu_{\varepsilon_k} \rightharpoonup \mu, \tag{5.5}$$

where μ is a non-negative Radon measure. Now recall the notation $u_\alpha = \Pi_{j=1}^N e^{im_j\theta_j}$, and let $\mu_{J(u_\alpha)}$ denote the total variation measure associated with the 1-current $J(u_\alpha)$, (cf. (2.3)). Combining (2.13), (2.16), (5.4) and (5.5) we observe that

$$\mu(\Omega) \leq \pi \sum_{j=1}^{N} |m_j|\, L_j = \mu_{J(u_\alpha)}(\Gamma) \leq \mu(\Gamma). \tag{5.6}$$

Hence, in particular, we have

$$\operatorname{supp} \mu = \Gamma. \tag{5.7}$$

We will now reach a contradiction of (5.7) under the assumption that (5.2) is false. To this end, assume that there a positive β and a positive δ_1 so that

$$S_{\varepsilon_k} \cap \Omega_\beta \not\subseteq \{x \in \Omega_\beta : \operatorname{dist}(x, \Gamma) < \delta_1\} \tag{5.8}$$

for some sequence $\varepsilon_k \to 0$. Then there exists a sequence $\{x_k\}$ such that $x_k \in S_{\varepsilon_k} \cap \Omega_\beta$ with $\operatorname{dist}(x_k, \Gamma) \geq \delta_1$. After passing to a subsequence (still denoted by $\{x_k\}$), we find $x_k \to x_0$, where x_0 satisfies

$$\operatorname{dist}(x_0, \partial\Omega) \geq \beta \text{ and } \operatorname{dist}(x_0, \Gamma) \geq \delta_1. \tag{5.9}$$

Let $\lambda = \min\{\frac{\beta}{2}, \frac{\delta_1}{2}\}$. Since $|u_{\varepsilon_k}(x_k)| < \frac{1}{2}$ for all k, we can invoke Proposition 5.1 to conclude that for k large enough one has

$$\frac{1}{\lambda}\mu_{\varepsilon_k}(B(x_0, \lambda)) \geq \frac{1}{\lambda}\mu_{\varepsilon_k}(B(x_k, \frac{\lambda}{2})) \geq \frac{1}{4}\eta. \tag{5.10}$$

Letting $k \to \infty$, we then get that $\mu(\bar{B}(x_0, \lambda)) > 0$. Hence,

$$\operatorname{dist}(x_0, \operatorname{supp} \mu) \leq \lambda,$$

a contradiction of (5.7) in light of (5.9). ∎

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