Local Minimizers of the Ginzburg-Landau Energy with Magnetic Field in Three Dimensions

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Abstract.

We establish the existence of locally minimizing vortex solutions to the full Ginzburg-Landau energy in three dimensional simply-connected domains with or without the presence of an applied magnetic field. The approach is based upon the theory of weak Jacobians and applies to nonconvex sample geometries for which there exists a configuration of locally shortest line segments with endpoints on the boundary.

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1 Introduction

Based on the interplay between the magnetic field and a complex-valued order parameter, the Ginzburg-Landau energy successfully captures a wide array of phenomena associated with the behavior of superconductors. Of primary interest in any model for superconductivity is the ability to predict the behavior of vortices, mathematically defined as the zero set of the order parameter and physically described as thin filaments holding magnetic flux within a superconducting sample that are encircled by supercurrent. The purpose of our investigation is to show how, in a certain asymptotic regime and for certain sample geometries, the Ginzburg-Landau energy possesses local minimizers whose presence corresponds to a somewhat surprising and intricate stable configuration of supercurrents and vortices.

Given a sample geometry $\Omega \subset \mathbb{R}^3$, the Ginzburg-Landau energy depends on an order parameter $u : \Omega \to \mathbb{C}$ and a magnetic potential $A : \mathbb{R}^3 \to \mathbb{R}^3$ and in a convenient non-dimensionalization, takes the form

$$G_\varepsilon(u,A) = \int_\Omega \frac{1}{2}\left|(\nabla - iA)u\right|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla\times A - H^\varepsilon_{ap}\right|^2\,dx \tag{1.1}$$

(cf. [12], [9], [35]). Here $H^\varepsilon_{ap} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes a given applied magnetic field and $\frac{1}{\varepsilon}$ denotes the Ginburg-Landau parameter, a material constant. We will take ε to be small, an assumption placing our work in the so-called "extreme Type-II" regime for superconductors. Physically measurable quantities in the model include $|u|^2$, which corresponds to the density of superconducting electron pairs, and $\nabla \times A$, which represents the effective magnetic field both within and outside the sample.

Another physically important quantity is the supercurrent, given by the quantity

$$\frac{i}{2}(\bar{u}\nabla u - u\nabla\bar{u}) - |u|^2 A.$$

In the case of no applied field, $H^\varepsilon_{ap} \equiv 0$, note that the global minimizer of G_ε is given simply by $u \equiv 1$, $A \equiv 0$, but physically interesting critical points of G_ε are those which at least locally minimize the energy and for which the supercurrent is nontrivial. The critical points we will construct in this article share these two properties.

Our approach is based upon the asymptotic connection, for $\varepsilon << 1$, between the reduced Ginzburg-Landau energy E_ε given by

$$E_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2\,dx \tag{1.2}$$

and the total length of the vortices. Stated loosely, this connection says that for sequences of functions $u_\varepsilon : \Omega \to \mathbb{C}$ having zero sets of bounded total length, one can show that

$$\frac{E_\varepsilon(u_\varepsilon)}{|\ln \varepsilon|} \approx C \cdot (\text{length of } \{x : u_\varepsilon(x) = 0\}) . \tag{1.3}$$

for some constant C. In the last few years, this relation has been developed and explored by many authors including [2], [5], [10], [16], [18], [25], [28] and [30].

Most closely related to this article is the work in [27], where (1.3) leads to the construction of local minimizers to E_ε for small values of ε provided there exists a collection of one or more line segments in Ω with endpoints on the boundary that locally minimize total length. This geometric requirement on the sample Ω can be verified for a general class of bounded, simply-connected, nonconvex domains in $\mathbb{R}^3$, and it is the main purpose of this paper to argue that the requirement is sufficient to produce local minimizers of G_ε as well, provided H_{ap}^ε is not too large. The precise result is stated as Theorem 4.2, which is the main result of this paper. In the situation where H_{ap} is independent of ε, we also derive a London-type equation satisfied asymptotically by the induced field, cf. Remark 4.4

Though technically it turns out to be the easiest case to handle, we wish to emphasize the significance of our results in the case of no applied field, $H_{ap}^\varepsilon \equiv 0$. The local minimizers that we find in this context correspond physically to what are known as "permanent currents," namely, stable supercurrents that can sustain themselves over very long periods without the impetus of external currents or fields. The existence of such currents has been well-known to physicists for years and their realization as local minimizers of G_ε was demonstrated in [20], [29] and [22]. However, in all three of these mathematical studies, and to our knowledge, in the treatments in the physics literature as well, the examination of permanent currents has always taken place in the context of non-simply-connected domains Ω. Furthermore, the local minimizers constructed in the previous three references contain no vortices; that is, the order parameter does not vanish. For these reasons, the case of stable vortex solutions locally minimizing G_ε in simply-connected domains in the absence of any applied field is particularly noteworthy. To our knowledge, the only previous such existence result is the one found in [23] where the authors use a perturbation of domain argument to obtain local minimizers in a simply-connected perturbation of a non-simply connected domain.

The primary tool in our analysis is the notion of weak Jacobians, a subject we review in Section 2 along with other relevant notions from the area of geometric measure theory. For our purposes, we view the Jacobian of a map $u : \Omega \to \mathbb{C}$ as a 1-current, a geometric measure theoretic generalization of a curve. More precisely, a 1-current is a bounded linear functional acting on differential 1-forms (cf. (2.5)), and in [15], [16], [17] and [18], and more recently in [1] and [2], this concept has been developed and applied to Ginzburg-Landau theory with the Jacobian acting on 1-forms of compact support in a domain Ω.

The most significant technical issue we must overcome in applying the theory of weak Jacobians to G_ε is the extension of various estimates and convergence results to the situation where the Jacobian acts on 1-forms whose tangential but not normal components vanish on $\partial\Omega$. This is accomplished in Section 3, but to explain briefly what necessitates this extension, consider the following relationship between G_ε and E_ε, first exploited (to our knowledge) in [6]:

$$\begin{aligned} G_\varepsilon(u,A) &= E_\varepsilon(u) - \int_\Omega j(u)\cdot A\,dx \\ &\quad + \frac{1}{2}\int_\Omega |u|^2\,|A|^2\,dx + \frac{1}{2}\int_\Omega \left|\nabla\times A - H_{ap}^\varepsilon\right|^2\,dx. \end{aligned}$$

Here $j(u) := \frac{1}{2i}(\bar{u}\nabla u - u\nabla\bar{u})$. Note that for a smooth function, the quantity $\frac{1}{2}\nabla\times j(u)$ corresponds to the classical Jacobian. Clearly control of the energy G_ε will lead to control of the energy E_ε provided one can handle the second term on the right, that is, the only term on the right of indeterminate sign. Recall now the Hodge decomposition that allows one to write an arbitrary vector field A as $A = \nabla\times B + \nabla\phi$ for some vector field B satisfying $B\times\nu = 0$ on $\partial\Omega$ and some scalar function ϕ (cf. Lemma 2.1). One observes upon substitution of this decomposition into the relation above that the key term to handle becomes

$$\int_\Omega j(u)\cdot\nabla\times B\,dx,$$

which after an integration by parts reduces to the Jacobian of u integrated against B. Hence understanding the action of the Jacobian on these purely normal vector fields becomes central to our investigation. This theory is developed in Section 3 and in addition to serving our purposes in proving the existence result of Section 4, we hope that the results of this section will be of independent interest in future applications of weak Jacobians as well.

The proof of the main result, given in Section 4, is reminiscent of the scheme first laid out in [24] for using Γ-convergence of energies to produce local minimizers in the context of scalar minimizers of E_ε . We refer readers interested in background on Γ-convergence to [7], though our treatment here is completely self-contained. Again, the key assumption we make is that for the Γ-limit, in this case total length, there is an isolated locally minimizing configuration of line segments spanning the sample $\Omega \subset \mathbb{R}^3$.

We suspect that no local minimizers of G_ε can be found in convex domains, for any value of ε. This is the case for E_ε, as was shown in [19], and it is the case for G_ε when $\Omega \subset \mathbb{R}^2$ in light of the results in [21]. In fact, concerning the two-dimensional setting, the recent results of [33] show that at least for small values of ε, one cannot find local minimizers for G_ε in *any* simply-connected domain. Finally, regarding the presence of the applied field, H^ε_{ap}, which we assume obeys the bound

$$\int_\Omega \left|H^\varepsilon_{ap}\right|^2 \, dx = o(|\ln \varepsilon|^2), \tag{1.4}$$

we view our results on 3-d local minimizers of G_ε as a contribution to the systematic mathematical analysis of the nature of stable critical points based upon the size of the applied magnetic field, much in the same vein as was carried out earlier for the two-dimensional problem in such works as [31] and [32]. In the scaling we have chosen for the Ginzburg-Landau energy, condition (1.4) corresponds to relatively weak fields, as opposed to the setting of bifurcation from the normal state $(u \equiv 0, \nabla \times A \equiv H^\varepsilon_{ap})$ in 3-d examined in such works as [14] and [26] where $H^\varepsilon_{ap} \sim \mathcal{O}(1/\varepsilon)$. Of course if H^ε_{ap} is too large, then the only critical point is the normal state, cf. [13].

2 Preliminaries

Throughout this article, $\Omega \subset \mathbb{R}^3$ will denote a bounded, simply-connected domain with smooth boundary. We begin by briefly introducing certain concepts and notation from the theory of currents. We refer the reader to [11] or [34] for more details. Throughout, we use $H^{(n)}$ to denote n-dimensional Hausdorff measure. We use $B(x, R)$ to denote the ball in $\mathbb{R}^n$ with center x and radius R. We will denote the characteristic function of a set S by χ_S.

2.1 Forms and currents

For integers $0 \le k \le n$, the space of Grassman k-covectors is denoted by $\wedge^k(\mathbb{R}^n)$ endowed with the usual Euclidean norm $|\cdot|$. A differential k-form ϕ on Ω is a mapping $\phi\colon \Omega \to \wedge^k(\mathbb{R}^n)$. The space of C^∞ k-forms compactly supported within Ω is denoted by $\mathcal{D}^k(\Omega)$. Its topology implies that $\phi_n \to 0$ in $\mathcal{D}^k(\Omega)$ if and only if there is a fixed compact set $K \subset \Omega$ with $\operatorname{spt}\phi_n \subset K$ and $\phi_n \to 0$ uniformly along with all derivatives of any order. We write $f^\#\phi$ to denote the pullback of the form ϕ by the mapping f. We employ the notation ϕ_T for the tangential part of ϕ on $\partial\Omega$, defined by $\phi_T = i^\#\phi$ where $i : \partial\Omega \to \bar{\Omega}$ denotes the usual injection. In particular, this means that $\phi_T(x) = 0$ if $\phi_T(x)(v_1 \wedge \ldots \wedge v_k) = 0$ whenever $v_1, \ldots, v_k$ are tangent to $\partial\Omega$ at x.

A $k-$current in Ω is a continuous linear functional on the space $\mathcal{D}^k(\Omega)$ and the space of such k-currents is denoted by $\mathcal{D}_k(\Omega)$. We recall that the boundary of a k-current T, denoted by ∂T, is the $(k-1)$-current defined by the relation

$$\partial T(\phi) = T(d\phi) \text{ for all } \phi \in \mathcal{D}^{k-1}(\Omega),$$

where $d\phi$ represents the k-form obtained by exterior differentiation of ϕ. In particular, we note that a k-current T has zero boundary relative to the set Ω if $T(d\phi) = 0$ for all $\phi \in \mathcal{D}^{k-1}(\Omega)$.

For $T \in \mathcal{D}_k(\Omega)$, we denote the mass of T in Ω by

$$M_\Omega(T) \equiv \sup_{\{\phi \in D^k(\Omega):\ \|\phi\|_{L^\infty(\Omega)} \le 1\}} |T(\phi)|\,. \tag{2.1}$$

In this paper we will typically write simply $M(T)$ rather than $M_\Omega(T)$ when no ambiguity can result.

A norm on k-forms whose dual will play a crucial role in what follows is the $C^{0,\alpha}$-norm, for any $\alpha \in (0,1]$, given by

$$\|\phi\|_{C^{0,\alpha}(\Omega)} \equiv \max\left\{ \sup_{x\in\Omega} |\phi(x)|\,,\ \sup_{x,y\in\Omega} \frac{|\phi(x) - \phi(y)|}{|x-y|^\alpha} \right\}.$$

We also write

$$\|\phi\|_{C_0^{0,\alpha}(\Omega)} \equiv \begin{cases} \|\phi\|_{C^{0,\alpha}(\Omega)} & \text{if } \phi = 0 \text{ on } \partial\Omega \\ +\infty & \text{if not,} \end{cases}$$

and similarly

$$\|\phi\|_{C_T^{0,\alpha}(\Omega)} \equiv \begin{cases} \|\phi\|_{C^{0,\alpha}(\Omega)} & \text{if } \phi_T = 0 \text{ on } \partial\Omega \\ +\infty & \text{if not.} \end{cases}$$

We will use the notation $C_0^{0,\alpha}(\Omega)^*$ to denote the dual space of k-currents with norm

$$\|T\|_{C_0^{0,\alpha}(\Omega)^*} \equiv \sup_{\{\phi:\bar{\Omega}\to\wedge^k(\mathbb{R}^n);\|\phi\|_{C_0^{0,\alpha}(\Omega)}\leq 1\}} |T(\phi)|\,. \tag{2.2}$$

and analogously $C_T^{0,\alpha}(\Omega)^*$. Note that $\|T\|_{C_0^{0,\alpha}(\Omega)^*} \leq \|T\|_{C_T^{0,\alpha}(\Omega)^*}$ for all currents T. One of the main technical points of this paper is to strengthen estimates involving the norm $\|\cdot\|_{C_0^{0,\alpha}(\Omega)^*}$ to obtain estimates in the $\|\cdot\|_{C_T^{0,\alpha}(\Omega)^*}$ norm.

Most prominent in our approach will be the class $\mathcal{R}_k(\Omega)$ of rectifiable, integer multiplicity k-currents. Especially crucial will be elements of $\mathcal{R}_1(\Omega)$, a geometric measure theoretic generalization of a Lipschitz curve. To describe this class, let us first recall that a set $\Gamma \subset \mathbb{R}^n$ is said to be 1-rectifiable if

$$\Gamma = \Gamma_0 \cup \left(\cup_{j=1}^{\infty} f_j(\gamma_j)\right)$$

where $H^{(1)}(\Gamma_0) = 0$, $\gamma_j \subset \mathbb{R}^1$ and $f_j : \gamma_j \to \mathbb{R}^n$ are Lipschitz functions. A 1-current T on Ω is said to be rectifiable and integer multiplicity if its action on a 1-form $\phi \in \mathcal{D}^1(\Omega)$ is given by

$$T(\phi) = \int_{\Gamma} \langle \phi(x), \tau(x)\rangle\, m(x)\, dH^{(1)}(x) \tag{2.3}$$

where Γ is a 1-rectifiable set, τ is a unit vector orienting the approximate tangent space $T_x\Gamma$ and m is an $H^{(1)}$-measurable, positive integer-valued function referred to as the multiplicity of the current. The notation $\langle\cdot,\cdot\rangle$ above refers to the dual pairing of a vector and co-vector.

2.2 Weak Jacobians as 1-Currents

For concreteness we now specialize to $\Omega \subset \mathbb{R}^3$, which is the setting of most of this paper. For a function $u \in W^{1,2}(\Omega;\mathbb{C})$, we write $J(u)$ to denote the two-form

$$J(u) = u^{\#}(dx) = du_1 \wedge du_2$$

where $u = u_1 + iu_2$, and dx denotes the standard area form on the target $\mathbb{C}$. It is often convenient to identify $J(u)$ with a 1-current, which we denote $\star J(u)$, and which is defined through its action on 1-forms ϕ by

$$\star J(u)(\phi) = \int \phi \wedge J(u).$$

The current $\star J(u)$ can still be defined for u in certain Sobolev spaces below $W^{1,p}$ for $p < 2$. To this end, we define the 1-form $j(u)$ via the formula

$$j(u) = \frac{1}{2i}\sum_{k=1}^{3}(\overline{u}u_{x_k} - u\overline{u}_{x_k})dx_k = \frac{1}{2i}(\bar{u}du - ud\bar{u}). \tag{2.4}$$

where $\bar{\cdot}$ denotes complex conjugation. We also define an associated 2-current $\star j(u)$ that acts on 2-forms ϕ via $\star j(u)(\phi) = \int \phi \wedge j(u)$. Note that $|j(u)| \in L^1(\Omega)$ for $u \in W^{1,2}(\Omega;\mathbb{C})$ or $W^{1,1}(\Omega;S^1)$. Then we define $\star J(u) = \frac{1}{2}\partial(\star j(u))$, so that

$$\star J(u)(\phi) = \frac{1}{2}\int_\Omega d\phi \wedge j(u) \tag{2.5}$$

for any $\phi \in \mathcal{D}^1(\Omega)$. One can check through integration by parts that this agrees with the previous definition of $\star J(u)$ when $u \in W^{1,2}(\Omega)$. This is a consequence of the identity $J(u) = \frac{1}{2}dj(u)$. In fact for $u \in W^{1,2}(\Omega)$, the two definitions of $\star J(u)(\phi)$ both make sense and coincide for smooth, not necessarily compactly supported 1-forms ϕ such that $\phi_T = 0$.

At times we will also use the symbol $\star$ to denote the usual Hodge operator $\star : \wedge^k(\mathbb{R}^3) \to \wedge_{3-k}(\mathbb{R}^3)$ defined by requiring that

$$\langle \star\phi, \omega\rangle\, dx_1 \wedge dx_2 \wedge dx_3 = \phi \wedge \omega \quad \text{for all } \omega \in \wedge^k(\mathbb{R}^3), \phi \in \wedge^{3-k}(\mathbb{R}^3).$$

With this notation, if $B = \sum_{i=1}^3 B^i dx^i$ is a 1-form, then $\star dB$ is the vector field

$$\star dB = (B^3_{x_2} - B^2_{x_3})e_1 + (B^1_{x_3} - B^3_{x_1})e_2 + (B^2_{x_1} - B^1_{x_2})e_3,$$

where $e_i, i = 1, \dots 3$ are the standard basis vectors.

Two simple but important consequences of (2.5) are that for any $u \in W^{1,2}(\Omega;\mathbb{C}) \cup W^{1,1}(\Omega;S^1)$ one has

$$\partial(\star J(u)) = 0 \quad \text{relative to } \Omega \quad (\text{since } d^2\phi = 0) \tag{2.6}$$

and for any $u, v \in W^{1,2}(\Omega;C) \cup W^{1,1}(\Omega;S^1)$ one has

$$\|\star J(u)\|_{C^{0,1}_T(\Omega)^*} \le 6\int_\Omega |j(u)|\, dx. \tag{2.7}$$

2.3 Divergence free vector fields and Hodge decomposition

In this section we describe a Sobolev space that will play an important role for us. We also describe a useful decomposition for $L^2(\Omega;\mathbb{R}^3)$ vector fields. Here, and in what follows, the norm of $f \in L^p(\Omega;\mathbb{R}^n)$ (resp. $f \in W^{k,p}(\Omega;\mathbb{R}^n)$) will be denoted by $\|f\|_{L^p(\Omega;\mathbb{R}^n)}$ (resp. $\|f\|_{W^{k,p}(\Omega;\mathbb{R}^n)}$).

Let $\mathcal{H}$ be the completion of the set

$$\{\phi \in C^\infty(\mathbb{R}^3;\mathbb{R}^3) : \ \phi \text{ has compact support}\}$$

with respect to the norm $\|\nabla\phi\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}$. It is clear then that $\mathcal{H}$ is a Hilbert space. Let us define $\mathcal{H}_0$ to be

$$\mathcal{H}_0 = \{\phi \in \mathcal{H} : \operatorname{div}\phi = 0\}.$$

We note here that $\mathcal{H}_0$ is strongly closed in $\mathcal{H}$, and convex. It follows that $\mathcal{H}_0$ is weakly closed in $\mathcal{H}$. Also note that, in $\mathcal{H}_0$, the norm inherited from $\mathcal{H}$ is equivalent to the norm $\|\nabla\times\phi\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}$. Finally, note that by the classical Sobolev embeddings, the following inequality holds for $\phi \in \mathcal{H}_0$:

$$\|\phi\|_{L^6(\mathbb{R}^3;\mathbb{R}^3)} \le C\,\|\nabla\times\phi\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\,. \tag{2.8}$$

In several instances in this paper we will need a decomposition of a vector field $A \in W^{1,2}(\Omega;\mathbb{R}^3)$ as $A = \nabla\times B + \nabla\phi$. In such circumstances the following lemma will be useful.

2.1 Lemma. *(cf. [5], Lemma A.5, Prop. A4) For any smooth, simply connected domain $\Omega \subset \mathbb{R}^3$ there is a constant $C = C(\Omega)$ such that for any $A \in W^{1,2}(\Omega;\mathbb{R}^3)$ there exists a unique $B \in W^{1,2}(\Omega;\mathbb{R}^3)$ and a function $\phi \in W^{1,2}(\Omega;\mathbb{R})$ which is unique up to an additive constant satisfying*

$$A = \nabla\times B + \nabla\phi \text{ in } \Omega, \tag{2.9}$$

$$\operatorname{div}B = 0 \text{ in } \Omega, \qquad B\times\nu = 0 \text{ on } \partial\Omega \quad \text{and} \tag{2.10}$$

$$\|B\|_{W^{2,2}(\Omega;\mathbb{R}^3)} \le C\,\|\nabla\times A\|_{L^2(\Omega;\mathbb{R}^3)}\,. \tag{2.11}$$

2.2 Definition. *We will often use the notation*

$$\mathbb{P}(A) := \nabla\times B = A - \nabla\phi. \tag{2.12}$$

2.3 Remark. Instead of writing $\mathbb{P}(A) = \nabla\times B$ for a vector field B such that $B\times\nu = 0$ on $\partial\Omega$, we can express $\mathbb{P}(A)$ in the equivalent form $\mathbb{P}(A) = \star dB$, where B is a 1-form such that $B_T = 0$. Note that then we have

$$\int\langle\mathbb{P}(A), j(u)\rangle = \int dB\wedge j(u) = 2\star J(u)(B) \tag{2.13}$$

for $u \in W^{1,2}(\Omega;\mathbb{C})$, where boundary terms vanish because $B_T = 0$. This is the form in which we will typically use the Hodge decomposition.

2.4 Remark. Note that ϕ minimizes the functional $F(\psi) := \int_\Omega |A - \nabla\psi|^2\, dx$. To see this, observe that the Euler-Lagrange equation and natural boundary conditions satisfied by the minimizer ϕ are $\operatorname{div}(A - \nabla\phi) = 0$ in Ω and $(A - \nabla\phi)\cdot\nu = 0$ on $\partial\Omega$. These are exactly the equations that characterize the function ϕ of the above lemma, where the boundary condition is a consequence of the fact that $B \times \nu = 0$ on $\partial\Omega$, which implies that $(\nabla \times B)\cdot\nu = 0$ on $\partial\Omega$.

We will now use Lemma 2.1 to rewrite the Ginzburg-Landau energy G_ε. Note that since $\operatorname{div} H^\varepsilon_{ap} = 0$, there exists a potential $A^\varepsilon_{ap} \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$ such that $\nabla \times A^\varepsilon_{ap} = H^\varepsilon_{ap}$. Since this condition only determines A^ε_{ap} up to the addition of a gradient of a scalar function, we take advantage of this freedom to make a convenient choice of the gradient so that A^ε_{ap} satisfies

$$\operatorname{div} A^\varepsilon_{ap} = 0 \quad \text{in } \Omega \quad \text{and} \quad A^\varepsilon_{ap}\cdot\nu = 0 \quad \text{on } \partial\Omega. \tag{2.14}$$

2.5 Lemma. *Let $v \in W^{1,2}(\Omega;\mathbb{C})$ and let $A : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy $A - A^\varepsilon_{ap} \in \mathcal{H}_0$, with $A = \mathbb{P}(A) + \nabla\phi$ as in Lemma 2.1. Then*

$$\begin{aligned} &G_\varepsilon(e^{i\phi}v, A) = \\ &E_\varepsilon(v) - \int_\Omega \langle \mathbb{P}(A), j(v)\rangle + \frac{1}{2}\int_{\mathbb{R}^3} |v|^2\, |\mathbb{P}(A)|^2 \chi_\Omega + \left|\nabla \times A - H^\varepsilon_{ap}\right|^2\, dx \\ &:= \mathcal{G}_\varepsilon(v, A). \end{aligned} \tag{2.15}$$

Proof. A direct calculation shows that

$$\left|(\nabla - iA)(e^{i\phi}v)\right|^2 = |\nabla v|^2 - 2\langle \mathbb{P}(A), j(v)\rangle + |v|^2\, |\mathbb{P}(A)|^2 .$$

The conclusion of the lemma follows by using this identity to rewrite the first integral in the definition (1.1) of G_ε, and then rearranging terms. □

We conclude this section with the following lemma that will allow us to seek local minimizers of the more convenient energy $\mathcal{G}_\varepsilon$ in order to establish the existence of local minimizers of G_ε:

2.6 Lemma. *Using the notation of Lemma 2.1, (v, A) is a local minimizer of $\mathcal{G}_\varepsilon$ if and only if $(e^{i\phi}v, A)$ is a local minimizer of G_ε.*

Proof. The proof is a straightforward verification. We need to check that $(v, A) \mapsto (e^{i\phi}v, A)$ is a homeomorphism of $W^{1,2}(\Omega;\mathbb{C}) \times \{A :\ A - A^\varepsilon_{ap} \in \mathcal{H}_0\}$

to itself. To see that it is one-to-one and onto, note that it has an inverse, i.e. the map $(v, A) \mapsto (e^{-i\phi}v, A)$. To check continuity, fix (v_1, A_1) and (v_2, A_2), and write $A_i = \mathbb{P}(A_i) + \nabla\phi_i$ for $i = 1, 2$. Note that by Sobolev embedding and (2.11) we have

$$\left\|e^{i\phi_1} - e^{i\phi_2}\right\|_{L^\infty}(\Omega;\mathbb{C}) \le C\left\|\phi_1 - \phi_2\right\|_{L^\infty}(\Omega) \le C\left\|\phi_1 - \phi_2\right\|_{W^{2,2}(\Omega)} \le C\left\|A_1 - A_2\right\|_{W^{1,2}(\Omega;\mathbb{R}^3)}.$$

Similarly, we have

$$\left\|v_1 - v_2\right\|_{L^6(\Omega;\mathbb{C})} \le C\left\|v_1 - v_2\right\|_{W^{1,2}(\Omega;\mathbb{C})} \quad \text{and} \quad \left\|\nabla(\phi_1 - \phi_2)\right\|_{L^6(\Omega;\mathbb{R}^3)} \le \left\|A_1 - A_2\right\|_{W^{1,2}(\Omega;\mathbb{R}^3)}$$

Using the assembled estimates with Hölder's inequality and the fact that Ω is bounded, one easily checks that

$$\begin{aligned}\left\|\nabla(v_1e^{i\phi_1} - v_2e^{i\phi_2})\right\|_{L^2} &\le \left\|\nabla(v_1 - v_2)e^{i\phi_1}\right\|_{L^2} + \left\|(v_1 - v_2)\nabla\phi_1 e^{i\phi_1}\right\|_{L^2} + \\ &\quad + \left\|\nabla v_2(e^{i\phi_1} - e^{i\phi_2})\right\|_{L^2} + \left\|v_2\nabla(\phi_1 - \phi_2)\right\|_{L^2} \\ &\le C\left\|v_1 - v_2\right\|_{W^{1,2}}(1 + \left\|A_1\right\|_{W^{1,2}}) \\ &\quad + C\left\|A_1 - A_2\right\|_{W^{1,2}}\ \left\|v_2\right\|_{W^{1,2}}.\end{aligned}$$

This shows the continuity of $(v, A) \mapsto (ve^{i\phi}, A)$. The continuity of the inverse follows by exactly the same estimates. □

3 Jacobian estimate and Γ-limit revisited

The purpose of this section is to extend various Jacobian estimates as well as the Γ-convergence results of [16] from the topology $C_0^{0,\alpha}(\Omega)^*$ to $C_T^{0,\alpha}(\Omega)^*$.

3.1 Extension of Jacobian estimates and Γ-convergence

The first main result of this section is the following:

3.1 Proposition. *Let $\Omega \subset \mathbb{R}^3$ be a smooth domain, and let $\alpha \in (0, 1]$. Then there are constants $\gamma > 0$ and $C(\alpha, \Omega) > 0$ such that for any $v \in W^{1,2}(\Omega;\mathbb{C})$ and any $\varepsilon \in (0, 1)$ one has*

$$\left\|J(v)\right\|_{C_T^{0,\alpha}(\Omega)^*} \le C(\alpha, \Omega)\left(\varepsilon^\gamma + \frac{E_\varepsilon(v)}{|\ln\varepsilon|}\right). \tag{3.1}$$

This is an extension of an estimate from [16], in which essentially the same result was established with the weaker $C_0^{0,\alpha}(\Omega)^*$ dual norm, rather than the $C_T^{0,\alpha}(\Omega)^*$ dual norm. The statement of that result, which is needed for the proof of Proposition 3.1, is given in Lemma 3.5 at the end of this section. We also give a sketch of the proof, since the exact estimate we need does not explicitly appear in [16].

The other main result we establish in this section has a similar character:

3.2 Proposition. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary. Suppose that $\{w_\varepsilon\}_{\varepsilon\in(0,1]} \subset W^{1,2}(\Omega;\mathbb{C})$ satisfies the uniform bound $E_\varepsilon(w_\varepsilon) \le C\,|\ln\varepsilon|$ for some $C > 0$. Then there is a sequence $\varepsilon_k \to 0$ and a rectifiable 1-current J such that $\partial J = 0$ relative to Ω, $\frac{1}{\pi}J$ is integer multiplicity and*

$$\lim_{k\to\infty} \|\star J(w_{\varepsilon_k}) - J\|_{C_T^{0,\alpha}(\Omega)^*} = 0, \tag{3.2}$$

$$\liminf_{\varepsilon\to 0} \frac{1}{|\ln\varepsilon|} E_\varepsilon(w_\varepsilon) \ge M(J). \tag{3.3}$$

Moreover, given any rectifiable 1-current J such that $\partial J = 0$ relative to Ω and $\frac{1}{\pi}J$ is integer multiplicity, there exists a sequence $\{v_\varepsilon\} \subset W^{1,2}(\Omega;\mathbb{C})$ with $|v_\varepsilon| \le 1$, such that

$$\lim_{\varepsilon\to 0} \|\star J(v_\varepsilon) - J\|_{C_T^{0,\alpha}(\Omega)^*} = 0 \qquad \textit{and} \qquad \lim_{\varepsilon\to 0} |\ln\varepsilon|^{-1} E_\varepsilon(v_\varepsilon) = M(J). \tag{3.4}$$

This is proved in [16] and [2], with the $C_0^{0,\alpha}(\Omega)^*$ norm instead of the $C_T^{0,\alpha}(\Omega)^*$ norm in (3.2) and (3.4). The point again is to show that the result remains true when we allow test 1-forms for which the normal part does not vanish on $\partial\Omega$. We remark that, while a general Γ-limit upper bound for the $C_0^{0,\alpha}(\Omega)^*$ norm was first established in [2], the upper bound in all particular cases needed for this paper was proved in [27].

3.3 Remark. We will normally omit the dependence of the energy on Ω, and write $E_\varepsilon(v)$, unless it becomes useful to distinguish between different domains. In such cases we will write $E_\varepsilon(v;\Omega)$. A useful and easy estimate can be derived as follows. Suppose we have two domains Ω_1, Ω_2 and a C^1 diffeomorphism $g : \Omega_1 \to \Omega_2$. Assume also that Jg and Jg^{-1}, the Jacobians for g and g^{-1} respectively, are bounded away from zero in their respective domains. A direct computation shows that there are constants $C_1, C_2 > 0$ that depend only on Jg and $J(g^{-1})$ such that for any $v \in W^{1,2}(\Omega;\mathbb{C})$ and any $\varepsilon > 0$ one has

$$C_1 E_\varepsilon(v;\Omega_1) \le E_\varepsilon(z;\Omega_2) \le C_2 E_\varepsilon(v;\Omega_1). \tag{3.5}$$

Here we are calling $z : \Omega_2 \to \mathbb{C}$ the function defined by $z(y) = v(g(y))$ for $y \in \Omega_2$.

We now present the proof of Proposition 3.1.

Proof of Proposition 3.1. Fix any $\alpha \in (0,1]$, $v \in W^{1,2}(\Omega;\mathbb{C})$ and smooth 1-form B satisfying $B_T = 0$ on $\partial\Omega$. The result (3.1) for Hölder continuous B will follow by density.

Case 1. We start by analyzing the case of $\Omega = B_+(0,1) \equiv B(0,1) \cap \mathbb{R}^3_+$, under the assumption that $B = 0$ in a neighborhood of $\{x \in \partial B_+(0,1) : x_3 > 0\}$. Once we obtain (3.1) in this setting, we will use a partition of unity and a flattening of the boundary to get the result for general smooth domains Ω.

First we define certain reflections of v and B. To this end, for $x = (x_1, x_2, x_3) \in B(0,1)$, set $\tilde{x} = (x_1, x_2, |x_3|)$. Then define

$$\tilde{v}(x) = v(\tilde{x}), \tag{3.6}$$

$$\tilde{B}(x) = \begin{cases} B(x) & \text{for } x \in B_+(0,1) \\ -B_1(\tilde{x})dx_1 - B_2(\tilde{x})dx_2 + B_3(\tilde{x})dx_3 & \text{otherwise} \end{cases} \tag{3.7}$$

It is clear that $\tilde{v} \in W^{1,2}(B(0,1);\mathbb{C})$. On the other hand, since $B_T = 0$ on $\partial\Omega$, B_1 and B_2 vanish on $\{x : x_3 = 0\}$, and this implies that $\tilde{B} \in C_0^{0,\alpha}(B(0,1); \wedge^1(\mathbb{R}^3))$. Now, a straightforward computation shows that if $x_3 < 0$ then $\tilde{B} \wedge J(\tilde{v})(x) = B \wedge J(v)(\tilde{x})$, and as a result,

$$\int_{B_+(0,1)} B \wedge J(v) = \frac{1}{2} \int_{B(0,1)} \tilde{B} \wedge J(\tilde{v}). \tag{3.8}$$

Since $\tilde{B} \in C_c^{0,\alpha}(B(0,1);\mathbb{R}^3)$, we then find that

$$\left| \int_{B_+(0,1)} B \wedge J(v) \right| \leq \frac{1}{2} \left\| \tilde{B} \right\|_{C_0^{0,\alpha}(B(0,1))} \left\| \star J(\tilde{v}) \right\|_{C_0^{0,\alpha}(B(0,1))^*}.$$

By (3.6), (3.7) and Lemma 3.5, one then arrives at the inequality

$$\left| \int_{B_+(0,1)} B \wedge J(v) \right| \leq C \left\| B \right\|_{C^{0,\alpha}(B_+(0,1))} \left(\varepsilon^\gamma + \frac{E_\varepsilon(v; B_+(0,1))}{|\ln \varepsilon|} \right), \tag{3.9}$$

which is (3.1) for the case under consideration.

Case 2. We now consider a general domain Ω with smooth boundary and any $B \in C^\infty(\Omega; \wedge^1(\mathbb{R}^3))$ such that $B_T = 0$ on $\partial\Omega$. Let $\{U_j\}_{j=1}^{n+1}$ be an open cover of $\overline{\Omega}$ with $U_{n+1} \subset\subset \Omega$ and $\partial\Omega \cap U_j$ nonempty for $j = 1, ..., n$. Let $\{\psi_j\}_{j=1}^{n+2}$ be a C^∞ partition of unity subordinate to the open cover of $\mathbb{R}^3$ consisting of $\{U_j\}_{j=1}^{n+1}$ and $\mathbb{R}^3 \setminus \overline{\Omega}$, and such that $\sum_{j=1}^{n+1} \psi_j \equiv 1$ in $\bar{\Omega}$. Assume further that the sets U_j are such that there there exist C^2 diffeomorphisms $g_j : B(0,1) \to U_j$ satisfying the condition $g_j(B_+(0,1)) = U_j \cap \Omega$ for $j =$

$1, ..., n$; this can be arranged, since $\partial\Omega$ is smooth. We can further require the g_j to be such that Jg_j and Jg_j^{-1} are bounded away from zero in $B(0,1)$ and U_j respectively and such that the first and second derivatives of g_j are uniformly bounded, $j = 1, \ldots, n$. Now we compute

$$\int_\Omega B \wedge J(v) = \sum_{j=1}^{n+1} \int_{\Omega\cap U_j} \psi_j B \wedge J(v). \tag{3.10}$$

For the term with $j = n+1$ we can apply Lemma 3.5 to find

$$\left| \int_{U_{n+1}} \psi_{n+1} B \wedge J(v) \right| \leq$$
$$C(\alpha, U_{n+1}) \, \|\psi_{n+1}B\|_{C_0^{0,\alpha}(U_{n+1})} \left\{ \varepsilon^\gamma + \frac{E_\varepsilon(v; U_{n+1})}{|\ln \varepsilon|} \right\} \tag{3.11}$$

since $\psi_{n+1}B$ is compactly supported within $U_{n+1} \subset \Omega$.

Turning to the terms with $j = 1, ..., n$, we fix any one j and reason as follows. For convenience, we will suppress the subscript j below and write for instance g for g_j, etc. We first define $z : B_+(0,1) \to \mathbb{C}$ by $z(y) = v(g(y))$. Writing $\Omega \cap U$ as $g(B_+(0,1))$, we recall that $J(v) = v^\#(dx)$, so that

$$\begin{aligned} \int_{g(B_+(0,1))} \psi B \wedge J(v) &= \int_{B_+(0,1)} g^\#(\psi B \wedge J(v)) \\ &= \int_{B_+(0,1)} g^\#(\psi B) \wedge J(z) \end{aligned} \tag{3.12}$$

since $g^\#(\psi B \wedge J(v)) = g^\#(\psi B) \wedge g^\# J(v)$ and $g^\# J(v) = g^\# v^\#(dx) = (v \circ g)^\#(dx) = J(z)$. Here dx denotes the standard area 2-form on $\mathbb{C}$.

We now claim $(g^\#(\psi B))_T = 0$ on the flat part of $\partial B_+(0,1)$. To see this, let $i_{\partial B_+} : \partial B_+(0,1) \to \overline{B_+(0,1)}$ denote the natural injection, and similarly $i_{\partial\Omega} : \partial\Omega \to \overline{\Omega}$. Recall that $(g^\#(\psi B))_T = i^\#_{\partial B_+} g^\#(\psi B)$. Since $g \circ i_{\partial B_+} = i_{\partial\Omega} \circ g$,

$$i^\#_{\partial B_+} g^\#(\psi B) = (g \circ i_{\partial B_+})^\#(\psi B) = (i_{\partial\Omega} \circ g)^\#(\psi B) = g^\# i^\#_{\partial\Omega}(\psi B) = g^\#(\psi B)_T.$$

However $(\psi B)_T = \psi B_T = 0$ by assumption, so our claim is established.

The additional fact that ψ is compactly supported in U allows us to apply (3.9) to (3.12) to obtain

$$\begin{aligned} \left| \int_{\Omega\cap U} \psi B \wedge J(v) \right| &\leq C \left\| g^\#(\psi B) \right\|_{C^{0,\alpha}(B_+(0,1))} \left(\frac{E_\varepsilon(z; B_+(0,1))}{|\ln \varepsilon|} + \varepsilon^\gamma \right) \\ &\leq C \left\| \psi B \right\|_{C^{0,\alpha}(\Omega\cap U)} \left(\frac{E_\varepsilon(v; \Omega \cap U)}{|\ln \varepsilon|} + \varepsilon^\gamma \right), \end{aligned} \tag{3.13}$$

where the last inequality follows by (3.5) and the assumed smoothness of g. Going back to (3.10) and using (3.11) we conclude that (3.1) is valid for all $B \in C^\infty(\Omega; \mathbb{R}^3)$ such that $B_T = 0$ on $\partial\Omega$. $\square$

We now prove the other main result of this section, the extension of the Γ-limit result of [16] and [2]. We employ arguments similar to those in the proof of Proposition 3.1 above.

Proof of Proposition 3.2. First note that in view of [16], Theorem 5.2 (see also [2]), whenever $\{v_\varepsilon\} \subset W^{1,2}(\Omega;\mathbb{C})$ is a sequence satisfying $E_\varepsilon(v_\varepsilon) \leq C|\ln\varepsilon|$, we can conclude that there exists a sequence ε_k and a rectifiable 1-current J satisfying all conclusions of Proposition 3.2 apart from (3.2), and such that $\|\star J(v_{\varepsilon_k}) - J\|_{C_0^{0,\alpha}(\Omega)^*} \to 0$ as $k \to \infty$. We must prove that (3.2) holds as well. In both cases we consider below we will assume that such a sequence ε_k and limiting current J have been selected.

Case 1. We again start by considering $\Omega = B_+(0,1)$. For every k we define $\tilde{v}_{\varepsilon_k} : B(0,1) \to \mathbb{C}$ by reflection, as in the proof of Proposition 3.1. Then $E_{\varepsilon_k}(\tilde{v}_{\varepsilon_k}; B(0,1)) \leq C|\ln\varepsilon_k|$, so again appealing to results of [16], [2], we can assume, after passing if necessary to a further subsequence (still labelled ε_k), that there exists a rectifiable 1-current $\tilde{J}$ in $B(0,1)$ such that

$$\left\|\star J(\tilde{v}_{\varepsilon_k}) - \tilde{J}\right\|_{C_0^{0,\alpha}(B(0,1))^*} \to 0 \quad \text{as } k \to \infty.$$

Define

$$\mathcal{A} := \big\{B \in C_T^{0,\alpha}(B_+(0,1); \wedge^1(\mathbb{R}^3)) \ \ : \ \ B = 0 \text{ near } \partial B_+(0,1) \cap \{x_3 > 0\}, \\ \|B\|_{C_T^{0,\alpha}(B_+(0,1))} \leq 1\big\}.$$

Given $B \in \mathcal{A}$, let $\tilde{B}$ denote the extension of B to a compactly supported 1-form on $B(0,1)$ defined in the proof of Proposition 3.1 We claim that

$$\tilde{J}(\tilde{B}) = 2J(B). \tag{3.14}$$

If B has compact support in $B_+(0,1)$, this follows directly from letting $k \to \infty$ in the identity $\star J(\tilde{v}_{\varepsilon_k})(\tilde{B}) = 2 \star J(v_{\varepsilon_k})(B)$, see (3.8). For general $B \in \mathcal{A}$, consider a sequence $\{\chi_k\}$ of smooth functions compactly supported in $B_+(0,1)$ that increase to the characteristic function of $B_+(0,1)$. Then applying (3.14) to the sequence $\{\chi_k B\}$ we have

$$2J(\chi_k B) = \tilde{J}(\tilde{\chi}_k \tilde{B}) = \tilde{J}(\tilde{B}) - \tilde{J}((1-\tilde{\chi}_k)\tilde{B}). \tag{3.15}$$

Now $J \in \mathcal{R}_1(B_+(0,1))$, so $2J(\chi_k B) \to 2J(B)$ as $k \to \infty$. Then, by representing $\tilde{J}$ as in (2.3), note that

$$\lim_{k\to\infty} \tilde{J}((1-\tilde{\chi}_k)\tilde{B}) = \int_{\Gamma_0} \langle \tilde{B}(x), \tau(x)\rangle \, m(x) \, dH^{(1)}(x) \tag{3.16}$$

where $\Gamma_0 := \text{spt } \tilde{J} \cap \{x : x_3 = 0\}$. We claim that this integral must vanish. To see this, note that Γ_0 is itself 1-rectifiable, so if $H^{(1)}(\Gamma_0) > 0$ then Γ_0 can be covered, up to a set of H^1-measure zero, by a countable union of C^1 curves, $\{\Gamma_j\}$, with each Γ_j contained in $\{x_3 = 0\}$. Away from a set of H^1-measure zero, the approximate tangent to Γ_0 at a point in $\Gamma_0 \cap \Gamma_j$ equals the tangent to Γ_j and so necessarily is tangent to the plane. Since $\tilde{B} = B_3 dx_3$ on the set $\{x : x_3 = 0\}$, it follows that $\langle \tau(x), \tilde{B}(x) \rangle = 0$, $H^{(1)}$ a.e. in Γ_0. Consequently, we may pass to the limit in (3.15) to obtain (3.14) for any $B \in \mathcal{A}$.

Since $\left\| \tilde{B} \right\|_{C_0^{0,\alpha}(B(0,1))} \leq 1$ for all $B \in \mathcal{A}$, we can deduce from (3.14) that

$$\begin{aligned} \sup_{B \in \mathcal{A}} (\star J(v_{\varepsilon_k}) - J)(B) &= \frac{1}{2} \sup_{B \in \mathcal{A}} (\star J(\tilde{v}_{\varepsilon_k}) - \tilde{J}))(\tilde{B}) \\ &\leq \left\| \star J(\tilde{v}_{\varepsilon_k}) - \tilde{J} \right\|_{C_0^{0,\alpha}(B(0,1))^*} \to 0 \qquad (3.17) \end{aligned}$$

as $k \to \infty$.

Case 2 For a general domain Ω with C^2 boundary, fix an open cover $\{U_j\}_{j=1}^{n+1}$ of $\bar{\Omega}$, a partition of unity $\{\psi_j\}_{j=1}^{n+1}$, and diffeomorphisms $g_j : B(0,1) \to U_j$ satisfying the same conditions as in the proof of Proposition 3.1. For any $B \in C_T^{0,\alpha}(\Omega)$,

$$(\star J(v_{\varepsilon_k}) - J)(B) = \sum_{j=1}^{n+1} (\star J(v_{\varepsilon_k}) - J)(\psi_j B)$$

and so it suffices to show that

$$\lim_{k \to \infty} \sup_{\|B\|_{C_T^{0,\alpha}} \leq 1} (\star J(v_{\varepsilon_k}) - J)(\psi_j B) \to 0$$

for every j. For $j = n+1$ this is immediate, since $\psi_{n+1} B$ has compact support in Ω and satisfies $\|\psi_{n+1} B\|_{C_0^{0,\alpha}} \leq C(\Omega) \|B\|_{C_T^{0,\alpha}}$. We therefore fix some $j \leq n$ and for notational simplicity, we drop the subscripts and write for example ψ instead of ψ_j. Let T denote the current on $B_+(0,1)$ characterized by

$$T(g^\# \omega) = J(\omega) \qquad (3.18)$$

for 1-forms ω on $U \cap \Omega$ with compact support. Since g is a diffeomorphism, T is a well-defined current on $B_+(0,1)$; in fact T is the image of J under g^{-1}, that is $T = (g^{-1})_\# J$, defined by $T(\phi) = J((g^{-1})^\# \phi)$ for 1-forms ϕ on $B_+(0,1)$.

We also define $z_\varepsilon := v_\varepsilon \circ g$, and by using (3.18) and arguing as in Proposition 3.1 we obtain

$$(\star J(z_{\varepsilon_k}) - T)(g^\#(\psi B)) = (\star J(v_{\varepsilon_k}) - J)(\psi B)$$

for all $B \in C_T^{0,\alpha}$. Since $\star J(v_{\varepsilon_k}) \to J$ in $C_0^{0,\alpha}(\Omega)^*$, one readily verifies that $\star J(z_{\varepsilon_k}) \to T$ in $C_0^{0,\alpha}(B_+(0,1))^*$. In addition, as seen in Proposition 3.1, the tangential part of $g^{\#}(\psi B)$ vanishes on the flat part of $\partial B_+(0,1)$, and so there exists some constant $C(\Omega)$, such that $C(\Omega)^{-1} g^{\#}(\psi B) \in \mathcal{A}$. Using (3.5) as in Proposition 3.1 to bound $E_{\varepsilon_k}(z_{\varepsilon_k})$, we can then deduce from (3.17) that

$$\sup_{\|B\|_{C_T^{0,\alpha}(\Omega)} \leq 1} (\star J(v_{\varepsilon_k}) - J)(\psi B) \leq C(\Omega) \sup_{\omega \in \mathcal{A}} (\star J(z_{\varepsilon_k}) - T)(\omega) \to 0$$

as $k \to \infty$, and this completes the proof of (3.2).

Finally, given a 1-current J such that $\frac{1}{\pi} J$ is integer multiplicity rectifiable and $\partial J = 0$ relative to Ω, Theorem 1.1 of [2] establishes the existence of a sequence $\{v_\varepsilon\}$ such that $\lim_{k\to\infty} \|\star J(v_\varepsilon) - J\|_{C_0^{0,\alpha}(\Omega)} = 0$ and $\lim |\ln \varepsilon|^{-1} E_\varepsilon(v_\varepsilon) = M(J)$. It follows from the compactness results established above that in fact $\star J(v_\varepsilon) \to J$ in the stronger $C_T^{0,\alpha}(\Omega)^*$ norm, and this proves (3.4). $\square$

We next give the version of Proposition 3.1 we will use later.

3.4 Corollary. *Let $\alpha \in (0,1]$ and $\delta > 0$ be given. Then there exist constants $C(\alpha, \Omega, \delta) > 0$ and $\varepsilon_0(\alpha, \Omega, \delta) > 0$ such that, for any $v \in W^{1,2}(\Omega; \mathbb{C})$ satisfying $\|\star J(v)\|_{C_T^{0,\alpha}(\Omega)^*} \geq \delta$, any $\varepsilon \in (0,1)$, we have*

$$\|\star J(v)\|_{C_T^{0,\alpha}(\Omega)^*} \leq C(\alpha, \Omega, \delta) \frac{E_\varepsilon(v)}{|\ln \varepsilon|} \tag{3.19}$$

Proof. Fixing any $\delta > 0$ and any $\alpha \in (0,1]$, inequality (3.19) follows from Proposition 3.1 once we obtain a lower bound on the quantity $\frac{E_\varepsilon(v)}{|\ln \varepsilon|}$ based on the assumption $\|\star J(v)\|_{C_T^{0,\alpha}(\Omega)^*} \geq \delta$. That is, we assert the existence of positive numbers $\varepsilon_0(\delta)$ and $\gamma(\delta)$ such that

$$\frac{1}{|\ln \varepsilon|} E_\varepsilon(v) \geq \gamma(\delta)$$

for any $v \in W^{1,2}(\Omega; \mathbb{C})$ with $\|\star J(v)\|_{C_T^{0,\alpha}(\Omega)^*} \geq \delta$ and any $0 < \varepsilon \leq \varepsilon_0(\delta)$. Were this not the case, we could find sequences $\gamma_n \to 0$, $\varepsilon_n \to 0$ and $v_n \in W^{1,2}(\Omega; \mathbb{C})$ with $\|\star J(v_n)\|_{C_T^{0,\alpha}(\Omega)^*} \geq \delta$ satisfying

$$\frac{1}{|\ln \varepsilon_n|} E_{\varepsilon_n}(v_n) \leq \gamma_n.$$

But then, by Proposition 3.2 (cf. (3.2)-(3.3)), we have $\star J(v_n) \to J$ in $(C_T^{0,\alpha})^*$, and necessarily $J = 0$. On the other hand $\delta \leq \|\star J(v_n)\|_{C_T^{0,\alpha}(\Omega)^*} \to \|J\|_{C_T^{0,\alpha}(\Omega)^*}$, which is a contradiction. $\square$

We end this section with the Lemma used in the proof of Proposition 3.1:

3.5 Lemma. *Let* $\Omega \subset \mathbb{R}^3$, $\alpha \in (0,1]$ *be as above. Then there are constants* $\gamma > 0$ *and* $C(\alpha,\Omega) > 0$ *such that for any* $v \in W^{1,2}(\Omega;\mathbb{C})$ *and any* $\varepsilon \in (0,1)$, *one has*

$$\| \star J(v)\|_{C_0^{0,\alpha}(\Omega)^*} \le C(\alpha,\Omega)\left(\varepsilon^\gamma + \frac{E_\varepsilon(v)}{|\ln\varepsilon|}\right). \tag{3.20}$$

In the proof of this lemma we use the $C_0^0(\Omega)^*$ dual norm on currents, where C_0^0 denotes the sup norm on continuous k-forms that vanish on $\partial\Omega$; note that this is just the mass norm.

Proof. We fix an $\alpha \in (0,1]$, $v \in W^{1,2}(\Omega;\mathbb{C})$ and a $B \in C_0^\infty(\Omega;\wedge^1(\mathbb{R}^3))$. The result will follow for $B \in C_0^{0,\alpha}(\Omega;\wedge^1(\mathbb{R}^3))$ by density. It suffices to prove (3.20) for B of the form $B = B_3 dx_3$ say, since the same arguments will establish the result for $B_1 dx_1$ or $B_2 dx_2$, and the general case then follows by linearity.

We start by noting if $B = B_3 dx_3$ and $v = v_1 + iv_2$ then

$$B \wedge J(v) = B_3(u_{1,x_1}u_{2,x_2} - u_{2,x_1}u_{1,x_2})dx$$

where the quantity in parentheses is simply the two-dimensional Jacobian of the restriction of v to the plane $x_3 = const.$ In light of this observation, we can integrate the inequality of Theorem 2.1 of [16] with respect to x_3 to obtain

$$\left|\int_\Omega B \wedge J(v)\right| \le C(\Omega)\left\{\frac{E_\varepsilon(v)}{|\ln\varepsilon|}\|B\|_\infty + C_\varepsilon \varepsilon^\beta \|\nabla B\|_\infty\right\} \tag{3.21}$$

for some $\beta \in (0,1]$. Here

$$C_\varepsilon = \varepsilon^\gamma + \frac{E_\varepsilon(v)}{|\ln\varepsilon|} \qquad \text{for some } \gamma \in (0,1]. \tag{3.22}$$

We now appeal to Proposition 3.2 of [16]. Following the proof of this proposition, but using (3.21) to keep explicit track of the constants arising in the estimates, we get a decomposition of $\star J(v) = J_0 + J_1$ where J_0 and J_1 are two 1-currents satisfying

$$\|J_0\|_{C_0^0(\Omega)^*} \le C(\Omega)C_\varepsilon \quad \text{and} \quad \|J_1\|_{C_0^{0,1}(\Omega)^*} \le C(\Omega)C_\varepsilon\varepsilon^\beta. \tag{3.23}$$

Consequently, we also have

$$\|J_0\|_{C_0^{0,1}(\Omega)^*} \le C(\Omega)C_\varepsilon \tag{3.24}$$

and

$$\begin{aligned}\|J_1\|_{C_0^0(\Omega)^*} &\leq \|J_0\|_{C_0^0(\Omega)^*} + \|\star J(v)\|_{C_0^0(\Omega)^*} \\ &\leq C\left\{C_\varepsilon + \int_\Omega |\nabla v|^2 \, dx\right\} \\ &\leq C\left\{C_\varepsilon + E_\varepsilon(v)\right\} \\ &\leq C \cdot C_\varepsilon \left\{1 + |\ln \varepsilon|\right\}. \end{aligned} \tag{3.25}$$

Then from an interpolation result (cf. [16], Lemma 3.3), one finds from (3.23), (3.24) and (3.25) that

$$\|J_0\|_{C_0^{0,\alpha}(\Omega)^*} \leq C\left(\|J_0\|_{C_0^0(\Omega)^*}\right)^{1-\alpha}\left(\|J_0\|_{C_0^{0,1}(\Omega)^*}\right)^{\alpha} \leq C \cdot C_\varepsilon$$

and

$$\|J_1\|_{C_0^{0,\alpha}(\Omega)^*} \leq C\left(\|J_1\|_{C_0^0(\Omega)^*}\right)^{1-\alpha}\left(\|J_1\|_{C_0^{0,1}(\Omega)^*}\right)^{\alpha} \leq C \cdot C_\varepsilon(1 + |\ln \varepsilon|)^{1-\alpha}\varepsilon^{\alpha\beta}.$$

Then combining these last two inequalities, it follows that for some $\varepsilon_0 > 0$ one has

$$\|\star J(v)\|_{C_0^{0,\alpha}(\Omega)^*} \leq \|J_0\|_{C_0^{0,\alpha}(\Omega)^*} + \|J_1\|_{C_0^{0,\alpha}(\Omega)^*} \leq C \cdot C_\varepsilon$$

for all $\varepsilon \in (0, \varepsilon_0)$. In light of (3.22), we obtain the desired conclusion. □

3.6 Remark. Proposition 3.1 and Proposition 3.2 remain valid in arbitrary dimensions $n \geq 3$, with essentially the same proof. In the general version of Proposition 3.2, the limiting current J is $n-2$ rectifiable. The proofs differ only in that one would define $\tilde{x} = (x_1, \ldots, x_{n-1}, |x_n|)$ and for $B = \sum_{1 \leq \alpha_1 < \cdots < \alpha_{n-2} \leq n} B_\alpha dx_\alpha$,

$$\tilde{B}(x) = -\sum_{\alpha_{n-2} < n} B_\alpha(\tilde{x}) dx_\alpha + \sum_{\alpha_{n-2} = n} B_\alpha(\tilde{x}) dx_\alpha$$

when $x_n < 0$.

3.2 Necessity of the condition $B_T = 0$ on $\partial\Omega$

In this section we have extended the bounds and compactness results of [16] to the case where the Jacobian acts on 1-forms that have no tangential part on the boundary. We conclude this section with an example to show that these results cannot be extended to the case where forms are allowed to have nonzero tangential part.

We first construct an example in a two-dimensional domain, and then use it to give similar examples in three and more dimensions. Note that for a map $u : \mathbb{R}^2 \to \mathbb{C}$, $Ju = du^1 \wedge du^2$ is a two-form on $\mathbb{R}^2$, so that $\star J(u)$ is a 0-form, that is a distribution acting on scalar functions. Hence, in two dimensions, the assumption $B_T = 0$ reduces simply to $B = 0$ on $\partial\Omega$ for a scalar function B.

For our example in the plane, let $\Omega = B_+(0,1) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0, |x| < 1\}$. More general domains can be reduced to this situation by a suitable diffeomorphism.

The example will be built out of scalings and translates of a function w that we now define. First, denote by p the point $(0,1) \in \mathbb{R}^2$, and let $w \in W^{1,2}(\mathbb{R}^2_+; \mathbb{C})$ be a function with the following properties:

$$|w| = 1 \text{ in } \mathbb{R}^2_+ \setminus B(p,1), \qquad w \equiv e^{i\beta} \quad \text{for some scalar } \beta \text{ in } \mathbb{R}^2_+ \setminus B(0,3),$$

$$w(p) = 0, \quad w(x) \neq 0 \text{ for } x \neq p,$$

$$\deg(w; \partial V) = 1 \text{ for any smooth open set } V \text{ such that } p \in V$$

For instance, an explicit example of such a function w is easily built using polar coordinates centered at p via $w(re^{i\theta}) = \rho(r)e^{i\phi(\theta)}$, where for example, $\rho = \min\{r, 1\}$, and ϕ is defined by

$$\phi(\theta) = \begin{cases} 4\theta & \text{if } -3\pi/4 \leq \theta \leq -\pi/4 \\ -\pi & \text{if not} \end{cases}$$

See Figure 1.

It is useful here to distinguish between the two-form Jw and the function $\eta := \det \nabla w$, where here we consider ∇w to be a 2×2 matrix. Thus $Jw = \eta \ dx_1 \wedge dx_2$. We claim that η is supported in $B(p,1)$ and that $\int Jw = \int \eta dx_1 \wedge dx_2 = \pi$. Indeed, outside of $B(p,1)$, we can differentiate the relation $|u|^2 = 1$ to find that $u^i_{x_j} u^i = 0$, which shows that u is an eigenvector of ∇u with eigenvalue 0, and hence that $\eta = \det \nabla u = 0$. Then writing w as $w = \rho e^{i\phi}$ (for ϕ multivalued) we can compute

$$\int_{B(p,1)} Jw = \frac{1}{2} \int_{B(p,1)} dj(u) = \frac{1}{2} \int_{\partial B(p,1)} j(u) = \frac{1}{2} \int_{\partial B(p,1)} \nabla\phi \cdot \tau = \pi$$

using the assumption on the degree of w.

Next define $w_\varepsilon(x_1, x_2) = w(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})$. Note that by a change of variables,

$$\begin{aligned}\int_{\mathbb{R}^2_+} \frac{1}{2}|\nabla w_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(|w_\varepsilon|^2 - 1)^2 \, dx &= \int_{\mathbb{R}^2_+} \frac{1}{2}|\nabla w|^2 + \frac{1}{4}(|w|^2 - 1)^2 \, dx \\ &= \int_{\mathbb{R}^2_+ \cap B(0,3)} \frac{1}{2}|\nabla w|^2 + \frac{1}{4}(|w|^2 - 1)^2 \, dx \\ &= \text{const.}\end{aligned}$$

for all $\varepsilon > 0$. Also, from the chain rule, $Jw_\varepsilon = \eta_\varepsilon dx_1 \wedge dx_2$, where $\eta_\varepsilon = \varepsilon^{-2}\eta(x/\varepsilon)$. Note that $\{\frac{1}{\pi}\eta_\varepsilon\}_{\varepsilon\in(0,1]}$ forms an approximate identity, and also that, since η is supported in $B(\varepsilon p, \varepsilon) \subset \mathbb{R}^2_+$, for any scalar function ζ such that $\|\zeta\|_{C^{0,1}} \leq 1$, one has

$$\begin{aligned}\int_{\mathbb{R}^2_+} \zeta J w_\varepsilon &= \int_{B(\varepsilon p,\varepsilon)} \zeta \eta_\varepsilon dx_1 dx_2 \\ &= \pi\zeta(0) + \int_{B(\varepsilon p,\varepsilon)} (\zeta - \zeta(0))\eta_\varepsilon dx_1 dx_2 \\ &= \pi\zeta(0) + O(\varepsilon).\end{aligned}$$

Thus, if we allow test functions that do not vanish on $\partial B_+(0,1)$, we observe that

$$\sup_{\|\zeta\|_{C^{0,1}(B_+(0,1))}\leq 1} \left(\int_{B_+(0,1)} \zeta J w_\varepsilon\right) \approx \int_{B_+(0,1)} e_\varepsilon(w_\varepsilon)\, dx,$$

(where we have denoted $e_\varepsilon(w) := \frac{1}{2}|\nabla w_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(|w_\varepsilon|^2 - 1)^2$), and there is no gain of a factor of $|\ln\varepsilon|$.

As a result, uniform energy bounds of the sort $\int_{B_+(0,1)} e_\varepsilon(w_\varepsilon) \leq C|\ln\varepsilon|$ do not imply any sort of compactness for Jw_ε, when acting on test functions ζ as above. To see this, let $k_\varepsilon := \lfloor |\ln\varepsilon| \rfloor$ (where $\lfloor x \rfloor$ denotes the integer part of x), and let

$$s_i^\varepsilon := \frac{1}{4} + \frac{i}{2k_\varepsilon}$$

for $i = 0, \ldots, k_\varepsilon$, and define

$$W_\varepsilon(x_1, x_2) := \prod_{i=0}^{k_\varepsilon} w_\varepsilon(x_1 - s_i^\varepsilon, x_2). \tag{3.26}$$

Then W_ε has $k_\varepsilon \approx |\ln\varepsilon|$ vortices, all of them centered a distance ε above the flat part of $\partial B_+(0,1)$, and evenly spaced between $x_1 = \frac{1}{4}$ and $x_1 = \frac{3}{4}$. (This interval along the x_1 axis is chosen for convenience in our later construction of examples in higher dimensions.)

Each factor in the product that defines W_ε has nontrivial behavior in a ball $B_+((s_i^\varepsilon,0),3\varepsilon)$. Since these balls are pairwise disjoint, it is easy to see that check that

$$\int_{B_+(0,1)} e_\varepsilon(W_\varepsilon)\,dx \;=\; \sum_{i=0}^{k_\varepsilon}\int_{B_+((s_i^\varepsilon,0),3\varepsilon)} e_\varepsilon(W_\varepsilon) = Ck_\varepsilon \le C\,|\ln\varepsilon|\,.$$

Also, $JW_\varepsilon = \sum_{i=0}^{k_\varepsilon}\eta_\varepsilon(x_1-s_i^\varepsilon,x_2)dx_1\wedge dx_2$, and so we have

$$\begin{aligned}
&\sup_{\|\zeta\|_{C^{0,1}(B_+(0,1))}\le 1}\int_{B_+(0,1)}\zeta\, JW_\varepsilon\;dx =\\
&\sup_{\|\zeta\|_{C^{0,1}(B_+(0,1))}\le 1}\sum_{i=0}^{k_\varepsilon}\int_{B_+((s_i^\varepsilon,\varepsilon),\varepsilon)}\zeta(x_1,x_2)\;\eta_\varepsilon(x_1-s_i^\varepsilon,x_2)dx_1dx_2 =\\
&\sup_{\|\zeta\|_{C^{0,1}(B_+(0,1))}\le 1}\pi\sum_{i=0}^{k_\varepsilon}\zeta(s_i^\varepsilon,0)+O(\varepsilon|\ln\varepsilon|)\ge\frac{\pi}{2}\,|\ln\varepsilon|\,.
\end{aligned}$$

We now demonstrate how to extend this counter-example to the three-dimensional setting. Our example will be constructed on the domain $\Omega = B_+(0,1) := \{x\in\mathbb{R}^3 : |x|<1, x_3>0\}$.

It is convenient to use cylindrical coordinates, so we define

$$U_\varepsilon(r\cos\theta, r\sin\theta, z) := W_\varepsilon(r,z)$$

where W_ε is the two-dimensional example defined by (3.26). Hence, U_ε has a zero set consisting of k_ε circles of radius s_i^ε, $i=1,\ldots,k_\varepsilon$, all centered at $(0,0,\varepsilon)$, lying parallel to the x_1x_2-plane. See Figure 1.

Then

$$\int_{B_+(0,1)} e_\varepsilon(U_\varepsilon)\,dx \;=\; \int_0^1\int_0^{2\pi}\int_0^{\sqrt{1-r^2}} e_\varepsilon(W_\varepsilon(r,z))r\,dz\,d\theta\,dr \le C|\ln\varepsilon|.$$

Moreover, since we can write

$$U_\varepsilon = W_\varepsilon\circ q \quad \text{ for } q((x_1,x_2,x_3)) \;=\; ((x_1^2+x_2^2)^{1/2},x_3) = (r,z),$$

we have (writing dy for the standard volume 2-form on $\mathbb{C}$):

$$JU_\varepsilon = U_\varepsilon^{\#}(dy) = q^{\#}W_\varepsilon^{\#}(dy) = q^{\#}(JW_\varepsilon) \;=\; \sum_{i=0}^{k_\varepsilon}\eta_\varepsilon(r-s_i^\varepsilon,z)dr\wedge dz.$$

Thus if we define $B = x_1 dx_2 - x_2 dx_1 = r^2 d\theta$, since $r\ dr\ d\theta dz$ equals the standard volume form dx on $\mathbb{R}^3$, it follows that

$$
\begin{aligned}
\int_{B_+(0,1)} B \wedge JU_\varepsilon &= \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{1-r^2}} \sum_{i=0}^{k_\varepsilon} r\eta_\varepsilon(r - s_i^\varepsilon, z) r dz\ d\theta\, dr \\
&= \sum_{i=0}^{k_\varepsilon} 2\pi^2 s_i^\varepsilon + O(\varepsilon |\ln \varepsilon|) \geq \frac{\pi^2}{4} |\ln \varepsilon| .
\end{aligned}
$$

Hence, again we see that a logarithmic bound on the energy E_ε does not induce a uniform bound on the Jacobian if we allow $B_T \neq 0$ on the boundary. See Figure 1. The example above can be readily extended to dimensions larger than three as well.

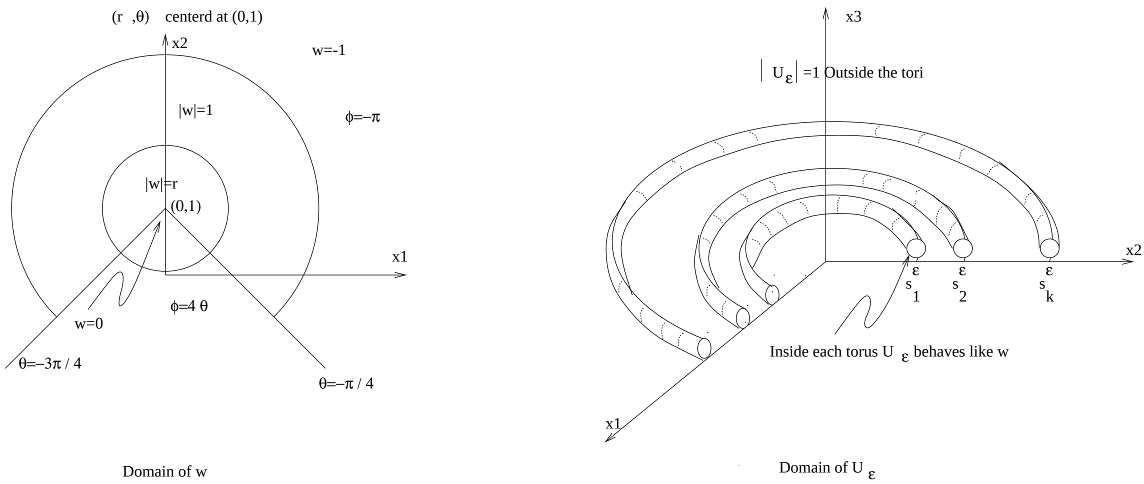

Figure 1: The functions w and U_ε

4 Existence of local minimizers

In this section we present our main result on existence of local minimizers to G_ε, based upon the asymptotic connection between the Ginzburg-Landau energy and the length of vortices as laid out in the previous section.

Up to now, we have taken $\Omega \subset \mathbb{R}^3$ to be any smooth, bounded, simply connected domain but now we make additional assumptions that will guarantee the existence of line segments serving as isolated local minimizers of the Γ-limit; that is, serving as local minimizers of length among curves with endpoints on the boundary. Specifically, we assume that for some positive integer N, there exist lines $l_1, l_2, \ldots, l_N$ and a positive number R such that the collection of infinite solid cylinders $\{\mathcal{C}_{R,j}\}_{j=1}^{N}$ with axis l_j and radius R satisfy the following conditions:

$$\mathcal{C}_{R,j} \cap \Omega \text{ has only one component,} \tag{4.1}$$

$$\mathcal{C}_{R,j} \cap \mathcal{C}_{R,k} \cap \Omega = \emptyset \text{ for all } j \neq k, \tag{4.2}$$

and in a coordinate system where the x_3-axis coincides with l_j one has

$$\mathcal{C}_{R,j} \cap \Omega = \{(x_1, x_2, x_3) : \, x_1^2 + x_2^2 < R^2, \, z_1^j(x_1, x_2) < x_3 < L_j + z_2^j(x_1, x_2)\}, \tag{4.3}$$

for Lipschitz functions z_1^j and z_2^j satisfying $z_1^j(0,0) = z_2^j(0,0) = 0$. We have introduced here the notation $L_j = H^{(1)}(l_j \cap \Omega)$. Condition (4.3) should be viewed as saying that l_j meets $\partial\Omega$ transversely.

In order to establish the existence of local minimizers we must further assume that the collection of line segments $\{l_j \cap \Omega\}$ locally minimizes length. To state this assumption more precisely, for each $j \in \{1, 2, \ldots, N\}$ let C_{R,L_j} denote the open solid cylinder of radius R and height L_j with axis consisting of $l_j \cap \Omega$. Let $a_j, b_j \in \partial\Omega$ denote the endpoints of the segments $l_j \cap \Omega$. Then we assume that for each j we have:

$$C_{R,L_j} \subset \Omega \text{ and } \bar{C}_{R,L_j} \cap \partial\Omega = \{a_j, b_j\}, \tag{4.4}$$

where $\bar{\cdot}$ denotes closure.

A crucial step in our approach is the contention that the union of oriented line segments joining the points a_j and b_j with arbitrarily assigned multiplicities m_j, viewed as a 1-current, is a local minimizer of mass in the $C_T^{0,1}(\Omega)^*$-topology among appropriate competitors in $\mathcal{R}_1(\Omega)$. This was accomplished in [27] in the topology $C_0^{0,1}(\Omega)^*$ which implies that the result holds in the topology $C_T^{0,1}(\Omega)^*$ as well. To state this precisely, for any $\alpha = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}^N$ we denote by T_j the above-mentioned multiplicity m_j 1-current supported on $l_j \cap \Omega$ and let $T_\alpha = \sum_{j=1}^{N} T_j$.

4.1 Proposition. *(cf. [27], Thm. 4.5) Assume a bounded, open domain Ω satisfies (4.1)–(4.4) for all $j \in \{1, 2, \ldots, N\}$ where N is any positive integer. For any $\alpha \in \mathbb{Z}^N$, let $T_\alpha \in \mathcal{R}_1(\Omega)$ be defined as above. Then there exists a positive number $\delta_0 = \delta_0(\alpha, L_1, \ldots, L_N, R)$ such that for all $T \in \mathcal{R}_1(\Omega)$ with $\partial T = 0$ relative to Ω one has*

$$0 < \|T - T_\alpha\|_{C_T^{0,1}(\Omega)^*} \leq \delta \implies M(T) > M(T_\alpha). \tag{4.5}$$

We are now in a position to state and prove our existence result for G_ε. We assume that for any $\varepsilon > 0$, $H_{ap}^\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies the condition

$$\limsup_{\varepsilon\to 0} \frac{1}{|\ln\varepsilon|^2} \int_\Omega \left|H_{ap}^\varepsilon\right|^2 \, dx = 0. \tag{4.6}$$

Recall that since H_{ap}^ε is divergence-free, there exists a divergence-free potential, denoted by A_{ap}^ε, in the sense that $H_{ap}^\varepsilon = \nabla \times A_{ap}^\varepsilon$ in $\mathbb{R}^3$ where we take A_{ap}^ε to satisfy (2.14). Then we can establish the following result:

4.2 Theorem. *Assume $\Omega \subset \mathbb{R}^3$ is a bounded, smooth, simply connected domain satisfying (4.1)–(4.4) for some positive integer N. Let $\alpha = (m_1, m_2, \ldots, m_N)$ be any element of $\mathbb{Z}^N$ and let T_α be the locally minimizing 1-current from Proposition 4.1. Assume $\{H_{ap}^\varepsilon\}$ satisfies (4.6). Then there exists an $\varepsilon_0 > 0$ and an open set $\mathcal{O} \subset W^{1,2}(\Omega;\mathbb{C}) \times \mathcal{H}_0$ such that, for each $\varepsilon < \varepsilon_0$, there exists $(U_\varepsilon, A_\varepsilon) \in \mathcal{O}$ satisfying*

$$G_\varepsilon(U_\varepsilon, A_\varepsilon + A_{ap}^\varepsilon) \le G_\varepsilon(u, A + A_{ap}^\varepsilon) \tag{4.7}$$

for all $(u, A) \in \mathcal{O}$. Furthermore, one has

$$\lim_{\varepsilon\to 0} \|\star J(U_\varepsilon) - \pi T_\alpha\|_{C_T^{0,1}(\Omega)^*} = 0. \tag{4.8}$$

Finally, if H_{ap} is independent of ε, then A_ε converges in $\mathcal{H}_0$ to a limit $A_0 \in \mathcal{H}_0$ which satisfies

$$\frac{1}{2}\int_\Omega \mathbb{P}(A_0 + A_{ap}) \cdot \mathbb{P}(A) + \frac{1}{2}\int_{\mathbb{R}^3} \nabla \times A_0 \cdot \nabla \times A = 2\pi T_\alpha(B) \tag{4.9}$$

for all $A \in \mathcal{H}_0$, where B denotes the unique solution of $\star dB = A$ in Ω, $B_T = 0$ on $\partial\Omega$.

4.3 Remark. If one assume smoothness of H_{ap}^ε, then the condition of local minimality (4.7) along with standard elliptic regularity imply that in particular, $(U_\varepsilon, A_\varepsilon)$ constitute classical solutions to the Ginzburg-Landau system:

$$\left(\nabla - i(A_\varepsilon + A_{ap}^\varepsilon)\right)^2 U_\varepsilon = \frac{1}{\varepsilon^2}(|U_\varepsilon|^2 - 1)U_\varepsilon \quad \text{in } \Omega, \tag{4.10}$$

$$\nabla \times \nabla \times A_\varepsilon = \begin{cases} \frac{i}{2}(\bar{U}_\varepsilon \nabla U_\varepsilon - U_\varepsilon \nabla \bar{U}_\varepsilon) - |U_\varepsilon|^2 (A_\varepsilon + A_{ap}^\varepsilon) & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^3 \setminus \bar{\Omega}, \end{cases} \tag{4.11}$$

along with the boundary condition $(\nabla - i(A_\varepsilon + A_{ap}^\varepsilon))U_\varepsilon \cdot \nu = 0$ on $\partial\Omega$ and the condition that A_ε is of class $C^{1,\alpha}$ (though not in general C^2) across $\partial\Omega$. (See, e.g. [21] for a regularity argument.)

4.4 Remark. Condition (4.9) is the weak form of the Euler-Lagrange equations and natural boundary conditions for A_0 given by

$$\begin{aligned} -\Delta H_0 + (H_0 + H_{ap}) &= \pi T_\alpha && \text{in } \Omega \\ -\Delta H_0 &= 0 && \text{in } \mathbb{R}^3 \setminus \bar{\Omega} \\ H_0 \times \nu &= 0 && \text{on } \partial\Omega \end{aligned}$$

where $H_0 := \nabla \times A_0$. Note in particular that the vortex line generates a nontrivial magnetic field H_0 even in the case of "permanents currents" where $H^\varepsilon_{ap} = 0$. This is the 3-d analog of the well-known 2-dimensional London's equation.

4.5 Remark. We note that for any vector field $A = (A_1, A_2, A_3)$ and $u : \Omega \to \mathbb{C}$, one can define a gauge-invariant analog of the Jacobian via the formula

$$J_A(u) = J(u) - \sum_{j,k=1}^{3} A_k(\frac{|u|^2}{2})_{x_j} dx_j \wedge dx_k,$$

so that $J_{A+\nabla\phi}(e^{i\phi}u) = J_A(u)$ for $A, u, \phi \in W^{1,2}$. Then one can easily check that (4.8) holds for $\{\star J_{A_\varepsilon}(U_\varepsilon)\}$ as well.

Proof. We will find a pair $(u_\varepsilon, A_\varepsilon)$ so that $(u_\varepsilon, A_\varepsilon + A^\varepsilon_{ap})$ is a local minimizer of $\mathcal{G}_\varepsilon$. If we then define

$$U_\varepsilon := e^{i(\phi_\varepsilon + \phi^\varepsilon_{ap})} u_\varepsilon, \tag{4.12}$$

where we are using the notation from Lemma 2.1, Lemma 2.6 will show that $(U_\varepsilon, A_\varepsilon + A^\varepsilon_{ap})$ is a local minimizer of G_ε.

We begin by defining

$$\delta = \frac{1}{2} \min \left\{ \delta_0, \|\pi T_\alpha\|_{C^{0,1}_T(\Omega)^*} \right\}$$

where δ_0 is the constant given in Proposition 4.1. Then let us define the sets

$$\begin{aligned} \mathcal{F} &= \{(u, A) \in W^{1,2}(\Omega; \mathbb{C}) \times \mathcal{H}_0 : \|\star J(u) - \pi T_\alpha\|_{C^{0,1}_T(\Omega)^*} \le \delta\} && (4.13) \\ \mathcal{O} &= \{(u, A) \in W^{1,2}(\Omega; \mathbb{C}) \times \mathcal{H}_0 : \|\star J(u) - \pi T_\alpha\|_{C^{0,1}_T(\Omega)^*} < \delta\} && (4.14) \end{aligned}$$

We will look for a local minimizer of $\mathcal{G}_\varepsilon$ in the set $\mathcal{O}$.

From ([27], proof of Thm. 4.2), $\mathcal{F}$ is weakly closed in $W^{1,2}(\Omega; \mathbb{C}) \times \mathcal{H}_0$ and $\mathcal{O}$ is open. One may easily apply the direct method to the problem

$$\inf_{(u,A)\in\mathcal{F}} \mathcal{G}_\varepsilon(u, A + A^\varepsilon_{ap}) \tag{4.15}$$

to obtain a solution that we call $(u_\varepsilon, A_\varepsilon)$. The remainder of the proof consists in showing that in fact, $(u_\varepsilon, A_\varepsilon) \in \mathcal{O}$, so that $(u_\varepsilon, A_\varepsilon + A^\varepsilon_{ap})$ is truly a local minimizer of $\mathcal{G}_\varepsilon$. Thus, we will proceed by contradiction and suppose for some subsequence (still denoted by $\{u_\varepsilon\}$) that the condition

$$\|\star J(u_\varepsilon) - \pi T_\alpha\|_{C_T^{0,1}(\Omega)^*} = \delta \tag{4.16}$$

holds.

We will reach a contradiction to (4.16) easily through an appeal to Proposition 3.2 once we establish a bound on the sequence $\{E(u_\varepsilon)/\left|\ln \varepsilon\right|\}$. With this goal in mind, we begin by estimating the energy of the sequence $\{(v_\varepsilon, 0)\}$, where $\{v_\varepsilon\}$ is the sequence whose existence is asserted in Proposition 3.2, satisfying (3.4) with $J = \pi T_\alpha$.

First we claim that for any $(u, A) \in \mathcal{F}$, one has

$$\left| \int_\Omega \langle \mathbb{P}(A + A^\varepsilon_{ap}), j(u) \rangle \right| \leq \frac{C(\alpha, \Omega, \delta) \left\| \nabla \times (A + A^\varepsilon_{ap}) \right\|_{L^2} E_\varepsilon(u)}{\left|\ln \varepsilon\right|}. \tag{4.17}$$

To see this, we invoke Lemma 2.13 to write $\mathbb{P}(A + A^\varepsilon_{ap}) = \star dB$ for B such that $B_T = 0$. Note that by our choice of δ, $(u, A) \in \mathcal{F}$ implies $\|\star J(u)\|_{C_T^{0,1}(\Omega)^*} \geq \delta$. This, (2.13) and Corollary 3.4 imply that

$$\begin{aligned} \left| \int_\Omega \langle \mathbb{P}(A + A^\varepsilon_{ap}), j(u) \rangle \right| &\leq \|\star J(u)\|_{C_T^{0,\alpha}(\Omega)^*} \|B\|_{C_T^{0,\alpha}(\Omega)} &(4.18)\\ &\leq C(\alpha, \Omega, \delta) \frac{E_\varepsilon(u)}{\left|\ln \varepsilon\right|} \|B\|_{C_T^{0,\alpha}(\Omega)}. &(4.19) \end{aligned}$$

From here, (2.11) and the Sobolev embedding theorem give (4.17). In applying Corollary 3.4 it is necessary to choose $\alpha < 1/2$, so that $W^{2,2}(\Omega; \mathbb{R}^3) \subset C^{0,\alpha}(\Omega; \mathbb{R}^3)$.

Using (4.17), one finds that

$$\begin{aligned} \mathcal{G}_\varepsilon(v_\varepsilon, 0 + A^\varepsilon_{ap}) &= E_\varepsilon(v_\varepsilon) - \int_\Omega \langle \mathbb{P}(A^\varepsilon_{ap}), j(v_\varepsilon) \rangle + \frac{1}{2} \int_\Omega |v_\varepsilon|^2 \left|\mathbb{P}(A^\varepsilon_{ap})\right|^2 dx \\ &\leq E_\varepsilon(v_\varepsilon) \left(1 + \frac{C(\alpha, \Omega, \delta) \left\| \nabla \times A^\varepsilon_{ap} \right\|_{L^2}}{\left|\ln \varepsilon\right|} \right) + \frac{1}{2} \int_\Omega \left|\mathbb{P}(A^\varepsilon_{ap})\right|^2 dx. \end{aligned}$$

Recalling from (2.11) that $\left\|\mathbb{P}(A^\varepsilon_{ap})\right\|_{W^{1,2}(\Omega)} \leq C \left\| \nabla \times A^\varepsilon_{ap} \right\|_{L^2(\Omega)}$, we then deduce from the inequality above, along with (3.4) and (4.6) that

$$\mathcal{G}_\varepsilon(v_\varepsilon, 0 + A^\varepsilon_{ap}) = o(\left|\ln \varepsilon\right|^2). \tag{4.20}$$

Also, observe from (3.4) that the sequence $\{(v_\varepsilon, 0)\}$ lies in $\mathcal{F}$ for ε sufficiently small and therefore we have

$$\mathcal{G}_\varepsilon(u_\varepsilon, A_\varepsilon + A^\varepsilon_{ap}) \leq \mathcal{G}_\varepsilon(v_\varepsilon, 0 + A^\varepsilon_{ap}). \tag{4.21}$$

Combining this with (4.20) we conclude, in particular, that

$$\lim_{\varepsilon\to 0} \frac{1}{|\ln\varepsilon|^2} \int_{\mathbb{R}^3} |\nabla \times A_\varepsilon|^2 \, dx = 0. \tag{4.22}$$

We now employ (2.15), together with (4.6), (4.17) and (4.22), to deduce that, whenever a function u satisfies $\|J(u)\|_{C_0^{0,\alpha}(\Omega)^*} \geq \delta$, one has

$$\mathcal{G}_\varepsilon(u, A_\varepsilon + A^\varepsilon_{ap}) = E_\varepsilon(u)(1+o(1)) + \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 |\mathbb{P}(A_\varepsilon + A^\varepsilon_{ap})|^2 \underset{\Omega}{\chi} + |\nabla \times A_\varepsilon|^2 \, dx$$

as $\varepsilon \to 0$, which we rewrite as

$$\begin{aligned}\mathcal{G}_\varepsilon(u, A_\varepsilon + A^\varepsilon_{ap}) &= E_\varepsilon(u)(1+o(1)) + \frac{1}{2}\int_\Omega (|u|^2 - 1)|\mathbb{P}(A_\varepsilon + A^\varepsilon_{ap})|^2 \, dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^3} |\mathbb{P}(A_\varepsilon + A^\varepsilon_{ap})|^2 \underset{\Omega}{\chi} + |\nabla \times A_\varepsilon|^2 \, dx.\end{aligned} \tag{4.23}$$

Note also that for any $u \in W^{1,2}(\Omega;\mathbb{C})$, Hölder's inequality and Sobolev embeddings imply that

$$\begin{aligned}&\int_\Omega (1-|u|^2)\left|\mathbb{P}(A_\varepsilon + A^\varepsilon_{ap})\right|^2 \, dx \\ &\qquad \leq \varepsilon \left(\frac{1}{\varepsilon^2}\int_\Omega (1-|u|^2)^2 \, dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|\mathbb{P}(A_\varepsilon + A^\varepsilon_{ap})\right|^4 \, dx\right)^{\frac{1}{2}} \\ &\qquad \leq C\varepsilon \left(E_\varepsilon(u)\right)^{\frac{1}{2}} \int_\Omega \left|\nabla \times A_\varepsilon + H^\varepsilon_{ap}\right|^2 \, dx.\end{aligned} \tag{4.24}$$

For u such that $(u, A_\varepsilon) \in \mathcal{F}$, the argument of the proof of Corollary 3.4 shows that $E_\varepsilon(u) \geq 1$ for ε sufficiently small, and then (4.6), (4.22), (4.23) and (4.24) imply that

$$\mathcal{G}_\varepsilon(u, A_\varepsilon + A^\varepsilon_{ap}) = E_\varepsilon(u)(1+o(1)) + \frac{1}{2}\int_{\mathbb{R}^3} |\mathbb{P}(A_\varepsilon + A^\varepsilon_{ap})|^2 \underset{\Omega}{\chi} + |\nabla \times A_\varepsilon|^2 \, dx \tag{4.25}$$

as $\varepsilon \to 0$. Since (4.25) applies, in particular, to the case $u = u_\varepsilon$, the inequality

$$\mathcal{G}(u_\varepsilon, A_\varepsilon + A^\varepsilon_{ap}) \leq \mathcal{G}(u, A_\varepsilon + A^\varepsilon_{ap}) \quad \text{for} \quad (u, A_\varepsilon) \in \mathcal{F}$$

yields that

$$E_\varepsilon(u_\varepsilon) \leq (1+o(1))E_\varepsilon(u) \quad \text{ for all } u \text{ such that } \|J(u) - \pi T_\alpha\|_{C_T^{0,\alpha}(\Omega)^*} \leq \delta, \tag{4.26}$$

where the $o(1)$ term is uniform for u satisfying the above condition. In particular this holds for the sequence v_ε from Proposition 3.2, since as remarked above, $(v_\varepsilon, A_\varepsilon) \in \mathcal{F}$ for ε sufficiently small. Thus, (3.4) implies that

$$E_\varepsilon(u_\varepsilon) \le (1+o(1))E_\varepsilon(v_\varepsilon) \ \le \ (1+o(1))M(\pi T_\alpha)\,|\ln\varepsilon| \tag{4.27}$$

as $\varepsilon \to 0$, and the desired bound on $\{E_\varepsilon(u_\varepsilon)/\,|\ln\varepsilon|\}$ is achieved.

Now we can apply Proposition 3.2 to find a subsequence u_{ε_k} such that

$$\star J(u_{\varepsilon_k}) \to J \text{ in } (C_T^{0,1})^*, \qquad M(J) \le \liminf_{\varepsilon_k \to 0} \frac{1}{|\ln\varepsilon_k|} E_{\varepsilon_k}(u_{\varepsilon_k}) \le M(\pi T_\alpha) \tag{4.28}$$

for some J such that $\frac{1}{\pi}J \in \mathcal{R}_1(\Omega)$ with $\partial J = 0$. Then the contradiction hypothesis (4.16) tells us that

$$\|J - \pi T_\alpha\|_{C_T^{0,1}(\Omega)^*} = \delta,$$

so that

$$M(J) > M(\pi T_\alpha)$$

by Proposition 4.1. In light of (4.28), this is impossible, so we have arrived at the desired contradiction.

Thus $(u_\varepsilon, A_\varepsilon) \in \mathcal{O}$ for all ε sufficiently small, and so by Lemma 2.6, $(U_\varepsilon, A_\varepsilon + A_{ap}^\varepsilon)$ is a local minimizer of G_ε. Since the argument can be repeated for any $\delta' < \delta$, we also deduce that

$$\lim_{\varepsilon\to 0} \|\star J(u_\varepsilon) - \pi T_\alpha\|_{C_T^{0,1}(\Omega)^*} = 0. \tag{4.29}$$

To derive (4.8) from (4.29), note that in light of (2.5),(4.12) and (4.26), a direct calculation (using the notation from Lemma 2.1) shows that

$$\begin{aligned}\|\star J(u_\varepsilon) - \star J(U_\varepsilon)\|_{C_T^{0,1}(\Omega)^*} &\le \left\| |u_\varepsilon|^2 - 1 \right\|_{L^2(\Omega)} \left(\|\nabla\phi_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} + \left\|\nabla\phi_{ap}^\varepsilon\right\|_{L^2(\Omega;\mathbb{R}^3)} \right) \\ &\le C\varepsilon\,|\ln\varepsilon|^{1/2} \left(\|\nabla\phi_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} + \left\|\nabla\phi_{ap}^\varepsilon\right\|_{L^2(\Omega;\mathbb{R}^3)} \right) \end{aligned} \tag{4.30}$$

As a consequence of Lemma 2.1, (2.8) and (4.22), we have

$$\begin{aligned}\|\nabla\phi_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} &\le \|A_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbb{P}(A_\varepsilon)\|_{L^2(\Omega;\mathbb{R}^3)} \\ &\le C\,\|\nabla\times A_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} = o(|\ln\varepsilon|). \end{aligned} \tag{4.31}$$

Then utilizing the identity $\nabla\phi_{ap}^\varepsilon = A_{ap}^\varepsilon - \mathbb{P}(A_{ap}^\varepsilon)$ and (2.14), we also see that

$$\left\|\nabla\phi_{ap}^\varepsilon\right\|_{L^2(\Omega;\mathbb{R}^3)}^2 = -\int_\Omega \mathbb{P}(A_{ap}^\varepsilon)\cdot\nabla\phi_{ap}^\varepsilon\,dx \le \left\|\mathbb{P}(A_{ap}^\varepsilon)\right\|_{L^2(\Omega;\mathbb{R}^3)} \left\|\nabla\phi_{ap}^\varepsilon\right\|_{L^2(\Omega;\mathbb{R}^3)}$$

Hence, by (2.11) and (4.6), we obtain

$$\left\|\nabla \phi_{ap}^{\varepsilon}\right\|_{L^2(\Omega;\mathbb{R}^3)} \leq C \left\|H_{ap}^{\varepsilon}\right\|_{L^2(\Omega;\mathbb{R}^3)} = o(|\ln \varepsilon|) \tag{4.32}$$

as well. Together, (4.31) and (4.32) applied to (4.30) allow us to conclude (4.8) from (4.29).

Finally, assume that H_{ap} is independent of ε. It is not hard to see that the first variation of $\mathcal{G}_\varepsilon$ in A_ε yields

$$\int_\Omega |u_\varepsilon|^2\, \mathbb{P}(A_\varepsilon + A_{ap})\cdot\mathbb{P}(A) dx + \int_{\mathbb{R}^3} \nabla\times A_\varepsilon \cdot \nabla \times A\, dx - 2\star J(u_\varepsilon)(B) = 0 \tag{4.33}$$

for all $A \in \mathcal{H}_0$, where B satisfies $\star dB = \mathbb{P}(A)$ and $B_T = 0$. We will obtain (4.9) by taking a limit as $\varepsilon \to 0$ in this last identity once we can establish the compactness of the sequence $\{A_\varepsilon\}$.

To this end, we use (2.15), (4.17), (4.24) and the fact that $E_\varepsilon(u_\varepsilon) \leq C|\ln \varepsilon|$ to obtain

$$\begin{aligned}
&\mathcal{G}_\varepsilon(u_\varepsilon, A_\varepsilon + A_{ap}) - E_\varepsilon(u_\varepsilon) \\
&\quad = -\int_\Omega \langle \mathbb{P}(A_\varepsilon + A_{ap}), j(u_\varepsilon)\rangle + \frac{1}{2}\int_{\mathbb{R}^3} \left\{ \chi_\Omega |u_\varepsilon|^2 \left|\mathbb{P}(A_\varepsilon + A_{ap})\right|^2 + |\nabla \times A_\varepsilon|^2 \right\} dx \\
&\quad = -\int_\Omega \langle \mathbb{P}(A_\varepsilon + A_{ap}), j(u_\varepsilon)\rangle + \frac{1}{2}\int_{\mathbb{R}^3} \left\{ \chi_\Omega |\mathbb{P}(A_\varepsilon + A_{ap})|^2 + |\nabla \times A_\varepsilon|^2 \right\} dx + o(1) \\
&\quad \geq -C \left\|\nabla \times (A_\varepsilon + A_{ap})\right\|_{L^2(\Omega)} + \frac{1}{2}\left\|\nabla \times A_\varepsilon\right\|_{L^2(\mathbb{R}^3)}^2 + o(1) \\
&\quad \geq -C \left\|\nabla \times A_\varepsilon\right\|_{L^2(\Omega)} + \frac{1}{2}\left\|\nabla \times A_\varepsilon\right\|_{L^2(\mathbb{R}^3)}^2 - C.
\end{aligned} \tag{4.34}$$

In addition, again appealing to (2.15), (4.17) and (4.24), we find that

$$\begin{aligned}
\mathcal{G}_\varepsilon(u_\varepsilon, A_\varepsilon + A_{ap}) - E_\varepsilon(u_\varepsilon) &\leq \mathcal{G}_\varepsilon(u_\varepsilon, A_{ap}) - E_\varepsilon(u_\varepsilon) \\
&= -\int_\Omega \langle \mathbb{P}(A_{ap}), j(u_\varepsilon)\rangle + \frac{1}{2}\int_\Omega |u_\varepsilon|^2 |\mathbb{P}(A_{ap})|^2 dx \\
&\leq C\|\nabla \times A_{ap}\|_{L^2(\Omega)} + C\|\nabla \times A_{ap}\|_{L^2(\Omega)}^2 + o(1) \\
&\leq C.
\end{aligned}$$

Together with (4.34), this implies that

$$\|\nabla \times A_\varepsilon\|_{L^2(\mathbb{R}^3)} = \|A_\varepsilon\|_{\mathcal{H}_0} \leq C \tag{4.35}$$

for all ε.

Thus after passing to a subsequence (still labelled A_ε) we can assume that $A_\varepsilon \rightharpoonup A_0$ weakly in $\mathcal{H}_0$. Furthermore, from (2.11) and (4.35), one sees that $\{\mathbb{P}(A_\varepsilon)\}$ is uniformly bounded in $W^{1,2}(\Omega;\mathbb{R}^3)$, allowing us to extract a strongly convergent subsequence in $L^2(\Omega;\mathbb{R}^3)$. Next observe that the bound $\mathcal{G}_\varepsilon(u_\varepsilon, A_\varepsilon + A_{ap}^\varepsilon) \leq C\,|\ln\varepsilon|$ yields the strong L^2-convergence of $|u_\varepsilon|^2$ to 1. This, plus the fact that $\star J(u_\varepsilon) \to \pi T_\alpha$ in $C_T^{0,\alpha}(\Omega)^*$, allow us to pass to the limit in (4.33) to obtain (4.9).

Note for future reference that, due to (2.11) and (4.35), the 1-forms B_ε satisfying $(B_\varepsilon)_T = 0$ on $\partial\Omega$ and $\star dB_\varepsilon = \mathbb{P}(A_\varepsilon)$ are uniformly bounded in $W^{2,2}$. Thus, for the chosen subsequence, they converge in $C^{0,\alpha}$ for all $\alpha < 1/2$ to a limit B_0, characterized by $(B_0)_T = 0$ on $\partial\Omega$ and $\star dB_0 = \mathbb{P}(A_0)$.

All that remains to prove is that the convergence $A_\varepsilon \rightharpoonup A_0$ is in fact strong in $\mathcal{H}_0$. To see this we first choose $A = A_\varepsilon$ in (4.33). Keeping in mind the convergences proved in the previous paragraph, we let $\varepsilon \to 0$ in (4.33) to obtain

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} |\nabla \times A_\varepsilon|^2 = 2\pi T_\alpha(B_0) - \int_\Omega \mathbb{P}(A_0 + A_{ap}) \cdot \mathbb{P}(A_0). \tag{4.36}$$

Next we choose $A = A_0$ in (4.33) and again let $\varepsilon \to 0$ in that equation. In this case we obtain

$$\int_{\mathbb{R}^3} |\nabla \times A_0|^2 = 2\pi T_\alpha(B_0) - \int_\Omega \mathbb{P}(A_0 + A_{ap}) \cdot \mathbb{P}(A_0). \tag{4.37}$$

Comparing now (4.36) with (4.37), we deduce that

$$\lim_{\varepsilon\to 0} \|A_\varepsilon\|_{\mathcal{H}_0} = \|A_0\|_{\mathcal{H}_0}.$$

Hence, $A_\varepsilon \to A_0$ strongly in $\mathcal{H}_0$ along a subsequence.

Finally, we note that the limit A_0 is uniquely determined by (4.9) as can readily be checked by assuming to the contrary that there are two solutions A_0 and A_1 and choosing $A = A_0 - A_1$ in (4.9). Hence, the convergence of $A_\varepsilon \to A_0$ occurs along the whole sequence as $\varepsilon \to 0$. □

We end this section with a proposition that yields information on the location of the vortices of the local minimizers $\{(U_\varepsilon, A_\varepsilon)\}$ to G_ε we just constructed. Its proof was suggested to us by G. Alberti and is very similar in spirit to that of Lemma 3.6 from [2]. For the purposes of this proposition let us introduce the notation

$$S_\varepsilon = \left\{ x \in \Omega : |U_\varepsilon(x)| < \frac{1}{2} \right\}, \tag{4.38}$$

$N(S_\varepsilon;\delta)$ for the δ-neighborhood of the set S_ε and $\Gamma_0 = \cup_{j=1}^N l_j \cap \bar{\Omega}$, where $l_j \cap \bar{\Omega}$ are the line segments constituting the support of T_α.

4.6 Proposition. *Let* $\{(U_\varepsilon, A_\varepsilon + A^\varepsilon_{ap})\}_{0<\varepsilon<\varepsilon_0}$ *be the family of local minimizers of* G_ε *given by Theorem 4.2. Then, for any* $\delta > 0$ *there is a number* $\varepsilon_1 \in (0, \varepsilon_0)$ *so that for* $0 < \varepsilon \le \varepsilon_1$ *one has*

$$\Gamma_0 \subseteq N(S_\varepsilon; \delta). \tag{4.39}$$

4.7 Remark. The proof below works equally well for the case of the local minimizers to E_ε constructed in [27]. Hence, condition (4.39), coupled with Theorem 5.2 of [27] establishes full Hausdorff convergence of the zero set to the union of line segments away from the boundary in that context.

Proof. Since by (4.12), we have $|U_\varepsilon| = |u_\varepsilon|$, it will suffice to establish (4.39) with U_ε replaced by u_ε in (4.38). We begin by defining $\rho_\varepsilon : \Omega \to \mathbb{R}$ as any sequence of smooth functions satisfying $0 \le \rho_\varepsilon \le 4$ in Ω and

$$\rho_\varepsilon(x) = \frac{1}{|u_\varepsilon(x)|} \quad \text{whenever} \quad |u_\varepsilon(x)| \ge \frac{1}{2}.$$

Recalling the definition of the 2-form $J(u)$ as

$$J(u) = u^\#(dx) = du_1 \wedge du_2,$$

it is easy to check that

$$J(\rho_\varepsilon u_\varepsilon) = J(u_\varepsilon) - \frac{1}{2} d\left((1-\rho_\varepsilon^2) j(u_\varepsilon)\right), \tag{4.40}$$

where $j(u_\varepsilon) = \frac{1}{2i}(\bar{u}_\varepsilon du_\varepsilon - u_\varepsilon d\bar{u}_\varepsilon)$ and d is the exterior derivative. We claim that

$$\|\star(J(u_\varepsilon) - J(\rho_\varepsilon u_\varepsilon))\|_{C_T^{0,1}(\Omega)^*} \to 0 \tag{4.41}$$

as $\varepsilon \to 0$. To see this note that (4.40) and (2.5) imply that

$$\|\star(J(u_\varepsilon) - J(\rho_\varepsilon u_\varepsilon))\|_{C_T^{0,1}(\Omega)^*} \le C \int_\Omega \left|1-\rho_\varepsilon^2\right| |j(u_\varepsilon)|\, dx. \tag{4.42}$$

We now estimate the integral on the right hand side of this last inequality as follows:

$$\begin{aligned}\int_\Omega \left|1-\rho_\varepsilon^2\right| |j(u_\varepsilon)|\, dx &\le \int_{\Omega\setminus S_\varepsilon} \left|1-\rho_\varepsilon^2\right| |j(u_\varepsilon)|\, dx + \int_{S_\varepsilon} \left|1-\rho_\varepsilon^2\right| |j(u_\varepsilon)|\, dx \\ &\le C\int_{\Omega\setminus S_\varepsilon} \left|1-|u_\varepsilon|^2\right| |\nabla u_\varepsilon|\, dx + \int_{S_\varepsilon} |\nabla u_\varepsilon|\, dx. \quad (4.43)\end{aligned}$$

We used here the fact that ρ_ε is uniformly bounded and that $|j(u_\varepsilon)| \le |\nabla u_\varepsilon|$, since $|u_\varepsilon| \le 1$. (The latter inequality follows from the maximum principle applied to the equation satisfied by $|U_\varepsilon|^2$ derived from (4.10).)

We now apply Hölder's inequality to (4.43) to obtain

$$\int_\Omega \left|1-\rho_\varepsilon^2\right| \left|j(u_\varepsilon)\right| \, dx \leq \varepsilon E_\varepsilon(u_\varepsilon) + \left(H^{(3)}\left(S_\varepsilon\right)\right)^{\frac{1}{2}} \left(E_\varepsilon(u_\varepsilon)\right)^{\frac{1}{2}}.$$

On the other hand, we can estimate $H^{(3)}\left(S_\varepsilon\right)$ using (4.27) as follows:

$$\begin{aligned} \frac{1}{\varepsilon^2} H^{(3)}\left(S_\varepsilon\right) &\leq \frac{2}{\varepsilon^2} \int_{S_\varepsilon} (1-|u_\varepsilon|^2)^2 \, dx \\ &\leq \frac{2}{\varepsilon^2} \int_\Omega (1-|u_\varepsilon|^2)^2 \, dx \\ &\leq 2E_\varepsilon(u_\varepsilon) \leq C\left|\ln \varepsilon\right|. \end{aligned}$$

From here we conclude that

$$\int_\Omega \left|1-\rho_\varepsilon^2\right| \left|j(u_\varepsilon)\right| \, dx \leq C\varepsilon \left|\ln \varepsilon\right|$$

and this implies (4.41) through (4.42).

We now proceed to prove (4.39). Were this not the case, there would be a $\delta > 0$, a sequence $\varepsilon_n \to 0$ and $x_n \in \Gamma_0$ with

$$x_n \in \Gamma_0 \setminus N(S_{\varepsilon_n}; \delta).$$

Then we can always find an $x_0 \in \Gamma_0$ and a subsequence of the x_n (still labeled x_n) with $x_n \to x_0$. By assumption,

$$\left\{B\left(x_n, \delta\right) \cap \Omega\right\} \cap S_{\varepsilon_n} = \emptyset.$$

Clearly then, for n sufficiently large, we have

$$\left\{B\left(x_0, \frac{\delta}{2}\right) \cap \Omega\right\} \cap S_{\varepsilon_n} = \emptyset,$$

which is the same as saying that $|u_{\varepsilon_n}(x)| \geq \frac{1}{2}$ for all $x \in B\left(x_0, \frac{\delta}{2}\right) \cap \Omega$. This implies that $|\rho_{\varepsilon_n} u_{\varepsilon_n}| \equiv 1$ in $B\left(x_0, \frac{\delta}{2}\right) \cap \Omega$. Now take ψ to be any 1-form supported in the set $B\left(x_0, \frac{\delta}{2}\right) \cap \Omega$ such that $T_\alpha(\psi) \neq 0$. This can be achieved, for instance, by taking ψ of the form $\psi = f\, dx_3$ where f is any non-negative function in $C_0^\infty\left(B(x_0, \frac{\delta}{2}) \cap \Omega\right)$ and the coordinate direction x_3 corresponds to the direction of one of the lines l_j. Then (4.41) implies that

$$\lim_{n\to\infty} \star\left(J(u_{\varepsilon_n}) - J(\rho_{\varepsilon_n} u_{\varepsilon_n})\right)(\psi) = 0.$$

However, $\star J(\rho_{\varepsilon_n} u_{\varepsilon_n})(\psi) = 0$ because $\rho_{\varepsilon_n} u_{\varepsilon_n}$ is smooth and has modulus one in $B\left(x_0, \frac{\delta}{2}\right) \cap \Omega$, whereas from Theorem 4.2 we have

$$\lim_{n\to\infty} \star J(u_{\varepsilon_n})(\psi) = \pi T_\alpha(\psi) \neq 0.$$

This contradiction then confirms (4.39). □

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