LENGTH/VOLTAGE PHASE DIAGRAM FOR A THIN SUPERCONDUCTING WIRE SUBJECTED TO AN APPLIED VOLTAGE

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Abstract. A thin superconducting wire (bridge) subjected to a voltage gradient is studied via the time-dependent Ginzburg-Landau system under bridge geometry boundary conditions. Our numerical experiments reveal a rich array of phase slip center behavior, period-doubling, period-tripling and quasi-periodic solutions. We show that the parameter plane \((L, V)\), where \(2L = \text{wire length}, V = \text{voltage}\), is partitioned into regimes where the solutions exhibit different periodicity. In particular we find that when \(L\) is below a certain critical value, the system always evolves to a state that has the basic Josephson period \(P = 2\pi/V\).

Key words: bridge, periodic solutions, phase slip center, PT- symmetry

1. Introduction

Consider a finite superconducting wire that is attached at both ends to bulk superconducting materials. The bulk domains are at equilibrium and under different electric potentials. As a consequence, a resistive state develops in the wire, in which a supercurrent and a normal current coexist. The problem is analyzed here using the time-dependent Ginzburg-Landau model with “bridge boundary conditions.”

An earlier study of this problem was reported in \([1]\). The authors described different types of solutions, including time-periodic solutions with the Josephson period \(P = 2\pi/V\), where \(V\) is the potential difference between the wire’s ends, solutions with a double period \(2P\), and even chaotic solutions. More recent papers \([2]\) and \([3]\) report on experiments revealing unusual I-V curves for this problem, as well some additional numerical tests.

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The purpose of the present paper is to provide further insight into the many fascinating phenomena associated with this problem. In particular we explore the parameter plane \((L, V)\) where \(L\) is the wire’s half length. We find that this plane can be divided into regions according to the nature of the solutions observed for different parameter values. An interesting prediction we make is that for sufficiently short bridges the solutions are always \(P\)-periodic. An intricate picture of phase slip centers is also shown to be correlated with these regions. Another observation is that anomalies in the I-V curves, such as those reported in \([2]\) are related to the periodicity of the solutions. In particular we give evidence showing that the loss of convexity in the I-V curve is related to the transition from \(P\)-periodic solutions to \(2P\)-periodic solutions.

2. Formulation

We model the problem with the time-dependent Ginzburg Landau equations. Thus, the order parameter \(\psi\) and electric potential evolve via

\[
\frac{\hbar^2}{2mD} \left( \psi_t + \frac{ie}{\hbar} \phi \psi \right) = \frac{\hbar^2}{2m} \psi_{xx} + a|\psi|^2 - b|\psi|^2 \psi,
\]

and

\[
\left( \frac{-4\pi \sigma}{c} \phi_x + \frac{4\pi e\hbar}{2im} (\psi^* \psi_x - \psi \psi_x^*) \right)_x = 0.
\]

Here \(D\) is a diffusion coefficient, \(e\) and \(m\) are the effective charge and effective mass of the Cooper pairs, \(\sigma\) is the conductivity and \(a\) and \(b\) are phenomenological parameters depending, among other things, on the temperature. The coherence length \(\xi\) is given by \(\xi^2 = \hbar^2/2ma\).

We scale time by \(\xi^2/D\), space by \(\xi\), \(\psi\) by \(\sqrt{(a/b)}\), and \(\phi\) by \(2mDa/\hbar e\). In addition we scale \(\sigma\) by \(e^2\hbar^2/2m^2Db\). Recalling that \(\kappa^2 = 2bc^2m^2/4\pi e^2\hbar^2\), this scaling of \(\sigma\) is the same as in \([4]\).

Under this scaling we obtain the dimensionless system

\[
\psi_t + i\phi \psi = \psi_{xx} + (1 - |\psi|^2) \psi,
\]

\[
\phi_{xx} = \frac{i}{2\sigma} (\psi \psi_x^* - \psi_x^* \psi^*).
\]

The problem is considered on an interval \(-L \leq x \leq L\) with so-called bridge geometry boundary conditions

\[
\psi(-L, t) = e^{-iVt/2}, \quad \psi(L, t) = e^{iVt/2}, \quad \phi(-L, t) = \frac{V}{2}, \quad \phi(L, t) = -\frac{V}{2}.
\]
We note that the total current $I$ is given by $I(t) = -\sigma \phi_x + \frac{i}{2} (\psi \psi^*_x - \psi_x \psi^*)$ with the first term representing the normal current and the second term corresponding to the supercurrent, so the condition of constancy of total current is enforced through (2.4). The system of equations above, along with initial conditions for $\psi$, will be denoted by us as time-dependent Ginzburg-Landau (TDGL). In most of what follows we set the conductivity $\sigma = 1$ in order to concentrate on two parameters: the wire length $2L$ and the potential difference $V$. In the discussion section at the end of this note, we comment on how some of our results vary with $\sigma$.

We should caution in scaling the dimensional length by the coherence length $\xi$, we have chosen a scaling that is temperature-dependent, and hence it is implicitly assumed that the temperature is kept constant in this setting. Finally, we point out that some authors, e.g. [2] and [3], use the Kramer–Watts-Tobin model instead of the TDGL. Wherever we checked, our results hold also for this model but for simplicity we present them for the TDGL.

In the opposite direction, one can simplify the TDGL by replacing the unknown time-dependent electric potential $\phi$ with the one coming from a purely normal state, i.e. $\Phi(x) = -\frac{V}{2L}x$. In this way equation (2.3) becomes

$$
\psi_t - i \frac{V}{2L} x \psi = \psi_{xx} + (1 - |\psi|^2) \psi,
$$

together with the boundary conditions (2.5) for $\psi$. This simplified version of (TDGL) has appeared in the work [4] and in this article we denote it by (STDGL). We shall discuss below the parameter regimes in which (STDGL) represents a good approximation to (TDGL) and those in which its solutions diverge in some significant way from the solutions to (TDGL).

3. Phase boundaries in the $L - V$ plane

While the boundary conditions (2.5) are $2P$-periodic, we point out that via the gauge transformation $(\psi, \phi) \rightarrow (\tilde{\psi}, \tilde{\phi}) := (\psi e^{i\nu t/2}, \phi - V/2)$, one arrives at a gauge-equivalent formulation of (TDGL) in which the boundary conditions (2.5) are replaced by $\tilde{\psi}(-L, t) = 1$, $\tilde{\psi}(L, t) = e^{i\nu t}$, $\tilde{\phi}(-L, t) = 0$, $\tilde{\phi}(L, t) = -V$. Since the boundary conditions in this formulation have period $P := 2\pi/V$, one might well anticipate that the gauge-invariant quantities such as $|\psi|$ and $I$ should be $P$-periodic for all solutions to (TDGL) with a similar expectation for (STDGL). However, as pointed out already by Webber and Malomed for the case of (TDGL), this is not
Figure 1. Phase diagram for (TDGL) showing regions of stability for period $P$ (denoted by $P_1$), period $2P$ (denoted by $P_2$) and more complicated solutions $C$ including those having period $kP$ with $k > 2$ and quasi-periodic solutions.

always the case. They report in [1] on numerical tests showing that when one keeps $L$ fixed and selects a small value for $V$, the total current $I(t)$ indeed tends to be periodic with period $P$. However, as one increases $V$, a critical voltage value $V_1(L)$ is reached at which the period doubles to $2P$. Moreover, as $V$ is increased further past a second critical value $V_2(L)$ new solutions emerge that contain many incommensurate frequencies. They term these solutions ‘chaotic.’

We carried out a more systematic numerical study of (TDGL) and the results are displayed in Figure 1 which depicts the different regimes in the $(L, V)$ plane. We remark that here and throughout the remainder of the paper, when we refer to a $P$-periodic, $2P$-periodic or in general $kP$-periodic solution, we are always referring to the period of such gauge-invariant quantities as the modulus of the order parameter $\psi$ or the total current $I$, unless otherwise specified.

The domain $P_1$ in the figure corresponds to $P$-periodic solutions. More precisely, when we work in this domain, any initial condition evolves in time under the flow (TDGL) towards a universal attracting $P$-periodic solution. A striking effect presented in the figure is that when the wire length is below a critical value, all the solutions are $P$-periodic. We also point
out that solutions tend to have period $P$ whenever the voltage is either sufficiently small or sufficiently large. These observations were given theoretical justifications in [5].

In the $P2$ regime, any initial condition evolves towards a $2P$-periodic attractor, so that a period-doubling event occurs as one crosses the lower $P1/P2$ phase boundary. Finally, in the $C$ regime initial conditions evolve in time either to functions $\psi(x, t), \phi(x, t)$ with period $kP$, where $k$ is an integer bigger than one, or to solutions which exhibit no periodicity. In fact, when these latter solutions $\psi$ are evaluated at the center of the wire, we find that the spectral decomposition of $\psi(0, t)$ has many incommensurate frequencies.

One of the most interesting phenomena associated with a wire under a potential difference is the bending of the I-V curve into an S-shaped curve. This bending was observed in [2] experimentally. The same authors verified such solutions numerically, at least for certain parameter values. We would like to draw attention to an intriguing relation between the potential value where the I-V curve bends and the issue of period doubling considered in the previous section. In Figure 2, we draw the scaled potential $V/2L$ vs. the current $I$ in the wire for $L = 5$ (left figure) and $L = 7$ (right figure). The curve starts bending at $V \approx 0.7$ for $L = 5$ and at $V \approx 0.38$ for $L = 7$. Referring back to the phase diagram for the $L-V$ plane given in Figure 1 and the values given later in the article in Table 1A, the transition from the P1 region to the P2 region (dashed curve in Figure 1) also occurs at these values of voltage for $L = 5$ and 7. Hence, it is evident that the bending of the I-V curve is associated with the P1 $\rightarrow$ P2 transition curve.

![Figure 2. I − V curves for (TDGL) when $L = 5$ and $L = 7$ exhibiting an S-shape as they pass through the period-doubling parameter regime.](image-url)
4. Phase slip centers

The notion of vortex is of course fundamental to superconductors under a magnetic field, and in the context of thin superconductors subjected to a forced current, the corresponding phenomenon of phase slip center (PSC) is well-documented numerically and well-studied theoretically. Indeed, Ivlev and Kopnin [6] argue that PSC’s behave just like vortices in space-time. The same can be said of PSC’s in the context of an applied voltage difference.

Being one of the prominent features of these so-called resistive superconducting states, PSC’s serve to largely characterize the solutions to (TDGL) or (STDGL). As such, it becomes significant to capture the number and location of these points in space-time for different parameter values of $L$ and $V$. In light of the afore-mentioned cascade from period $P$ to period-doubling, period tripling and more complicated behavior, one might be led to the expectation that there are $k$ PSC’s for a $kP$-periodic solution. In fact, our computations reveal that while this is generically true, there can be surprising exceptions.

In Figure 3 we show a sampling of numerical findings for (TDGL). Figure 3A shows a typical PSC configuration when the voltage is small enough, with one PSC per period occurring at the center of the wire. Once period-doubling occurs a typical PSC configuration is depicted in Figure 3B, with 2 PSC’s per period again occurring at $x = 0$. However, for some parameter values in the P2 regime, one can also find PSC’s occurring simultaneously in time in pairs placed symmetrically about $x = 0$ (not pictured). Then, perhaps most surprisingly, for certain values of $L$ and $V$, one finds a 2P-periodic solution with only one PSC per period, as in Figure 3C. Finally, in Figure 3D one sees a typical PSC configuration in the 3P-periodic regime, with 3 PSC’s per period, one at the origin and the other two symmetrically placed and occurring simultaneously in time.

The name phase slip center originates from the sharp change undergone by the phase as one moves across a zero in the order parameter. Specifically, if $\psi = 0$ at some point $(x, t)$, then generically $\chi(x^L, t) - \chi(x^R, t) = \pm \pi$, where $\chi$ is the phase of the order parameter, and $x^L$ and $x^R$ are points just to the left and right of $x$, respectively. It is often argued that PSCs are needed to relieve high phase gradients that accelerate the superconducting electrons. However, we point out that the phase can change dramatically, but continuously, even if $|\psi|$ is just very
small; namely, $\psi$ does not need to vanish for the phase to change drastically. Therefore, energetic arguments do not suffice to prove existence of PSCs or even vortices in magnetic-driven time-independent setups. In fact, one must apply some topological considerations to establish the existence of a PSC (or a vortex). For instance, Abrikosov [7] discovered his famous vortex lattice solution in a periodic setup where a topological argument alone applies. Similarly, Sandier and Serfaty [8] established vortical solutions for more arbitrary geometries using a combination of energetic and topological considerations. As an example related directly to the PSC phenomena of interest here we mention the derivation in references

Figure 3. Phase slip centers for solutions to (TDGL)
(A) $\frac{2\pi}{V}$-periodic solution for $L = 5, V = 0.3$ with 1 PSC per period at origin.
(B) $\frac{4\pi}{V}$-periodic solution for $L = 10, V = 0.22$ with 2 PSC's per period at origin.
(C) $\frac{4\pi}{V}$-periodic solution for $L = 5, V = 1.16$ with 1 PSC per period at origin.
(D) $\frac{6\pi}{V}$-periodic solution for $L = 10, V = 0.8$ with 3 PSC's per period: 1 at the origin and 2 more at $x \approx \pm 4$
[9, 10] of PSCs in certain parameter regimes when a given current is forced into a wire. Here, an approximation to the order parameter was first derived, and then a topological constraint was extracted from it to prove the existence of a PSC for the exact solution.

We now establish the existence of at least one PSC per period for any $kP$-periodic solution $\psi$ to the present voltage driven problems (TDGL) and (STDGL). To justify this assertion, recall the gauge-equivalent formulation of (TDGL) using $(\tilde{\psi}, \tilde{\phi}) := (\psi e^{iVt/2}, \phi - V/2)$, which leads via (2.5) to the boundary conditions $\tilde{\psi}(-L, t) = 1$ and $\tilde{\psi}(L, t) = e^{iVt}$, and note, of course, that $\psi(x, t) = 0$ if and only if $\tilde{\psi}(x, t) = 0$. Then consider for each fixed $x$ along the wire the mapping $t \mapsto \tilde{\psi}(x, t)$, which we denote by $h^x(t)$. As $t$ changes from 0 to $2\pi k/V$, the function $h^x(t)$ traces a closed orbit in the complex plane $(Re(h^x), Im(h^x))$. This orbit consists of a single point $(1, 0)$ for $h^{-L}(t)$ and of $k$ coverings of the unit circle for $h^L(t)$. However, as $x$ varies from $L$ to $-L$, the trace of the curve $h^x(t)$ changes continuously. Clearly any continuous deformation in the complex plane of the unit circle into the point $(1, 0)$ must cross the origin. Therefore there exists at least one $x$ for which $h^x(t)$ vanishes at some point $t$; that is, $\psi$ has at least one PSC per period.

We recall now the case of $L = 5$ and $V = 1.16$ as depicted in Figure 3C showing that although the solution has period $2P$, there is only one PSC per period so that this result is sharp. In terms of the homotopy of curves, what transpires in this non-generic case is that the point $(1, 0)$ corresponding to location $x = -L$ deforms to a double covering of the unit circle at $x = L$ as $t$ varies in such a way that the one curve, $h^0$, that passes through the origin does so with a cusp singularity.

5. PT-symmetry for (TDGL) and (STDGL)

The systems (TDGL) and (STDGL) both share an interesting structural property, namely they are PT-symmetric, cf. e.g. [11]. What this means is that both systems are invariant under the joint transformation of $x \rightarrow -x$ and complex conjugacy. Introducing the notation $f^\dagger$ to denote the transformation $f(-x, t)^*$ for any (perhaps) complex-valued function $f$, a consequence of this symmetry is that if a pair $(\psi, \phi)$ solves (TDGL) then so does the pair $(\psi^\dagger, -\phi^\dagger)$, or if a function $\psi$ solves (STDGL), then so does $\psi^\dagger$. In light of this property,
it is natural to ask whether or not solutions to (TDGL) or (STDGL) tend towards PT-symmetry as time evolves. For example, in the case of TDGL for a wire subject to a prescribed current rather than voltage, it is shown in [9] that small amplitude solutions generically do exhibit such an asymptotic PT-symmetry. Consequently, in the setting of prescribed current this implies that all small amplitude stable periodic states are PT-symmetric. The question gains significance when considered alongside the problem of characterizing PSC behavior since clearly the property $\psi = \psi^\dagger$ forces any PSC’s not appearing at the origin to emerge in spatially symmetric pairs.

With these notions in mind, we have checked for PT-symmetry in periodic and quasi-periodic solutions to (TDGL) and (STDGL) in different parameter regimes by computing the quantity $E = \int_{-L}^{L} \left( |\psi(x,t) - \psi^\dagger(x,t)|^2 \right) dx$. We find that the value of this measure of PT-symmetry is not very sensitive to the particular $t$-value chosen. A typical set of values for this integral is given in Table 1A using (TDGL) and in Table 1B for (STDGL). Examining the values of $E$ in these two tables, a striking dichotomy emerges. For (TDGL), the property of PT-symmetry is extremely robust, surviving the cascade of period-doubling and period-tripling bifurcations and even being preserved through the chaotic regime. On the other hand, for (STDGL), PT-symmetry of solutions is generally lost once one exits the period $P$ (denoted by dots) parameter regime. In light of this divergence of behaviors, we conclude that use of (STDGL) as an approximation for (TDGL) should be done with care.

6. Discussion

We presented a rich collection of patterns and solution types for the superconducting bridge problem. Our main result is Figure 1 which depicts the partitioning of the $(L, V)$ parameter plane into regions with different periodicity behavior. In particular our main theoretical prediction is the existence of a critical length $L_c$, such that for wires shorter than $2L_c$ all solutions converge in the long run to a universal attractor that is $2\pi/V$ periodic. We have examined the dependence of this value $L_c$ on the conductivity $\sigma$ and find it to be rather weak: for $\sigma$ lying between 0.25 and 1.0, $L_c \approx 2.88$, while in varying $\sigma$ up to 10.0, the value of $L_c$ decreases by just 10%. It was further shown that the different regions of Figure 1 are
Table 1. PT-symmetry in (TDGL) and (STDGL)

(A) Left table depicts value of the integral $E$ for solutions to (TDGL) and illustrates a robust PT-symmetry. Here dots denote the period $P$ regime, white background denotes period $2P$ and gray shading denotes all other solutions.

(B) Right table depicts value of the integral $E$ for solutions to (STDGL) and suggests PT-symmetry is largely limited to period $P$ solutions. The shading has the same meaning as in Table 1A.

associated with different phase slip center arrangements. Moreover, the change of convexity in $I-V$ curves was also shown to be correlated with the transition between different regions in Figure 1.

Some of the features of Figure 1 were justified theoretically in [5]. Specifically, it is shown there that the solutions are in the P1 region for $L$ below a critical value, and for $V$ large enough or $V$ small enough. An intriguing question that is still open is to find a theoretical tool to study the P2 or C regions of Figure 1.
REFERENCES


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