Topologically driven local minimizers of the Oseen-Frank energy

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1. INTRODUCTION

The analogy between certain liquid crystals and superconductivity has been recognized and explored by a number of scientists. In particular, mathematical techniques first developed within the Ginzburg-Landau theory of superconductivity have proved useful when adapted to the setting of liquid crystals (see e.g. [1]). Here we pursue nontrivial stable liquid crystal configurations, motivated by the approach in the paper [8] where the authors use the Ginzburg-Landau model to produce persistent currents in topologically nontrivial domains. Our starting point is the Oseen-Frank energy for a nematic ([4], [10]). We add to the standard model a term that penalizes deviation of the director from a given plane. Therefore, the energy takes the form

$$\begin{aligned} F_\alpha(\mathbf{n}) = \int_\Omega \Big\{ k_1|\mathrm{div}\,\mathbf{n}|^2 \quad &+ \quad k_2|\mathbf{n}\cdot\mathrm{curl}\,\mathbf{n}|^2 + k_3|\mathbf{n}\times\mathrm{curl}\,\mathbf{n}|^2 \\ &+ \quad (k_2+k_4)[\mathrm{tr}\,(\nabla\mathbf{n})^2 - (\mathrm{div}\,\mathbf{n})^2] + \alpha|\mathbf{n}\cdot\mathbf{e_3}|^2 \Big\}\, dx\ . \end{aligned}$$

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Here $\Omega \subset \mathbb{R}^3$ is a bounded, smooth domain occupied by the liquid crystal sample, $\mathbf{n}$ is a Frank director $\mathbf{n} : \Omega \to S^2$, the factors k_i are material constants, and $\mathbf{e_3}$ is the standard unit vector in the z direction in $\mathbb{R}^3$. The constants $\{k_i\}_{i=1}^4$ are generally assumed to satisfy

$$k_1 > k_2 + k_4 > 0,\ k_3 > k_2 + k_4,\ k_4 \leq 0. \tag{1}$$

We have included the last term with positive α to describe the interaction with an external electric or magnetic field ([4], [9]). The fact that α is positive means that the model is only relevant to certain materials ([4], [9]). We take the field to be uniform in the $\mathbf{e_3}$ direction, and in the case of an electric field we neglect the dipole contribution of the director. In this article we will explore the asymptotic regime $\alpha >> 1$ so that alignment of the director perpendicular to $\mathbf{e_3}$ is heavily favored, whence we are led to consider the functional

$$\begin{aligned} F_\infty(\mathbf{n}) := \int_\Omega [k_1|\text{div } \mathbf{n}|^2 &+ k_2|\mathbf{n}\cdot \text{curl } \mathbf{n}|^2 + k_3|\mathbf{n}\times \text{curl } \mathbf{n}|^2] \\ &+ (k_2+k_4)[\text{tr } (\nabla \mathbf{n})^2 - (\text{div } \mathbf{n})^2]\, dx\ . \end{aligned}$$

One should view F_∞ as arising formally in the limit $\alpha \to \infty$ where the extreme cost of out-of-plane alignment for the director leads to consideration of this functional over directors $\mathbf{n}$ taking values on the unit circle S^1 in the xy-plane rather than on S^2.

Our main result is the assertion that, under certain conditions on Ω, there exist nontrivial local minimizers of the energy F_α for α sufficiently large. We wish to emphasize that these stable critical points locally minimize the energy among all nearby competitors in a suitable topology, without the imposition of Dirichlet boundary conditions as is often the case in the literature. As such, wherever these solutions are smooth, they will satisfy the so-called ‘natural boundary conditions’ for the problem associated with setting the first variation of energy to zero and integrating by parts.

The technique, which we borrow from [8], involves looking for local minimizers for F_α near local minimizers of F_∞. The fact that one can find nontrivial local minimizers of F_∞ is not obvious given that one is not forcing the director into a non-constant configuration through a Dirichlet condition. It relies crucially on the assumption that the sample Ω occupies a region of nontrivial topology. We now make this hypothesis explicit:

Henceforth we assume that Ω *has the topology of a one-holed torus in* $\mathbb{R}^3$.

We should note that more exotic local minimizers can be found by the same methods if one takes Ω to be a many-holed torus but we will not pursue that here. The effect of this topological assumption is to allow us to introduce a notion of winding number for S^1-valued functions defined on Ω. We will make this notion precise in the next section. We can then minimize F_∞ among S^1-valued competitors of a given winding number m to obtain interesting solutions. Through a perturbative variational calculus, for each m we will then find local minimizers of the full energy F_α for large values of α near these local minimizers of F_∞.

In the final section we include some further observations about the limiting energy F_∞ in the case where $\Omega \subset \mathbb{R}^2$ is an annulus and the director $\mathbf{n} : \Omega \to S^1$.

2. EXISTENCE OF LOCAL MINIMIZERS

We begin by recalling the notion of winding number appropriate to our setting, namely the 1-homotopy type of an S^1-valued map. For smooth functions mapping Ω into S^1, one can simply consider the restriction of such a map to any oriented simple closed loop encircling the hole in Ω and measure the classical winding number. By a standard continuity argument, this number will not depend on the particular loop chosen. However, the natural space of competitors for both F_α and more crucially for F_∞ is not smooth maps but rather maps whose first derivatives are all square integrable, namely the Sobolev spaces $H^1(\Omega; S^2)$ and $H^1(\Omega; S^1)$ respectively, cf. [5].

Let us then recall the definition of the H^1-norm of a function $\mathbf{n} : \Omega \to \mathbb{R}^k$, $k \geq 1$:

$$\|\mathbf{n}\|^2_{H^1(\Omega;\mathbb{R}^m)} = \sum_{j=1}^{k} \left\|n^{(j)}\right\|^2_{L^2(\Omega)} + \sum_{j=1}^{k} \left\|\nabla n^{(j)}\right\|^2_{L^2(\Omega)} =$$

$$\sum_{j=1}^{k} \left\{ \int_\Omega \left|n^{(j)}\right|^2 dx + \int_\Omega \left|\nabla n^{(j)}\right|^2 dx \right\}$$

where $n^{(j)}$ denotes the j^{th} component of the vector-valued function $\mathbf{n}$.

It is well-known that H^1 functions defined on a set $\Omega \subset \mathbb{R}^3$ need not be continuous so some care must be taken to define the 1-homotopy type of such a Sobolev map. This difficulty was overcome in [11], where the author studies the harmonic map problem for functions of a given 1-homotopy type. A key tool is the following lemma:

Lemma 1. *([11]) For each $K > 0$ there is an $\varepsilon > 0$ such that if f_1 and f_2 are Lipschitz continuous mappings from Ω into S^1 satisfying the conditions*

$$\|f_1 - f_2\|_{L^2(\Omega)} < \varepsilon, \qquad \|\nabla f_i\|_{L^2(\Omega)} \leq K \quad \text{for } i = 1, 2, \tag{2}$$

then f_1 and f_2 have the same 1-homotopy type.

Armed with this information one can for instance appeal to the density of smooth maps taking Ω into S^1 (see e.g. [2]) to partition the space $H^1(\Omega; S^1)$ according to 1-homotopy type by defining for each integer m the set $H^1_m(\Omega; S^1) \subset H^1(\Omega; S^1)$ as the closure under the H^1-norm of smooth maps of 1-homotopy type m. Equally important from the standpoint of invoking the direct method in the calculus of variations, one has the following consequence of Lemma 1 and the weak compactness of bounded sets in H^1 (cf. [5]): Suppose a sequence $\{\mathbf{n}_j\} \subset H^1_m(\Omega; S^1)$ satisfies the uniform bound

$$\|\mathbf{n}_j\| \leq M$$

for some positive constant M. Then there exists a subsequence $\{\mathbf{n}_{j_k}\}$ and a function $\mathbf{n} \in H^1_m(\Omega; S^1)$ such that

$$\mathbf{n}_{j_k} \rightharpoonup \mathbf{n} \text{ in } H^1 \text{ as } k \to \infty.$$

In particular, the set $H^1_m(\Omega; S^1)$ is closed under weak H^1-convergence.

With these preliminaries in hand, we now establish the existence of minimizers for the limiting functional F_∞ within each 1-homotopy class. To this end, for each integer m we introduce the notation

$$\mu_m := \inf_{\mathbf{n} \in H^1_m(\Omega; S^1)} F_\infty(\mathbf{n}) \tag{3}$$

and

$$\mathcal{A}_m := \{\mathbf{n} \in H^1_m(\Omega; S^1) : \, F_\infty(\mathbf{n}) = \mu_m\} \tag{4}$$

and establish:

Proposition 1. *For every integer m, the set of minimizers $\mathcal{A}_m$ is nonempty.*

Proof. Existence will follow from the direct method. Indeed, if $\{\mathbf{n}_j\}$ denotes a minimizing sequence for the problem (3), then the identity

$$|\nabla \mathbf{n}|^2 = \operatorname{tr}(\nabla \mathbf{n})^2 + |\operatorname{curl} \mathbf{n}|^2 \tag{5}$$

(cf. [10], Lemma 3.3) and our assumptions (1) readily imply the bound

$$\overline{k}\int_\Omega |\nabla \mathbf{n}_j|^2\,dx \le 2\mu_m, \tag{6}$$

where $\overline{k} := \min\{k_3, k_2+k_4\} > 0$. Hence by Lemma 1 and the discussion following it, there exists a subsequence $\{\mathbf{n}_{j_k}\}$ and a function $\mathbf{n}\in H^1_m(\Omega;S^1)$ such that

$$\mathbf{n}_{j_k} \rightharpoonup \mathbf{n} \quad \text{in} \quad H^1 \quad \text{and} \quad \mathbf{n}_{j_k} \to \mathbf{n} \quad \text{in} \quad L^2 \quad \text{as } k\to\infty. \tag{7}$$

Rearranging the terms in F_∞ as

$$F_\infty(\mathbf{n}_{j_k}) = \int_\Omega \Big[(k_1-k_2-k_4)\,|\operatorname{div}\mathbf{n}_{j_k}|^2 \quad - \quad k_4\,|\mathbf{n}_{j_k}\cdot \operatorname{curl}\mathbf{n}_{j_k}|^2$$
$$+(k_3-k_2-k_4)\,|\mathbf{n}_{j_k}\times \operatorname{curl}\mathbf{n}_{j_k}|^2 \quad + \quad (k_2+k_4)\,|\nabla \mathbf{n}_{j_k}|^2\Big]\,dx,$$

we observe through (1) that the integrand is a convex function of the partial derivatives of $\mathbf{n}_{j_k}$ and so (7) implies that the energy is lower-semi-continuous (cf. [3]):

$$\liminf_{k\to\infty} F_\infty(\mathbf{n}_{j_k}) \ge F_\infty(\mathbf{n}).$$

Hence $\mathbf{n}\in\mathcal{A}_m$. □

Remark. We leave open the question as to whether the minimizer of F_∞ in $H^1_m(\Omega;S^1)$ is unique. Resolving this does not seem to be easy but in any event, it will not be needed for the main result.

Before proceeding to the main result we require a few simple consequences of Lemma 1. For these we introduce notation for the H^1-distance to the set $\mathcal{A}_m$:

$$d(\mathbf{n},\mathcal{A}_m) := \inf_{\tilde{\mathbf{n}}\in\mathcal{A}_m} \|\mathbf{n}-\tilde{\mathbf{n}}\|_{H^1}\,. \tag{8}$$

First we note some compactness properties of the set $\mathcal{A}_m$:

Lemma 2. *Let m be any integer.*

(i) For any sequence $\{\tilde{\mathbf{n}}_j\} \subset \mathcal{A}_m$ *there exists a subsequence* $\{\tilde{\mathbf{n}}_{j_k}\}$ *and a function* $\tilde{\mathbf{n}} \in \mathcal{A}_m$ *such that* $\tilde{\mathbf{n}}_{j_k} \to \tilde{\mathbf{n}}$ *strongly in* H^1*.*
(ii) For any $\mathbf{n} \in H^1(\Omega; S^2)$ *and any positive constant* C*, the condition* $d(\mathbf{n}, \mathcal{A}_m) \le C$ *implies that there exists a function* $\tilde{\mathbf{n}} \in \mathcal{A}_m$ *such that*

$$\|\mathbf{n} - \tilde{\mathbf{n}}\|_{H^1(\Omega;\mathbb{R}^3)} \le C.$$

Proof. To prove (i), we apply the bound (6) to the sequence $\{\tilde{\mathbf{n}}_j\}$ to obtain a weak H^1-limit $\tilde{\mathbf{n}}$ of a subsequence $\{\tilde{\mathbf{n}}_{j_k}\}$. As was noted in the proof of Proposition 1, the energy F_∞ is lower-semi-continuous under weak H^1-convergence so with the aid of Lemma 1 we immediately conclude that $\tilde{\mathbf{n}} \in \mathcal{A}_m$. However, using that in fact each term in F_∞ is weakly-lower-semicontinuous we can further assert that

$$\lim_{k\to\infty} \int_\Omega \left|\tilde{\mathbf{n}}_{j_k} \cdot \operatorname{curl} \tilde{\mathbf{n}}_{j_k}\right|^2 dx = \int_\Omega \left|\tilde{\mathbf{n}} \cdot \operatorname{curl} \tilde{\mathbf{n}}\right|^2 dx,$$
$$\lim_{k\to\infty} \int_\Omega \left|\tilde{\mathbf{n}}_{j_k} \times \operatorname{curl} \tilde{\mathbf{n}}_{j_k}\right|^2 dx = \int_\Omega \left|\tilde{\mathbf{n}} \times \operatorname{curl} \tilde{\mathbf{n}}\right|^2 dx,$$

and

$$\lim_{k\to\infty} \int_\Omega \operatorname{tr}(\nabla \tilde{\mathbf{n}}_{j_k})^2 \, dx = \int_\Omega \operatorname{tr}(\nabla \tilde{\mathbf{n}})^2 \, dx,$$

for otherwise $F_\infty(\tilde{\mathbf{n}}) < \mu_m$, a contradiction. Hence, via (5) we see that $\|\nabla \tilde{\mathbf{n}_{j_k}}\|_{L^2} \to \|\nabla \tilde{\mathbf{n}}\|_{L^2}$ and the convergence of the subsequence is indeed strong in H^1.

The proof of (ii) follows immediately from the definition (8) and property (i). □

The next lemma says that functions H^1-close to the set $\mathcal{A}_m$ must lie in $H^1_m(\Omega; S^1)$.

Lemma 3. *For each* m *there is a positive number* γ_m *such that if* $\mathbf{n} \in H^1(\Omega; S^1)$ *satisfies* $d(\mathbf{n}, \mathcal{A}_m) < \gamma_m$*, then* $\mathbf{n}$ *belongs to* $H^1_m(\Omega; S^1)$*.*

Proof. First note that if $\tilde{\mathbf{n}} \in \mathcal{A}_m$ then using (6) we have

$$\int_\Omega |\nabla \tilde{\mathbf{n}}|^2 \le \frac{2\mu_m}{\underline{k}}.$$

Now let $K = \frac{4\mu_m}{k} + 1$ and let $\varepsilon = \varepsilon(K)$ be the value guaranteed by Lemma 1. Set $\gamma_m = \min\{\frac{1}{\sqrt{2}}, \frac{1}{2}\sqrt{\varepsilon}\}$.

Then suppose

$$d(\mathbf{n}, \mathcal{A}_m) < \gamma_m, \tag{9}$$

and pick an element $\tilde{\mathbf{n}}$ in $\mathcal{A}_m$ such that $\|\mathbf{n} - \tilde{\mathbf{n}}\|_{H^1(\Omega;\mathbb{R}^3)} < \gamma_m$. It follows that

$$\|\nabla \mathbf{n}\|^2_{L^2(\Omega;S^1)} \leq 2\left(\|\nabla \tilde{\mathbf{n}}\|^2_{L^2(\Omega;S^1)} + \gamma_m^2\right) < K.$$

Similarly, (9) implies that $\|\mathbf{n} - \tilde{\mathbf{n}}\|_{L^2(\Omega;S^1)} < \frac{\varepsilon}{4}$. Taking smooth sequences $\{f_j\}$ and $\{\tilde{f}_j\}$ in $H^1(\Omega;S^1)$ satisfying $f_j \to \mathbf{n}$ and $\tilde{f}_j \to \tilde{\mathbf{n}}$ in H^1, we see from Lemma 1 that both sequences lie in $H^1_m(\Omega;S^1)$ for j large. Hence $\mathbf{n}$ does as well. $\square$

We can now state our main result:

Theorem 1. *For each m there exists a positive number α_m such that for all $\alpha > \alpha_m$, the functional F_α possesses an H^1-local minimizer $\mathbf{n}^m_\alpha$. Furthermore, $d(\{\mathbf{n}^m_\alpha\}, \mathcal{A}_m) \to 0$ as $\alpha \to \infty$.*

Remark. By an H^1-local minimizer we mean a function $\mathbf{n}^m_\alpha \in H^1(\Omega;\mathbb{R}^3)$ such that

$$F_\alpha(\mathbf{n}^m_\alpha) \leq F_\alpha(\mathbf{n}) \quad \text{whenever} \quad \|\mathbf{n}^m_\alpha - \mathbf{n}\|_{H^1(\Omega;S^2)} < \delta$$

for some $\delta > 0$. Such a function will in particular be a critical point of the functional F_α and so will weakly satisfy the associated Euler-Lagrange equation and natural boundary conditions. The regularity theory for F_α originally developed in [6] for Dirichlet boundary conditions says that minimizers with be smooth off of a set of finite zero-dimensional Hausdorff measure. We have not checked but we suspect that such a theory would apply to this present setting of natural boundary conditions as well. The technical issue would be handling the last (null-Lagragian) term carrying coefficient k_2+k_4 which in this non-Dirichlet setting cannot be equated to a constant.

Remark. In modeling a nematic liquid crystal, the "head" and "tail" of the director are indistinguishable. One way to capture this is to pose the minimization of F_α over unit vectors taking values in $\mathbb{P}^2$, two-dimensional projective space where antipodal points on S^2 are associated. The theorem above can readily be phrased with $\mathbb{P}^2$ replacing $\mathbb{S}^2$. In fact, if one takes this tack, then local minimizers can be found corresponding to all of the half-integer winding numbers as well. The proof below is unchanged by the substitution of $\mathbb{P}^2$ for S^2 and the consideration of the limiting energy F_∞ for mappings $\mathbf{n} : \Omega \to \mathbb{P}^1$.

Proof. Fix an integer m and consider the variational problem

$$\inf_{\{\mathbf{n}\in H^1(\Omega;S^2):\, d(\mathbf{n},\mathcal{A}_m)\leq \frac{1}{2}\gamma_m\}} F_\alpha(\mathbf{n}), \tag{10}$$

where γ_m is the value from Lemma 3. Exploiting the convexity of the various terms in F_α as well as the uniform H^1-bound valid for any minimizing sequence $\{\mathbf{n}_j\}$, the direct method succeeds here in a manner similar to that used in solving (3). The only new issue is that one must invoke Lemma 2 and the lower-semi-continuity of the H^1-norm under weak convergence to assure that the weak limit $\mathbf{n}_\alpha^m$ of a subsequence $\{\mathbf{n}_{j_k}\}$ will satisfy the admissibility condition $d(\mathbf{n}_\alpha^m, \mathcal{A}_m) \leq \frac{1}{2}\gamma_m$. We would like to argue that the minimizer $\mathbf{n}_\alpha^m$ to the constrained problem (10) is in fact an H^1-local minimizer of F_α. This would follow if one could show that the constraint is not exhausted, that is, if one could demonstrate that $d(\mathbf{n}_\alpha^m, \mathcal{A}_m) < \frac{1}{2}\gamma_m$. We will achieve this for α sufficiently large by in fact arguing that

$$d(\mathbf{n}_\alpha^m, \mathcal{A}_m) \to 0 \text{ as } \alpha \to \infty. \tag{11}$$

We proceed by contradiction and suppose that for some subsequence $\{\alpha_j\} \to \infty$ there exists a positive number η such that

$$d(\mathbf{n}_{\alpha_j}^m, \mathcal{A}_m) \geq \eta \text{ for all } j. \tag{12}$$

Note that necessarily, $\eta \leq \frac{1}{2}\gamma_m$.

Since all elements of $\mathcal{A}_m$ are uniformly bounded in H^1 by some constant depending on m, the condition $d(\mathbf{n}_\alpha^m, \mathcal{A}_m) \leq \frac{1}{2}\gamma_m$ implies an H^1-bound on the sequence $\{\mathbf{n}_\alpha^m\}$ that is independent of α. Hence, after passing to another subsequence (whose notation we suppress), we can conclude that there exists a function $\mathbf{\Phi} \in H^1(\Omega; S^2)$ satisfying

$$\mathbf{n}_{\alpha_j}^m \rightharpoonup \mathbf{\Phi} \quad \text{in} \quad H^1(\Omega; S^2) \quad \text{as} \quad \alpha_j \to \infty. \tag{13}$$

Now observe that for any element $\tilde{\mathbf{n}} \in \mathcal{A}_m$ one has

$$F_{\alpha_j}(\mathbf{n}_{\alpha_j}^m) \leq F_{\alpha_j}(\tilde{\mathbf{n}}) = F_\infty(\tilde{\mathbf{n}}) = \mu_m,$$

so that

$$\int_\Omega (\mathbf{n}_{\alpha_j}^m \cdot \mathbf{e_3})^2 \, dx \leq \frac{1}{\alpha_j}\mu_m \to 0 \text{ as } \alpha_j \to \infty.$$

Hence, $\int_\Omega (\mathbf{\Phi} \cdot \mathbf{e_3})^2 \, dx = 0$ and $\mathbf{\Phi} \in H^1(\Omega; S^1)$.

Next we apply Lemma 2 (ii) and note that for each j there exists an element $\tilde{\mathbf{n}}_j \in \mathcal{A}_m$ such that $\left\|\mathbf{n}_{\alpha_j}^m - \tilde{\mathbf{n}}_j\right\|_{H^1(\Omega;\mathbb{R}^3)} \leq \frac{1}{2}\gamma_m$. Then by Lemma 2 (i), there exists a strong limit $\tilde{\mathbf{n}} \in \mathcal{A}_m$ of a subsequence of $\{\tilde{\mathbf{n}}_j\}$ and the lower-semi-continuity of the H^1-norm under weak convergence implies that $\|\mathbf{\Phi} - \tilde{\mathbf{n}}\|_{H^1(\Omega;\mathbb{R}^3)} \leq \frac{1}{2}\gamma_m$ as well. Applying Lemma 3, we conclude that $\mathbf{\Phi} \in H_m^1(\Omega; S^1)$.

Finally, we once again exploit the convexity of the integrand of F_α and (13) to see that for any element $\tilde{\mathbf{n}} \in \mathcal{A}_m$ one has

$$\begin{aligned} F_\infty(\mathbf{\Phi}) &\leq \liminf_{\alpha_j \to \infty} F_\infty(\mathbf{n}_{\alpha_j}^m) \leq \limsup_{\alpha_j \to \infty} F_\infty(\mathbf{n}_{\alpha_j}^m) \\ &\leq \limsup_{\alpha_j \to \infty} F_{\alpha_j}(\mathbf{n}_{\alpha_j}^m) \leq \limsup_{\alpha_j \to \infty} F_{\alpha_j}(\tilde{\mathbf{n}}) = F_\infty(\tilde{\mathbf{n}}) = \mu_m. \end{aligned} \tag{14}$$

Consequently, $F_\infty(\mathbf{\Phi}) = \mu_m$ since $\mathbf{\Phi}$ is admissible in (3) so that $\mathbf{\Phi} \in \mathcal{A}_m$ and all of the inequalities in (14) are in fact equalities. In particular, we have

$$\lim_{\alpha_j \to \infty} F_\infty(\mathbf{n}^m_{\alpha_j}) = F_\infty(\mathbf{\Phi})$$

and the lower-semi-continuity of each term of F_∞ separately under the convergence (13), along with the identity (5) implies that $\mathbf{n}^m_{\alpha_j} \to \mathbf{\Phi}$ strongly in H^1. Hence, $d(\mathbf{\Phi}, \mathcal{A}_m) \geq \eta$ by (12), which is impossible since $\mathbf{\Phi} \in \mathcal{A}_m$. $\square$

3. Remarks on the two-dimensional case

The entire analysis above is valid also for the case where Ω is an annulus in $\mathbb{R}^2$, or any other smooth planar domain with nontrivial topology. In this section we discuss aspects of the limiting problem of minimizing F_∞ in this 2d setting. When Ω is planar, the limit functional F_∞ takes a rather simple form

$$F_\infty(\mathbf{n}) = \int_\Omega \left[k_1 |\text{div } \mathbf{n}|^2 + k_3 |\text{curl } \mathbf{n}|^2\right] dx \tag{15}$$

for $\mathbf{n} : \Omega \to S^1$. In particular, a calculation shows that the null-Lagrangian term involving $(\nabla \mathbf{n})^2 - (\text{div } \mathbf{n})^2$ integrates to zero in this setting. We note that F_∞ in the form (15) has also arisen in the literature ([7]) as a simplification of the Oseen-Frank energy even without an external field under the special geometric assumption that Ω lies in $\mathbb{R}^2$ and the assumption that the director has zero third component.

One immediate conclusion is that in the one-constant case, i.e. when $k_1 = k_3$, the phase φ of any critical point is a harmonic function. Moreover, the second variation is positive there and one can show that such a critical point is an isolated local minimizer (up to a constant phase shift). Of course, this is not surprising, since in such a special case the functional F_∞ is proportional to $\int_\Omega |\nabla \mathbf{n}|^2 \, dx$, which was considered in [8] and [11]. The only novelty in the liquid crystal problem is the existence of extra solutions since $\mathbf{n}$ actually takes values in one-dimensional projective space, $\mathbb{P}^1$ (see the second Remark following Theorem 1). We point out though that unlike the one-constant case, in the general case $k_1 \neq k_3$, the energy is *not* invariant to a constant shift of the phase.

Returning to the general case $k_1 \neq k_3$, we ask whether particular solutions can be found in the special geometry where Ω is a symmetric annulus (i.e. a domain bounded by concentric circles). Using polar coordinates (r, θ), we claim:

Proposition 2. *Assume that Ω is a symmetric annulus. Then the minimizer of (15) in the class of homotopy type $m = 1$ is $\mathbf{n} = (\cos\theta, \sin\theta)$ if $k_3 > k_1$, or $\mathbf{n} = (-\sin\theta, \cos\theta)$ if $k_1 > k_3$.*

Proof. Assume that $k_3 > k_1$. Then for every $\mathbf{n}$

$$F_\infty(\mathbf{n}) = \int_\Omega k_1 |\nabla \mathbf{n}|^2 \, dx + (k_3 - k_1) \int_\Omega |\mathrm{curl}\, \mathbf{n}|^2 \, dx.$$

It is well known that $\mathbf{n} = (\cos\theta, \sin\theta)$ minimizes $\int_\Omega k_1 |\nabla \mathbf{n}|^2 \, dx$ in the class of homotopy class 1. It also follows from a simple calculation that curl $(\cos\theta, \sin\theta) = 0$. This establishes the proposition for $k_3 > k_1$, and the other case follows similarly. □

The success in finding a simple solution for the case $m = 1$, motivates the search for solutions of the form $\mathbf{n} = \mathbf{n}(\theta)$ for other homotopy types as well. Indeed Landau and Lifshitz ([7], Chapter VI) propose such a solution in the entire plane. They introduce the notation ψ to denote the angle between the director $\mathbf{n}$ with the radius vector to the point $(\cos\theta, \sin\theta)$. Assuming an ansatz $\psi = \psi(\theta)$, they look for a minimizer $\mathbf{n}$ of F_∞ in the restricted family

$$\mathbf{n} = \big(\cos(\theta + \psi(\theta)), \sin(\theta + \psi(\theta)) \big). \tag{16}$$

Under this restriction, the functional F_∞ can be written as

$$\frac{1}{4}(k_1 + k_3) \int (1 - \gamma \cos 2\psi)(1 + (\psi')^2) \frac{1}{r} \, d\theta \, dr, \tag{17}$$

where $\gamma = (k_3 - k_1)/(k_3 + k_1)$, and primes mean differentiation with respect to θ. The problem of minimizing the functional $\int_0^{2\pi}(1 - \gamma\cos 2\psi)(1 + (\psi')^2)\, d\theta$ leads to the Euler Lagrange equation

$$(1 - \gamma\cos 2\psi)\psi^{''} = \gamma\sin 2\psi(1 - (\psi')^2). \tag{18}$$

Since the special case $m = 1$ was already established above, one can assume $m \neq 1$. A short calculation shows that the Euler-Lagrange equation (18) can be integrated in the form

$$\theta = \int_0^{\psi} q\left(\frac{1 - \gamma\cos 2\varphi}{1 - \gamma q^2\cos 2\varphi}\right)^{1/2} d\varphi, \tag{19}$$

where the integration constant q is determined by the condition

$$(m - 1)q\int_0^{\pi}\left(\frac{1 - \gamma\cos 2\psi}{1 - \gamma q^2\cos 2\psi}\right)^{1/2} d\psi = \pi. \tag{20}$$

However, if the solution of reference [7] is considered over the infinite plane, as proposed by the authors, the energy density has a nonintegrable singularity at the origin. Landau and Lishitz propose to remedy the problem by introducing a small cutoff radius. Instead of using an artificial cutoff radius, one may ask whether the symmetric solution can be used in an annulus, where the origin is kept well outside the domain. Unfortunately, it turns out that the case $m = 1$ is special - it is the only homotopy class in which a minimizer can have the symmetry $\mathbf{n} = \mathbf{n}(\theta)$ when $k_1 \neq k_3$. One can see this by calculating the natural boundary conditions for the functional (15). A short computation gives

$$k_1\nabla\mathbf{n}\cdot\nu + (k_1 - k_3)(\nabla\times\mathbf{n})\times\nu = \lambda\mathbf{n}, \tag{21}$$

where ν is the unit normal to the boundary and λ is a proportionality constant. Substituting the ansatz (16), and using the symmetry of the annulus, the condition

(21) becomes

$$(k_1 - k_3)(1 + \psi^{'}) \sin(\psi)(\sin\theta, -\cos\theta) = \lambda(\cos(\theta+\psi), \sin(\theta+\psi)).$$

Clearly, the only solutions to the equation above are $\psi = 0,\ \lambda = 0$ and $\psi = \pi/2,\ \lambda = 1$. Both solutions correspond to a degree 1 director field.

Finally, we make some observations regarding the functional F_∞ of (15) for a general two-dimensional domain having the topology of an annulus. For this purpose we express the director $\mathbf{n}$ as $\mathbf{n} = (\cos\varphi, \sin\varphi)$ for some phase function $\varphi(x, y)$. The functional can be written explicitly in terms of φ in the form

$$\begin{aligned} F_\infty(\varphi) &= \int L(\varphi, \nabla\varphi) \\ &= \int k_1(\varphi_y \cos\varphi - \varphi_x \sin\varphi)^2 + k_3(\varphi_x \cos\varphi + \varphi_y \sin\varphi)^2 \end{aligned}$$

Varying the phase φ by a function $\beta(x, y) \in H^1(\Omega)$, one can compute the first and second variations:

$$\begin{aligned} \delta F_\infty = \quad & \textstyle\int (k_3 - k_1)\beta \left\{(\varphi_y^2 - \varphi_x^2)\frac{\sin 2\varphi}{2} + \varphi_x\varphi_y \cos 2\varphi\right\} \\ & +(k_1 \cos\varphi^2 + k_3 \sin\varphi^2)\beta_y\varphi_y + (k_3 - k_1)\beta_y\varphi_x \sin\varphi\cos\varphi \\ & +(k_3 - k_1)\beta_x\varphi_y \sin\varphi\cos\varphi + (k_1 \sin\varphi^2 + k_3 \cos\varphi^2)\beta_x\varphi_x, \end{aligned}$$

and

$$\begin{aligned} \delta^2 F_\infty = \ & \textstyle\int k_1(\beta_x \sin\varphi - \beta_y \cos\varphi)^2 + k_3(\beta_x \cos\varphi + \beta_y \sin\varphi)^2 \\ & +2(k_3 - k_1)\beta_x\beta(\varphi_y \cos 2\varphi - \varphi_x \sin 2\varphi) \\ & +2(k_3 - k_1)\beta_y\beta(\varphi_x \cos 2\varphi + \varphi_y \sin 2\varphi) \\ & +(k_3 - k_1)\beta^2 \left\{(\varphi_y^2 - \varphi_x^2)\cos 2\varphi - 2\varphi_x\varphi_y \sin 2\varphi\right\}. \end{aligned}$$

Consider the special case where the variation β is constant. Since the second variation must be nonnegative at a minimum, we conclude that

$$(k_3 - k_1)\int_\Omega \left\{(\varphi_y^2 - \varphi_x^2)\cos 2\varphi - 2\varphi_x\varphi_y \sin 2\varphi\right\} =$$

$$(k_3 - k_1)\int_\Omega \left(|\text{div}\,\mathbf{n}|^2 - |\text{curl}\,\mathbf{n}|^2\right) dx \geq 0. \tag{22}$$

Fix now a positive k, and set $k_3 = k + \delta,\ k_1 = k - \delta$, for some $|\delta| < k$. For each δ define $E(\delta) = \min F_\infty$ in the homotopy class m (we suppress the m-dependency of E).

Proposition 3. *The function E is even, i.e. $E = E(|\delta|)$, and moreover, E decreases as a function of $|\delta|$.*

Proof. To prove the evenness of E, observe that a shift of $\pi/2$ in the phase φ swaps (up to a sign) the div and curl of a director $\mathbf{n}$. Hence, if $\mathbf{n}_\delta = (\cos\psi_\delta, \sin\psi_\delta)$ is a minimizer achieving $E(\delta)$ and we let $\mathbf{n}_\delta^\perp = (-\sin\psi_\delta, \cos\psi_\delta)$ then we have

$$E(\delta) = F_\infty^\delta(\mathbf{n}_\delta) = F_\infty^{-\delta}(\mathbf{n}_\delta^\perp) \geq E(-\delta)$$

and the reverse inequality follows by similar reasoning.

To see the monotonicity of E, we introduce the notation $\mathbf{n}_i$ for a minimizer of F_∞ when $\delta = \delta_i$ (we fix k and m). Assume without loss of generality that $\delta_1 > 0$. Then, the inequality (22) implies $\int_\Omega \left(|\text{div}\,\mathbf{n}_1|^2 - |\text{curl}\,\mathbf{n}_1|^2\right)\, dx \geq 0$. Thus, for $\delta_2 > \delta_1$:

$$F_\infty^{\delta_2}(\mathbf{n}_1) = E(\delta_1) - (\delta_2 - \delta_1)\int_\Omega \left(|\text{div}\,\mathbf{n}_1|^2 - |\text{curl}\,\mathbf{n}_1|^2\right)\, dx.$$

Therefore $E(\delta_2) \leq F_\infty^{\delta_2}(\mathbf{n}_1) \leq E(\delta_1)$.

□

Finally, inspecting the first variation (22) above for the case of constant variation β, we observe that at a minimizer $\mathbf{n}$ the quantities curl $\mathbf{n}$ and div $\mathbf{n}$ are orthogonal:

$$\int_\Omega \operatorname{div} \mathbf{n} \operatorname{curl} \mathbf{n} \, dx = 0.$$

We hope to study the minimization of F_∞ over a given 1-homotopy type further since much remains to be understood. In particular, the limiting cases where either k_1 or k_2 vanish look intriguing and we suspect that for $m \neq 1$, the infimum may not even be attained.

4. SUMMARY

We proved that liquid crystal samples that occupy a topologically nontrivial domain can exhibit nontrivial local minimizers when they are subjected to a strong external field penalizing deviation from a given plane. These patterns are the liquid crystal analogue of the well-known persistent current phenomenon in superconductivity.

There are several differences between the homotopy solutions derived here and the persistent current solutions in superconductivity. First, the set of homotopy types in the liquid crystal problem is twice as large as the corresponding set in superconductivity, since the director is a headless vector. Another difference is in the way in which the local minimizers are generated. To generate a persistent current, one subjects the superconducting toroidal sample to a large magnetic field. After a supercurrent circulates in the sample, the external field is abruptly shut down trapping the supercurrent. In the liquid crystal setup, a way to generate the local minimizers is to apply boundary conditions that force a director distribution with a given degree, and then to eliminate abruptly these boundary conditions. A third difference is that the limit problem (where the director $\mathbf{n}$ in liquid crystals, or the wave function in superconductivity, are confined to the unit circle S^1) is more general and more difficult in the liquid crystal setup. In fact, we are able to say very little on this limit problem. Finally, we point out that while the phenomenon of persistent currents has been extensively observed experimentally, we are not aware of an experimental demonstration of the patterns we predict here in the liquid crystal problem.

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