Formation and stability of phase slip centers in nonuniform wires with currents

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Abstract

The problem of phase slip centers in superconducting wires with forced current is studied. It is shown that when the wire is uniform and the temperature and current are in a certain regime in the parameter plane, phase slip centers appear periodically in time at the center of the wire. When the wire is not uniform, the phase slip centers are not stationary, but rather they move through convection by the thickness variation.

Key words: phase slip centers, superconducting wires, PT symmetry
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We study superconducting wires with current flowing through them. This is a classical problem that has received a lot of attention in the last half century. It has been found through numerical integration of the time-dependent Ginzburg Landau (TDGL) equations that the solution vanishes at certain points in space-time [1]. Moreover, Ivlev and Kopnin [2] pointed out that these zeros can be considered as vortices in space-time. Such zeros are called phase slip centers (PSCs). In addition to the numerical evidence, PSCs were identified in numerous experiments. We refer the reader to [3] for a recent extensive numerical and experimental study of PSC appearance under various conditions. See also [4]. In spite of the large number of investigations into this phenomenon, the formation and motion of PSCs was not established rigorously. Moreover, PSCs form only in certain regimes of the problem's parameters (temperature, external current, wire length, etc.) and it is still not clear what these regimes are. The goal of this paper is to offer partial answers to these outstanding questions.

We thus consider a wire extending along $[-1, 1]$. A current I is fed into the wire at the left end $x = -1$. The nondimensional TDGL equations are

$$\psi_t + i\varphi\psi = \psi_{xx} + \Gamma\psi - |\psi|^2\psi, \tag{1}$$

$$\frac{i}{2}\left(\psi\psi_x^* - \psi_x\psi^*\right) - \sigma\varphi_x = I. \tag{2}$$

Here ψ is the complex-valued order parameter, φ is the electric potential and Γ is proportional to $T_c - T$. To simplify the analysis we fix the normal conductivity σ to be $\sigma = 1$. In addition to equations (1)-(2) we shall use the boundary conditions $\psi(\pm 1, t) = 0$, along with initial conditions.

In [5] we showed that a lot of information on the solution to the TGDL model can be gained by examining the spectral problem obtained through linearization of the model about the normal state $\psi = 0,\ \varphi = -Ix$. The stability of this state is determined by whether the temperature is low enough, or, using the notation introduced here, whether Γ is larger than the real part of the first eigenvalue $\lambda(I)$ of the following operator:

$$M[u] = u_{xx} + ixIu = -\lambda u, \quad u(\pm 1) = 0. \tag{3}$$

For future reference we express $\lambda = \lambda_r + i\lambda_i$. The spectral problem (3) is not self-adjoint; still it enjoys a special property called PT-symmetry, namely, it is invariant under the joint transformation of $x \to -x$ (parity) and complex conjugacy (time reversal). It has been shown that there is a value $I_c \approx 12.31$ such that the first eigenvalue λ is real for $I \leq I_c$ but not for $I > I_c$ [6], [5].

Consider now an experiment where I is fixed and Γ is increased (that is the temperature is lowered) adiabatically above $\lambda_r(I)$. To find the bifurcating solution and its stability one needs to proceed into the nonlinear regime, namely to take into account the nonlinear terms in the TGDL. It turns out that the nature of the solution depends crucially on whether I is smaller or larger than I_c. When $I < I_c$ the solution ψ tends (after some temporary transients die out) to a stationary state that is characterized by a single complex function of x. Since the generic zero set of a complex valued function has codimension 2, it follows that ψ generically has no zeros in one dimensional wires. The situation is different for $I > I_c$. The PT-symmetry implies that the two leading eigenvalues are complex conjugates, with corresponding eigenfunctions w_1 and w_2 related via $w_2^*(-x) = w_1(x)$. Consequently we face a Hopf bifurcation there. Moreover, since the entire TDGL model is also PT-symmetric, we expect the long term solution to be PT-symmetric as well.

The spectral analysis above and some numerical tests guided by it led us to classify in [5], [7] the different types of solutions in the different regimes in the parameter plane (I, Γ). The important point for us in the present context is that there exists a finite region in the (I, Γ) plane where $\psi(x, t)$ evolves, following initial transients, into a periodic function of t. This periodic function can be constructed explicitly near the bifurcation curve by expanding the full TDGL model about the leading order Hopf solution. Solving the amplitude equations for the coefficients of the Hopf bifurcating solution gives

$$\psi(x,t) = \epsilon A\bigg(\exp\big(i(\lambda_i + \omega\epsilon^2)t\big)\, w_1(x) + \exp\big(-i(\lambda_i + \omega\epsilon^2)t\big)\, w_2(x)\bigg) + O(\epsilon^3). \quad (4)$$

Here ϵ is the bifurcation parameter defined by $\Gamma = \lambda_r + \epsilon^2$, while w_1 and w_2 are the two eigenfunctions associated with the eigenvalue pair at I, normalized arbitrarily by $w_i(0) = 1$. The functions $A(I)$ and $\omega(I)$ are computed numerically by projecting the TDGL equations on the eigenspace spanned by w_1 and w_2.

The key point for our purpose here is that at $x = 0$ the PT-symmetry implies that $\psi(0, t)$ is proportional to a real-valued function, and therefore its zero set is of codimension 1. Moreover, expanding the solution as $\psi \sim \epsilon\psi_0 + O(\epsilon^3)$, the leading order is given explicitly from (4) $\psi_0(0, t) = 2\epsilon A \cos\big((\lambda_i + \omega\epsilon^2)t\big)$. Therefore, to leading order, the solution indeed vanishes periodically at $x = 0$.

This explicit computation establishes the zero set only at the leading order of the expansion (4). Following the argument of [7], we next argue that the entire solution $\psi(x, t)$ must also have a discrete array of zeros in the (x, t) plane. To see this we compute the local structure of the leading order term of ψ. For this purpose, we first Taylor expand w_1 near $x = 0$ as $w_1 = 1 + ax + ibx + O(x^2)$, where a and b are constants. It is easy to check numerically that $a = \operatorname{Re} w_1'(0) \neq 0$. The PT-symmetry then implies that $w_2 = 1 - ax + ibx + O(x^2)$. Writing $\psi_0 = \exp(i\nu t)w_1(x) + \exp(-i\nu t)w_2(x)$, (ignoring the scaling factor A and writing $\nu = \lambda_i + \omega\epsilon^2$) and translating time via the substitution $t = \pi/2\nu + \tau$, we obtain the local form $\psi_0(x, \tau) \sim 2iax - 2\nu\tau$, thus revealing a local vortex about $(x, \tau) = (0, 0)$.

Now in [7] we proved that the expansion (4) is valid in a strong norm. More precisely, we show there that

$$|\psi(x,t) - \epsilon\psi_0(x,t)| = O(\epsilon^3). \quad (5)$$

Consider then a small rectangle around a zero of $\psi_0(x, t)$. The expansion above shows that the degree, or winding number, of ψ_0 on the boundary of this rectangle is ± 1. Thanks to the estimate (5) we conclude that the degree of ψ itself is also nonzero and thus, ψ must have a zero inside this rectangle.

So far we dealt with uniform wires which enabled us to exploit the PT-symmetry of the problem. Although we use a one-dimensional model, the actual wire is three dimensional. The one-dimensional model is obtained by passing to an appropriate asymptotic limit of small wire having uniform thickness. Alternatively, in passing to the limit we can take into account the thickness variation in the form of a smooth positive weight function $d(x)$ which is the cross-sectional area of the wire. When the wire's thickness is not uniform, the symmetry is broken. Since in practice one can never achieve a uniform wire, we consider the effect of symmetry breaking by a nonuniform thickness profile. Moreover, it is known [8], [9] that a symmetry breaking due to geometric nonuniformities has important influence on the formation and location of PSCs in rings under

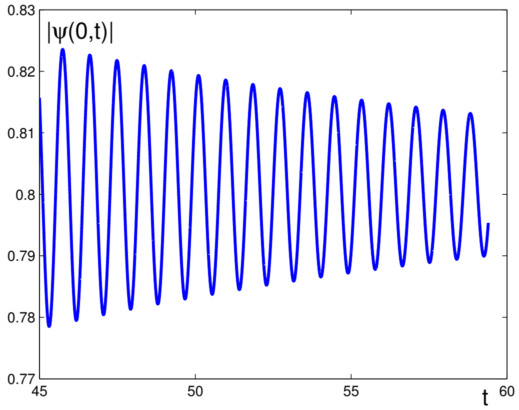

Fig. 1. The graph of $|\psi(0,t)|$ for the thickness profile $d(x) = 1 + 0.05(2x - x^2)$. The graph shows damped oscillation towards an eventual constant long term value.

an applied magnetic field. Our goal here is to examine the effect of the nonuniformity on the PSC dynamics.

The appropriate one-dimensional TDGL model takes the form

$$\psi_t + i\varphi\psi = \psi_{xx} + d_x\psi_x/d + \Gamma\psi - |\psi|^2\psi, \qquad (6)$$

where φ is determined by the current conservation equation $\frac{i}{2}\left(\psi\psi_x^* - \psi_x\psi^*\right) - \sigma\varphi_x = I/d$

Notice that the only contribution of the nonconstant thickness to equation (6) is the term $d_x\psi_x/d$. This takes the form of a convection term with $-d_x/d$ being the velocity. This fact, in addition to the breaking of the PT-symmetry, leads us to expect a zero set structure that differs from the uniform wire. Indeed we shall see that the phase slip centers in the non-uniform case are not fixed at $x = 0$; rather they move, and their motion is induced by the varying thickness.

The key idea behind establishing PSCs in nonuniform wires is again the stability of the topological degree of the vortex to small perturbations. Therefore a small deviation from uniform thickness implies that the vortices in space-time will be slightly perturbed. However, this is valid only for a finite domain in space time. The vortices will in general appear at different x locations and they might disappear altogether after a certain critical time t^*. Another way to analyze the effect of non-uniformity is to consider the new bifurcation picture. Since the problem is not PT-symmetric anymore, the eigenvalues for large I do not come in conjugate pairs. Therefore we cannot expect anymore to have a Hopf bifurcation. Instead of a pair of eigenvalues of the form $\lambda_{1,2} = \lambda_r \pm i\lambda_i$ that was present in the uniform wire case, the first pair of eigenvalues split into $\lambda_1 = \lambda_{r,1} + i\lambda_{i,1}$ and $\lambda_2 = \lambda_{r,2} + i\lambda_{i,2}$, ordered such that $\lambda_{r,1} < \lambda_{r,2}$. The associated eigenfunctions are denoted by $w_1(x)$ and $w_2(x)$, respectively. If $d(x)$ is nearly a constant, then $\lambda_{r,2}$ is larger than but very close to $\lambda_{r,1}$. Therefore, for $\Gamma > \lambda_{r,2}$ the bifurcating solution is a combination of the two eigenfunctions w_1 and w_2. Since the non-uniformity is weak and the perturbation is regular, there should be zeros near the zeros of the uniform wire.

Taking $\Gamma = \lambda_{r,2} + \epsilon^2$, for a time scale t such that $t \ll 1/(\lambda_{r,2} - \lambda_{r,1})$, we expect that for generic initial conditions the solution will develop into an approximate superposition of the two dominant eigenmodes:

$$\psi \approx \epsilon\big(A_1 \exp(-i\lambda_{i,1}t)\exp\left((\lambda_{r,2} - \lambda_{r,1})t\right) w_1(x) + A_2 \exp(-i\lambda_{i,2}t) w_2(x)\big). \qquad (7)$$

In the early stage of the evolution during which (7) is valid, we expect, as in the uniform case, that a combination of the two terms will lead to the presence of PSC's, though not, as before, at the origin. Eventually, of course, this expansion loses validity as the first dominant eigenmode takes over and one expects that the zero set of the solution will be empty. An analytic way of seeing why no PSC will be detected after some time, is to observe that once $\exp\left((\lambda_{r,2} - \lambda_{r,1})t\right)$ exceeds the maximum of $|A_2 w_2(x)/A_1 w_1(x)|$, the right hand side of equation (7) cannot vanish.

We now verify through numerical tests the theoretical analysis above. Consider first the case $d(x) = 1 + \delta(2x - x^2)$. The parameter δ measures the level of nonuniformity. We already know [5] that when $I = 18,\ \Gamma = 8.5,\ \delta = 0$ the solution is time-periodic (following brief transients). Since the Hopf bifurcation is destroyed by a nonzero δ, we expect that a single eigenfunction will prevail in the long run. Moreover, for this choice of d, we compute $d_x/d = 2\delta(1 - x)/(1 + \delta(2x - x^2))$. Therefore, the convection term $-d_x/d$ should push the PSC to the left and eventually out of the domain.

Although oscillations in the solution ψ persist for all time, these oscillations are in a sense spurious and are simply traceable to an oscillatory pre-factor in the solution related to the imaginary part of the first eigenvalue. We can alleviate these oscillations in the long term by studying the evolution of the modulus of ψ. In Figure 1 we depict the graph of $|\psi(0,t)|$ for

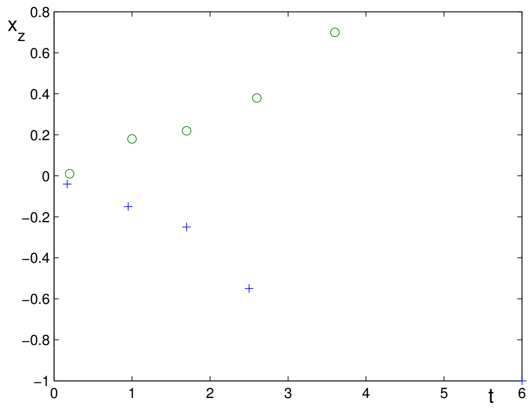

Fig. 2. The location $x_z(t)$ of the PSC. The points marked by $+$ depict the PSC location for the case $d(x) = 1+0.05(2x-x^2)$. The PSC clearly moves to the left in this case and it disappears after $t \approx 2.7$. The points marked by o give the PSC location for the case $d(x) = \exp(-0.1x)$. Now the PSC moves to the right and it disappears after $t \approx 3.6$.

the choice $\delta = 0.05$. Rather than being periodic as in the uniform wire case, it converges (albeit in an oscillatory fashion) towards a constant value, i.e. to its stationary value. Figure 2 provides information on the zero set (PSC). We draw there the location $x_z(t)$ of the PSCs. To identify a PSC we searched for points in the (x,t) plane where the absolute value of $\psi(x,t)$ is sufficiently small. To verify that there is indeed a PSC at such a point, we computed the degree of ψ around the point and verified that it is nonzero. Indeed we identified a few PSCs for $t < 2.7$. However, no PSC was detected at later time, as anticipated by the theoretical argument above.

As $t \to \infty$ the solution tends to a stable stationary state. For $d(x) = 1 + 0.05(2x - x^2)$, we depict the modulus of $\psi(x,\infty)$ in this state in Figure 3.

In a second numerical example we used $d(x) = \exp(-0.1x)$. Here the convection coefficient $-d_x/d$ is a positive constant and so in this example we expect the PSC to move to the right. The locations $x_z(t)$ of the PSC for this case are marked by the circles in Figure 2. Indeed the PSCs move to the right, and none are detected for time t larger than 3.6.

To summarize, we have rigorously established the existence of phase slip centers in superconducting wires with forced current. When the wire is uniform, we showed that for a certain regime of the parameters (temperature and current) PSCs appear at the center of the wire periodically in time. However, when the wire is not uniform, the PSCs move

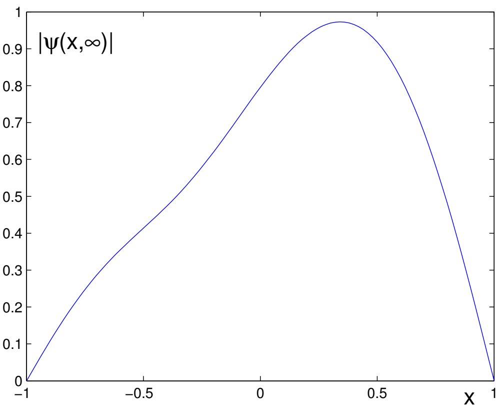

Fig. 3. The steady state distribution of $|\psi(x,\infty)|$ obtained in the large t limit for $I = 18$, $\Gamma = 8.5$ and $d(x) = 1 + 0.05(2x - x^2)$.

around, convected by the thickness variation. This means that periodic phase slips, as described in [2] or [10] practically never occur. They arise only in the ideal PT-symmetric case. In reality, there will be a finite number of PSCs, and the actual number depends on the degree of deviation from PT symmetry and on chance. After a sufficiently long time a steady state such as the one described in [11] will be reached.

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