Supercurrents in networks with constricted junctions

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Abstract. *We consider a system of superconducting networks with constrictions in the junctions. It is argued that for a family of canonical scalings of the junction size the network can be modeled by a deGennes-Alexander energy functional plus an extra term that depends on the jump of the order parameter across the junction. The current equations for such junctions are a natural generalization of the Josephson condition for a standard weak link in a wire. The model is solved for a number of special network geometries.*

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1. Introduction

About 20 years ago deGennes [1] and Alexander [2] proposed a model for describing currents in superconducting networks. Exner and his coworkers used the model to study the spectrum of quantum graphs [3], [4]. A number of authors (e.g. [5] and [6]) studied the patterns that emerge in superconducting narrow networks. An additional important mechanism in superconductivity is provided by the Josephson junction where the current is modulated by a phase difference across it. The purpose of this paper is to derive and study a model that will combine both of these features. Specifically, we construct a model for a superconducting or quantum network where the edges meet at the junctions by forming geometric weak links there. We show, in particular, that the currents flowing through the link are determined by an interference function of the different combination of phase differences across it. Therefore our model enjoys both the rich interaction between magnetic fluxes threading the holes of the network and the fascinating effect of phase-modulated currents.

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FIGURE 1. A constricted network of thin wires

We use the Ginzburg-Landau (GL) functional for a superconducting sample $\tilde{D}$:

$$\tilde{G}(\tilde{u},\tilde{\mathbf{A}}) = \int_{\tilde{D}} \left\{ \frac{1}{2m} \left| (i\hbar\tilde{\nabla} - \frac{e}{c}\tilde{\mathbf{A}})\tilde{u} \right|^2 - \alpha|\tilde{u}|^2 + \frac{\beta}{2}|\tilde{u}|^4 \right\} d\tilde{x} + \int_{R^3} \frac{1}{8\pi} |\tilde{\nabla} \times \tilde{\mathbf{A}} - \tilde{\mathbf{H}}_e|^2 d\tilde{x}. \tag{1}$$

Here $\tilde{u}$ and $\tilde{A}$ are the order parameter and the magnetic vector potential, respectively, m is the half-mass of an electron, H_e is the applied magnetic field and $d\tilde{x}$ is a volume element. We assume as usual that the parameter α has the form $\alpha = \alpha_0(T_c - T)$, where T is the temperature, and T_c is the critical temperature at zero applied field, and that β is essentially constant. To write down a nondimensional formulation we express the penetration length λ and the coherence length ξ as $\lambda = \lambda_0(1 - T/T_c)^{-1/2}$ and $\xi = \xi_0(1 - T/T_c)^{-1/2}$. We shall use $\lambda_0 = \sqrt{\frac{m\beta c^2}{4\pi\alpha_0 e^2}}$ as our length unit, and scale the magnetic vector potential according to $A = \frac{2\pi\lambda_0}{\Phi_0}\tilde{A}$, where Φ_0 is the fundamental flux quantum. We thus obtain the nondimensional formulation of the GL energy:

$$G(u,\mathbf{A}) = \int_D \left(|(i\nabla - \mathbf{A})u|^2 + \frac{1}{2}(|u|^2 - \mu^2)^2 \right) dx + \kappa^2 \int_{R^3} |\nabla \times \mathbf{A} - \mathbf{H_e}|^2 \, dx, \tag{2}$$

where $\mu = \kappa(T_c - T)^{1/2}$, $\kappa = \lambda/\xi$ is the GL parameter and D is the rescaled sample domain.

At this point, we introduce a small dimensionless parameter $\varepsilon > 0$ to facilitate our description of the superconducting sample D. Thus, we will write D_ε for D in (2) and we will assume D_ε consists of a network of thin constricted wires joined at an arbitrary number of nodes. (See Figure 1.)

To be more specific, let us focus our attention on any one of these wires. Let us suppose the wire is of length L, with cross-sectional area given by a function $g_\varepsilon(s)$ where s represents arclength along the wire so that $0 \leq s \leq L$. For simplicity let us assume that away from the constricted ends of the wire, we have a uniform cross-sectional area so that, say, $g_\varepsilon(s) \equiv \varepsilon^2$ for $s_\varepsilon < s < L - s_\varepsilon$ where s_ε is a small positive value that vanishes as $\varepsilon \to 0$. Then, near the endpoints we will assume that $g_\varepsilon(s) << \varepsilon^2$ in a way to be made precise shortly. Before we proceed further, however, we wish to emphasize two points. First, we are not making any assumption about the geometry of cross-sections (circular, square, etc.), only about the area. Second, we will work in the regime where $\varepsilon << 1$ so that, in particular, the (dimensional) diameter of a cross-section is assumed to be much smaller than λ_0 and ξ_0.

Now if one were to assume that the network of wires has no constrictions so that the cross-sectional area equals ε^2 all the way down to $s = 0$ and all the way up to $s = L$ in any one wire, then it is well known [1], [2], [7] that the three-dimensional GL energy for uniform narrow wires could be replaced by a one-dimensional model supported on the wires' skeleton, say M. That is, we could then replace G in (2) by the deGennes-Alexander (dGA) model

$$F(\psi) = \int_M \left\{ |\frac{d\psi}{ds} - iA^t\psi|^2 + \frac{1}{2}(\psi^2 - \mu^2)^2 \right\} ds, \tag{3}$$

where ψ is the average of the order parameter u across the cross-sections of the wires, and A^t is the tangential component of the magnetic potential associated with the applied field. The dGA model is supplemented by continuity conditions at the junctions of M. In particular, the order parameter ψ is continuous at the vertices, and the sum of the outgoing magnetic derivatives $\frac{d\psi}{ds} - iA^t\psi$ at a node is zero. We further point out that since the volume of the network is $\mathcal{O}(\varepsilon^2)$, the minimal energy of the original GL functional is $\mathcal{O}(\varepsilon^2)$ as well, so that in fact the correct asymptotic relation would be $\varepsilon^{-2}G \sim F$ in this uniform situation.

Instead of this assumption of uniformity, however, we now impose constrictions near the ends of the wire, that is on the arclength intervals $0 < s < s_\varepsilon$ and $L - s_\varepsilon < s < L$ where the wire being described meets the other wires at a node. To model a weak link or constriction, we will assume that on these intervals the cross-sectional area tapers dramatically, i.e. $g_\varepsilon(s) << \varepsilon^2$. Specifically, we make the assumption that as $s_\varepsilon \to 0$ one has

$$\int_0^{s_\varepsilon} \frac{1}{a_\varepsilon(s)}\, ds \to \frac{1}{b} \quad \text{and} \quad \int_{L-s_\varepsilon}^{L} \frac{1}{a_\varepsilon(s)}\, ds \to \frac{1}{b} \quad \text{as } \varepsilon \to 0 \tag{4}$$

for some nonzero positive constant b, where we have introduced

$$a_\varepsilon(s) = \frac{g_\varepsilon(s)}{\varepsilon^2}.$$

We stress that (4) implies a tapering of the wire which is significant yet is not too severe: we wish the constant b to be finite but nonzero. One can contrast this with other studies such as [7, 8, 9] in which there is either no tapering or more severe tapering at the nodes. These investigations ultimately lead to limiting problems in which critical point must satisfy other boundary conditions such as Kirchhoff-type or Dirichlet. As we shall see, assumption (4) allows for transmission across a node in a more subtle way via (10).

We also note that for simplicity alone, we have taken the constant b to be the same at each node. The factor, related to the specific rate of tapering near a node, can in fact be taken to differ at each constriction without difficulty.

Starting from (2), the assumption (4) will allow us to obtain a canonical limit model as ε approaches zero that, while still defined on the skeleton M of limiting curves, will differ from (3). We now sketch the heuristic derivation of this new limit problem. The precise mathematical proof of its validity is quite technical and involves the mathematical theories of functions of bounded variation and Γ-convergence. It can be found in [10] and [11]. However, we feel the argument below yields the main idea behind the analysis and requires only elementary notions from calculus. Once we derive it, we will proceed to demonstrate the consequences of minimizing this one-dimensional energy for a variety of configurations of wires and nodes. The end result will be formulas for maximal currents through various constricted networks expressed in terms of the magnetic fluxes passing through the holes in the network.

The main point behind the derivation below is that the constricted nodes in the network allow for a very large gradient of the minimizing order parameter $u = u_\varepsilon$ near the weak links without much expenditure of energy. Hence, in the asymptotic regime $\varepsilon \to 0$, one expects *discontinuities* to develop in $\psi = \lim \psi_\varepsilon$ where again $\psi_\varepsilon = \psi_\varepsilon(s)$ denotes the cross-sectional average of u_ε.

Derivation of one-dimensional model with constrictions

To fix notation, consider a three-dimensional constricted network of K wires $D_\varepsilon = D_\varepsilon^1 \cup \ldots \cup D_\varepsilon^K$ with skeleton M and cross-sectional area $g_\varepsilon(s)$ as given above. The graph M consists of K curves in space, $\Gamma_1, \ldots, \Gamma_K$, joined at N nodes $p_1, \ldots, p_N$. We focus on the energy associated with a particular wire D_ε^k whose skeleton is the curve Γ_k and whose endpoints are, say, the two nodes p_j and p_{j+1} so that $s = 0$ corresponds to p_j and $s = L$ corresponds to p_{j+1}. Focusing on the energy of a minimizing pair $(u_\varepsilon, \mathbf{A}_\varepsilon)$ in this particular wire, we decompose the (rescaled) energy into contributions near and away from the two nodes serving as endpoints:

$$\frac{1}{\varepsilon^2}\int_{D_\varepsilon^i}\left(|(i\nabla - \mathbf{A}_\varepsilon)u_\varepsilon|^2 + \frac{1}{2}(|u_\varepsilon|^2 - \mu^2)^2\right)dx + \frac{\kappa^2}{\varepsilon^2}\int_{\mathbb{R}^3}|\nabla \times \mathbf{A}_\varepsilon - \mathbf{H_e}|^2\,dx \sim$$

$$\frac{1}{\varepsilon^2}\int_{\{s_\varepsilon < s < L - s_\varepsilon\}} \cdot \quad + \frac{1}{\varepsilon^2}\int_{\{0 < s < s_\varepsilon\}} \cdot \quad + \frac{1}{\varepsilon^2}\int_{\{L - s_\varepsilon < s < L\}} \cdot \quad , \tag{5}$$

where we have dropped the magnetic energy integrated over $\mathbb{R}^3$ since in this limit, the effective magnetic field matches the applied magnetic field.

Now the first term on the right-hand side of (5) will converge as before to the deGennes-Alexander model (3) since the cross-sectional area of the wire is constant there. Regarding the remaining two integrals, notice that for small ε, one has $u_\varepsilon \approx \psi_\varepsilon$ so that these integrals take the form

$$\int_0^{s_\varepsilon} + \int_{L-s_\varepsilon}^{L} a_\varepsilon(s) \left\{ |\frac{d\psi_\varepsilon}{ds} - iA^t\psi_\varepsilon|^2 + \frac{1}{2}(\psi^2 - \mu^2)^2 \right\} ds,$$

and due to the smallness of a_ε, the only contribution from these integrals to the limiting energy comes the most singular terms involving the square of $\frac{d\psi_\varepsilon}{ds}$:

$$\int_0^{s_\varepsilon} + \int_{L-s_\varepsilon}^{L} a_\varepsilon(s)|\frac{d\psi_\varepsilon}{ds}|^2\, ds.$$

Now since ψ_ε has minimal energy, it should approximately minimize these integrals in the intervals $0 < s < s_\varepsilon$ and $L - s_\varepsilon < s < L$ respectively. Focusing on the first of these integrals, we then obtain from the associated Euler-Lagrange equation that

$$a_\varepsilon \psi_\varepsilon' \approx C_\varepsilon$$

for some constant C_ε. Integrating this from 0 to s_ε and using (4) yields

$$\psi_\varepsilon(s_\varepsilon) - \psi_\varepsilon(0) \approx C_\varepsilon \int_0^{s_\varepsilon} \frac{1}{a_\varepsilon(s)} ds \approx \frac{C_\varepsilon}{b}.$$

Hence,

$$\int_0^{s_\varepsilon} a_\varepsilon(s)|\frac{d\psi_\varepsilon}{ds}|^2\, ds \approx \int_0^{s_\varepsilon} \frac{C_\varepsilon^2}{a_\varepsilon(s)} ds \to b|\psi_k(p_j) - a_j|^2 \tag{6}$$

$\psi_k(p_j)$ is the value of the limiting order parameter ψ as it approaches the node p_j along the k^{th} curve Γ_k and a_j is the asymptotic value of order parameter right at the node. Again, these two values will not be the same due to the development of a discontinuity at the nodes. Similarly, for the integral over $L - s_\varepsilon < s < L$ one obtains

$$\int_{L-s_\varepsilon}^{L} a_\varepsilon(s)|\frac{d\psi_\varepsilon}{ds}|^2\, ds \to b|\psi_k(p_{j+1}) - a_{j+1}|^2. \tag{7}$$

Combining (5), (6) and (7) one sees that the asymptotic contribution to the total energy from one wire D_ε^k is given by

$$\int_{\Gamma_k} \left\{ |\frac{d\psi_\varepsilon}{ds} - iA^t\psi_\varepsilon|^2 + \frac{1}{2}(\psi^2 - \mu^2)^2 \right\} ds + b|\psi_k(p_j) - a_j|^2 + b|\psi_k(p_{j+1}) - a_{j+1}|^2.$$

Adding in the contribution from all of the wires comprising D_ε we then obtain a limiting energy of the form

$$F(\psi) = \sum_{k=1}^{K} \int_{\Gamma_k} \left\{ |\frac{d\psi}{ds} - iA^t\psi|^2 + \frac{1}{2}(\psi^2 - \mu^2)^2 \right\} ds + \min_{a_1, a_2, \ldots, a_N} \sum_{j=1}^{N} \sum_{k=1}^{K_j} b|\psi_k(p_j) - a_j|^2. \tag{8}$$

Here K_j denotes the number of curves in the collection $\{\Gamma_k\}$ that converge at a node p_j.

For any choice of the order parameter values $\psi_k(p_j)$ at a given node p_j, elementary minimization shows that the optimal value of a_j is the average of $\psi_1(p_j), \ldots, \psi_{K_j}(p_j)$. Then if we let say $\tilde{\psi}$ be a function that is nonzero only along the k^{th} wire, where it is arbitrary except for the requirement that it vanish at the node other than p_j, we can easily compute the first variation condition of criticality, $\frac{dF(\psi + t\tilde{\psi})}{dt} = 0$ at $t = 0$. This leads us to the associated Euler-Lagrange equation:

$$\left(\frac{d}{ds} - iA^t\right)^2 \psi_k = (\mu^2 - |\psi_k|^2)\psi_k \quad \text{on each } \Gamma_k, \tag{9}$$

as well as the natural boundary conditions at each junction

$$\left(\frac{d}{ds} - iA^t\right)\psi_k(p_j) = b\big(\psi_k(p_j) - \frac{1}{K_j}\sum_{m=1}^{K_j} \psi_m(p_j)\big). \tag{10}$$

Of particular interest is the current through each junction. It is convenient to express it by using the polar representation for the order parameter corresponding to the k^{th} wire near the j^{th} node, i.e. $\psi_k(p_j) = \rho_{k,j} e^{i\varphi_{k,j}}$. Denoting the current through the curve Γ_k near node p_j by $J_{k,j}$, we find through (10) that

$$J_{k,j} := \mathrm{Im}\left[\psi_k(p_j)^* \left(\frac{d}{ds} - iA^t\right)\psi_k(p_j)\right] = \frac{b}{K_j} \sum_{l=1}^{K_j} \rho_{k,j}\rho_{l,j} \sin(\varphi_{k,j} - \varphi_{l,j}). \tag{11}$$

The analysis in the next section is based on (11) and can serve as a guide for experimentalists in preparing constricted samples exhibiting interesting interference patterns in the current/magnetic flux relationship. While only a few wires are shown in Figure 1 for the sake of clarity, we emphasize that the model is valid for any number of wires.

2. Examples

We proceed to calculate the current associated with various networks of constricted junctions. For this purpose we consider the model (8) in several geometries. In each case the analysis will be simplified by assuming the London limit where the absolute value of the order parameter is a constant along each wire. Without loss of generality we take this constant to be 1, so that in particular, all $\rho_{k,j} = 1$ and

Figure 2. The figure 8 network

thus the current is determined solely by phase variations along the curves $\Gamma_1, \ldots, \Gamma_K$ and in particular by the limiting values of these phases, $\varphi_{k,j}$, $k = 1, 2, \ldots, K$, $j = 1, 2, \ldots, N$ as one approaches the j^{th} junction along the k^{th} wire. We therefore replace the energy (8) by

$$F(\varphi) = \sum_{k=1}^{K} \int_{\Gamma_k} \left(\frac{d\varphi}{ds} - A^t \right)^2 ds + \min_{a_1, a_2, \ldots, a_N} \sum_{j=1}^{N} \sum_{k=1}^{K_j} b|e^{i\varphi_{k,j}} - a_j|^2. \tag{12}$$

It will be useful to observe that the second term on the right hand side of (12) can be written alternatively as

$$F_2 = \sum_{j=1}^{N} b \left(K_j - 1 - \frac{2}{K_j} \sum_{l>k} \cos(\varphi_{l,j} - \varphi_{k,j}) \right). \tag{13}$$

A further simplification that we use is to assume that the geometric parameter b is small. This means that the current is of $O(b)$. Therefore to leading order the current is zero, and in that approximation the phase gradient is exactly the vector potential. Moreover, under our assumption the first term in the energy (12) is of $O(b^2)$, and the second term, involving the jumps in the phase of the order parameter at the junction, $\varphi_{l,j} - \varphi_{k,j}$, is of $O(b)$. Therefore, to leading order the energy is determined by this term.

The figure 8 network. Consider the network drawn in Figure 2. The network consists of two loops, not necessarily of the same size. The magnetic fluxes Φ_1 and Φ_2 drive the currents in the network. The incoming and outgoing curves at the junction are indexed as in the drawing. The curve forming the boundary of loop j will be denoted by j. The phases at the two ends of this curve are denoted by φ_j and $\varphi_{\bar{j}}$. The assumption $b \ll 1$ implies

$$\varphi_{\bar{1}} = \varphi_1 + \Phi_1, \quad \varphi_{\bar{2}} = \varphi_2 + \Phi_2. \tag{14}$$

Thanks to the current relation (11) applied to both sides of loop 1 we can write

$$J_1 = \frac{b}{4}(\sin\Delta_{\bar{1}1} + \sin\Delta_{21} + \sin\Delta_{\bar{2}1}) = b(\sin\Delta_{\bar{1}1} + \sin\Delta_{\bar{1}2} + \sin\Delta_{\bar{1}\bar{2}}), \tag{15}$$

where we use the notation $\Delta_{ij} = \varphi_i - \varphi_j$. Simple trigonometric identities imply then the relation

$$\varphi_1 + \varphi_{\bar{1}} = \varphi_2 + \varphi_{\bar{2}} + 2\pi n, \tag{16}$$

for some integer n. Notice that equation (16) is valid, in fact, for all b. The last equation enables us to write

$$J_1 = \frac{b}{4}(\sin\Phi_1 + 2\sin\frac{\Phi_1}{2}\cos\pi n\cos\frac{\Phi_2}{2}). \tag{17}$$

A similar formula can be derived for the current in the other loop.

It remains to determine the unknown integer n. Clearly it is enough to consider n in the set $\{0,1\}$. The value of n in this set depends on the fluxes Φ_1 and Φ_2. To find n, we substitute (14) into the energy (13). Using further (16) it is seen that n is such that the expression

$$\cos\Delta_{12} + \cos(\Delta_{12} + \Phi_1) + \cos(\Delta_{12} - \Phi_2) + \cos(\Delta_{12} + \Phi_1 - \Phi_2) \tag{18}$$

is maximized under the constraint

$$\Delta_{12} + \frac{\Phi_1 - \Phi_2}{2} = \pi n. \tag{19}$$

We shall now argue that n is determined by

$$n(\Phi_1,\Phi_2) = \begin{cases} 0 & \{0<\Phi_1<\pi, 0<\Phi_2<\pi\}\bigcup\{\pi<\Phi_1<2\pi, \pi<\Phi_2<2\pi\} \\ 1 & \{0<\Phi_1<\pi, \pi<\Phi_2<2\pi\}\bigcup\{\pi<\Phi_1<2\pi, 0<\Phi_2<\pi\} \end{cases} \tag{20}$$

Notice that (20) can be alternatively expressed using a logic notation. For this purpose a flux Φ is considered as *True* is $\pi < \Phi < 2\pi$, and *False* is $0 < \Phi < \pi$. Then

$$n(\Phi_1,\Phi_2) = \Phi_1 \text{ xor } \Phi_2. \tag{21}$$

To justify (20) it is useful to observe a geometric interpretation of the optimization problem. The last term in the energy F (see (12)), which is the object that needs to be minimized, is the sum of the square of the distances between the points

$$\exp(i\varphi_1), \exp(i(\varphi_1 + \Phi_1)), \exp(i\varphi_2), \exp(i(\varphi_2 + \Phi_2))$$

on the unit circle from their center of mass. Denote by θ the angle on the unit circle. Since the problem is invariant to the rotation of all the phases φ by a constant angle, we can choose an orientation in which φ_1 and $\varphi_{\bar{1}}$ are placed symmetrically around $\theta = 0$. The constraint (16) implies that the phases $\varphi_2, \varphi_{\bar{2}}$ are then also symmetric about $\theta = 0$. The case $n = 0$ means that $\varphi_{\bar{2}} - \varphi_{\bar{1}} = \varphi_1 - \varphi_2$, while when $n = 1$, one has $\varphi_{\bar{1}} - \varphi_{\bar{2}} = 2\pi - (\varphi_1 - \varphi_2)$. Consider for instance the case $0 < \Phi_1 < \pi$. In this case, $-\pi/2 < \varphi_1 \leq 0 \leq \varphi_{\bar{1}} < \pi/2$. Assume further that $0 < \Phi_2 < \pi$. If $n = 0$, then $-\pi/2 < \varphi_2, \varphi_{\bar{2}} < \pi/2$, while the case $n = 1$ corresponds to $\pi/2 < \varphi_2, \varphi_{\bar{2}} < 3\pi/2$.

We now invoke a simple geometric principle that is pivotal to the analysis here and in the sequel: $\{(\xi_i, \eta_i)\}$ be N points on the unit circle, and denote their center of mass by (ξ, η). Then a direct calculation yields

$$\sum_{i=1}^{N} (\xi_i - \xi)^2 + (\eta_i - \eta)^2 = N\left(1 - (\xi^2 + \eta^2)\right). \tag{22}$$

It follows that the sum on the left hand side of (22) is a decreasing function of the distance of (ξ, η) from the origin. It then follows that when $0 < \Phi_1, \Phi_2 < \pi$, the choice $n = 0$ is optimal. Applying the same argument to all the four squares on the right hand side of (20) that constitute the $2\pi \times 2\pi$ square leads to (20).

Consider the current J_1 given by (17) for a fixed Φ_1. The first term on the right hand side of (17) is the current in loop 1 due to the flux through the hole bounded by that loop. The second term represents the current through loop 1 induced by the flux through the *other hole*. Notice that while the problem is 2π periodic in each flux separately, then for a fixed value of Φ_1, the second term on (17) has the same sign for *all* values of Φ_2.

The lasso network. Consider next the network depicted in Figure 3. A current I passes along the base wire. The junction at the point P consists of four constricted pipes. The edges are numbered as in the drawing. Notice that our assumptions above on the order parameter's absolute value being constant and the current being weak are consistent with the standard modeling of Josephson junctions and SQUID devices [12], [13], [14]. In fact, we would like to consider the Lasso network as a flexible analog of a SQUID. Proceeding as in the figure 8 network, we write

$$J_2 = b(\sin \Delta_{21} + \sin \Delta_{2\bar{1}} + \sin \Delta_{2\bar{2}}) = b(\sin \Delta_{1\bar{2}} + \sin \Delta_{\bar{1}\bar{2}} + \sin \Delta_{2\bar{2}}), \tag{23}$$

and similarly,

$$J_1 = b(\sin \Delta_{\bar{1}1} + \sin \Delta_{21} + \sin \Delta_{\bar{2}1}) = b(\sin \Delta_{\bar{1}1} + \sin \Delta_{\bar{1}2} + \sin \Delta_{\bar{1}\bar{2}}). \tag{24}$$

Just as in the figure 8 example, the phase $\varphi_{\bar{2}}$ is determined by $\varphi_{\bar{2}} = \varphi_2 + \Phi$, where Φ is the flux through the hole. The difference is that the phase difference $\Delta_{1\bar{1}}$ is determined now indirectly by the

FIGURE 3. The lasso network. The plus sign indicates the direction of magnetic flux.

current I. For the time being, consider this phase difference as a parameter and write $\varphi_{\bar{1}} = \varphi_1 + \Delta_{1\bar{1}}$. The relation (16) holds in the lasso case as well by the same reasoning as in the figure 8 geometry.

At this point there are still two unknowns: The phase difference $\Delta_{\bar{1}1}$ and the binary integer n. Before considering the energy, we must satisfy the constraint $J_1 = I$. Using (24) and (16), this constraint can be expressed as

$$I/b = 2\sin\left(\frac{\Delta_{\bar{1}1}}{2} + \pi n\right)\cos\frac{\Phi}{2} + \sin\Delta_{\bar{1}1} = 2\sin\frac{\Delta_{\bar{1}1}}{2}\left(\cos\frac{\Phi}{2}\cos\pi n + \cos\frac{\Delta_{\bar{1}1}}{2}\right). \tag{25}$$

One could solve (25) for $\Delta_{\bar{1}1}$. If there exist two solutions (one for each choice of n), one can proceed to determine the optimal n from the energy functional, just as in the preceding example. If there exists only a solution for one value of n, it can be checked that this value corresponds to a minimal point of the energy. However we shall not pursue this computation here.

An object of physical interest is the maximal supercurrent through the junction. Observe that $\sin\frac{\Delta_{\bar{1}1}}{2}$ is never negative. Thus, if, for instance, $I > 0$, then the maximal current will be achieved at some value of $\Delta_{\bar{1}1}$ in the interval $[0, \pi]$, and by the rule $n =$ nearest integer to $\left(\frac{\Phi}{2\pi}\right) \mod(2)$. Under this choice, the optimal phase difference is

$$\cos\frac{\Delta_{\bar{1}1}}{2} = \frac{1}{4}\left(-\cos\pi n\cos\frac{\Phi}{2} + \sqrt{8 + \cos^2\frac{\Phi}{2}}\right). \tag{26}$$

Similarly one can find the maximal current for negative I. Recall that the maximal current through a SQUID is given in our notation by $I_M^S = 2|\cos\Phi/2|$. In Figure 4 we compare the maximal currents through a SQUID and through the lasso with a constricted junction. Notice that since the base wire is superconducting, there is always some net maximal current for *any* flux Φ. In other words, the loop

Figure 4. The solid curve denotes the maximal current through the junction in the lasso device (vertical axis) as a function of the flux Φ trapped in the loop (horizontal axis). The dashed line is the maximal current in a corresponding standard SQUID as a function of flux.

is inactive when there is a half-integer number of fluxes, but otherwise it helps to pass current through the junction. This is why the solid line in the figure has a base value of 1.

Network with three leaves.

We next consider a three leaves network. Some features of this network are similar to the figure 8 case we looked at earlier, but, as we shall see, some new effects are present as well. The network is depicted in Figure 5. The currents are driven by the fluxes threading the three holes. The phase of the order parameter at the junction will be denoted similarly to the figure 8 convention.

The assumption of weak current ($b \ll 1$) leads just as before to

$$\varphi_{\bar{j}} = \varphi_j + \Phi_j. \tag{27}$$

One way to proceed is to write the currents and compare them just as we did in (15). An alternative way is to substitute (27) in the leading term in the energy (13). The current equation (15) is equivalent to equating the first derivatives (in the parameters φ_i) of the energy (13) to zero; namely, the goal is to maximize $\sum_{j<k} \cos(\varphi_j - \varphi_k)$ subject to (27), where the summation is also on the barred indices. Equating the first variation to zero gives three equations. Because of the phase shift invariance, only

two of them are independent:

$$\cos\frac{\Phi_2}{2}\sin(\Delta_{12}+\frac{\Phi_1-\Phi_2}{2})+\cos\frac{\Phi_3}{2}\sin(\Delta_{13}+\frac{\Phi_1-\Phi_3}{2})=0, \tag{28}$$

$$\cos\frac{\Phi_1}{2}\sin(-\Delta_{12}+\frac{\Phi_2-\Phi_1}{2})+\cos\frac{\Phi_3}{2}\sin(\Delta_{13}-\Delta_{12}+\frac{\Phi_2-\Phi_3}{2})=0. \tag{29}$$

Indeed, one possible solution is similar to (16), say

$$\zeta_{12}:=\Delta_{12}+\frac{\Phi_1-\Phi_2}{2}=\pi n_{12},\ \ \zeta_{13}:=\Delta_{13}+\frac{\Phi_1-\Phi_3}{2}=\pi n_{13}, \tag{30}$$

where n_{12}, n_{13} are binary integers. There exist, however, additional solutions. One way to proceed is to compute the other solutions numerically, and to examine the stability of each solution. We show, however, that the problem can be fully solved by an analytical (or rather geometrical) method. In particular it will be shown that the solutions other than (30) are never stable.

The geometrical analysis is based on the center of mass argument that was used previously in the case of the figure 8. Denote the center of mass of each pair $\{\exp(i\varphi_i), \exp(i(\varphi_i+\Phi_i)\}$ by (x_i, y_i). Let us fix any two of the three pairs and denote the center of mass of these four points by (x, y). Let us further denote the remaining pair by $\{\exp(i\theta), \exp(i(\theta+\Phi))\}$. The square of the length of the global center of mass of all three pairs can thus be written as

$$\frac{1}{9}\left(2x+\cos\frac{\Phi}{2}\cos(\theta+\frac{\Phi}{2})\right)^2+\frac{1}{9}\left(2y+\cos\frac{\Phi}{2}\sin(\theta+\frac{\Phi}{2})\right)^2.$$

Therefore, up to an irrelevant additive constant, it remains to look for maximizers of

$$L(\theta)=x\cos\frac{\Phi}{2}\cos(\theta+\frac{\Phi}{2})+y\cos\frac{\Phi}{2}\sin(\theta+\frac{\Phi}{2}). \tag{31}$$

The equation $\frac{dL}{d\theta}=0$ implies $y/x=\tan(\theta+\frac{\Phi}{2})$. Therefore the radius vector to (x, y) bisects the sector $(\theta, \theta+\Phi)$. Since this argument applies to any combination of two pairs, it follows that the center of masses $\{(x_i, y_i)\}$ must be arranged in one of two ways. In the first one all three centers of mass are arranged on one line (which can be chosen without loss of generality to be the x axis), and in the other one the three radius vectors from the circle's center to the three centers of mass form equal angles of $2\pi/3$ between each radius vector and the preceding one.

Consider first the second case. Notice from the argument above that the global center of mass (of all 3 pairs) must lie on the radius vector connecting the origin to each of the centers of mass. The only way this can be achieved, when the centers of mass of the individual pairs are not colinear, is that the global center of mass would be *at* the circle's center. Therefore L is minimal for such an arrangement, and the identity (22) implies that such a configuration is actually a maximum point for the energy.

Figure 5. The three leaves network

Curiously we notice that in this case ζ_{ij} is either $2\pi/3$ or $4\pi/3$. To see this, observe from (30) that ζ_{12}, say, is the angular difference between the middle of the sector $\{\varphi_1, \varphi_1 + \Phi_1\}$ and the middle of the sector $\{\varphi_2, \varphi_2 + \Phi_2\}$.

When the centers of mass are colinear, all the pairs are symmetric about the angle $\theta = 0$. It easy to check that all such configurations correspond to the binary integer $n_i = 0$ or $n_i = 1$. The actual value for a minimizer depends on the three fluxes. Suppose, for instance, that $0 < \Phi_1, \Phi_2 < \pi$, and $\pi < \Phi_3 < 2\pi$. Then simple geometric considerations of the kind we already used extensively imply that $(n_{12}, n_{13}) = (0, 1)$ is a minimum point, while all other critical points are not stable. The general formula for the stable critical points can be written using the notation of (21) as

$$n_{1j} = \Phi_1 \text{ xor } \Phi_j. \tag{32}$$

The currents in each loop is conveniently expressed symmetrically in terms of the fluxes and of the binary parameters n_{ij}. For example, the current in loop 1 is

$$\frac{1}{2}(J_1 + J_{\bar{1}}) = \sin\Phi_1 + 2\sin\frac{\Phi_1}{2}\left(\cos\frac{\Phi_2}{2}\cos\pi n_{12} + \cos\frac{\Phi_3}{2}\cos\pi n_{13}\right). \tag{33}$$

3. Conclusions

A model for supercurrents in networks of quantum wires with constricted junctions was derived. The geometry comprises of narrow pipes and even narrower junctions connecting the pipes. The model consists of a dGA energy functional for the edges of the one-dimensional limit graph, plus a term that depends on the jump of the order parameter wave function across the junctions. The energy function is presented in (8). The actual network is described very explicitly, both in the pipes and in the junctions. Therefore experimentalists can use our construction to prepare actual samples exhibiting the behavior that we computed and hopefully other phenomena as well.

The Euler Lagrange equations were solved in the London limit for a number of particular geometries to demonstrate the peculiar properties of supercurrents in such constricted networks.

Finally, in the study of quantum wire networks with regular junctions it is known that the equations for the phase of the order parameter in the network can be integrated [15]. The integration yields an equivalent energy functional in terms of the absolute value of the order parameter, the fluxes through the holes bounded by the networks, the circulation of the phase along a basis of closed orbits in the network and the network topology. A similar integration can be performed for the model (8). The computations are similar to [15], so we do not spell out the details. Again, it turns out the magnetic vector potential enters the problem only through the fluxes through the holes.

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