The Onset Problem for a Thin Superconducting Loop in a Large Magnetic Field

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Abstract

We present a rigorous analysis of the eigenvalue problem associated with the onset of superconductivity for a thin domain in the presence of a large applied magnetic field. We prove the validity of the formal result of [20] revealing that in this double limit of thin domain and large field, the appropriate Rayleigh quotient differs from the standard one for order 1 applied fields through the addition of a potential depending on the field.

1 Introduction

When cooled below a certain critical temperature, a superconductor undergoes a phase transition from the normal state to the superconducting state, in which it can support electric currents without resistance. In the presence of an applied magnetic field acting on the superconducting material, the critical temperature associated with this phase transition decreases. The problem we study is one example of this phase transition between normal and superconducting states. In recent years, numerous authors have undertaken the mathematical study of this phase transition on a fixed sample in

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the asymptotic limit of large field starting from the Ginzburg-Landau model [26], including, for example, [1, 2, 10, 11, 16, 17, 18]. In another vein, there have also been many studies of this phase transition taking place on a thin domain subjected to a fixed order 1 applied field, i.e. in the asymptotic regime where a two-dimensional or three-dimensional domain collapses to a one-dimensional curve. This interest goes back to the observation by physicists, [15], that when a thin superconducting ring is subjected to an applied magnetic field, the critical temperature/applied field relationship is an oscillatory one. The behavior is now referred to as Little-Parks oscillations. Mathematical studies of Ginzburg-Landau is this asymptotic regime include [21, 22, 24]. More generally, there has been interest in the question of how elliptic variational problems behave asymptotically as domains collapse onto one-dimensional graphs, see e.g. [13, 14].

The situation we treat here, however, stems from the formal work of [20] in what is perhaps the first and only mathematical study of the asymptotics associated with this phase transition in the double limit of thin domains subjected to large applied magnetic fields. Using formal asymptotic expansions to analyze the Ginzburg-Landau system in this setting, the authors of [20] predict that the eigenvalue problem associated with this transition is altered by the addition of a potential related to the applied field. The purpose of this paper is to make rigorous some of the results in [20]. A second article, [25], will pursue the problem of full Γ-convergence of the associated energy.

To arrive at the relevant eigenvalue problem one begins with a non-dimensionalized version of the Ginzburg-Landau energy

$$\begin{aligned} G(\psi,\tilde{\mathbf{A}}) &= \int_U \left(\left|(\nabla - i\tilde{\mathbf{A}})\psi\right|^2 + \frac{\sigma}{2}(|\psi|^2-\mu^2)^2 \right) dx \\ &+ \sigma^{-1}\kappa^2 \int_{\mathbb{R}^3} |\nabla\times\tilde{\mathbf{A}} - \mathbf{B_e}|^2\, dx. \end{aligned} \tag{1.1}$$

cf. e.g. [23]. Here $U \subset \mathbb{R}^3$ denotes the region occupied by the sample (where we have non-dimensionalized with respect to a characteristic diameter of the sample so that U is of diameter $\mathcal{O}(1)$), $\psi : U \to \mathbb{C}$ is the complex-valued order parameter, $\mathbf{B_e} : \mathbb{R}^3 \to \mathbb{R}^3$ is the applied magnetic field and $\tilde{\mathbf{A}} : \mathbb{R}^3 \to \mathbb{R}^3$ is the magnetic potential whose curl corresponds to the effective magnetic field. The parameter κ is the Ginzburg-Landau parameter (not to be confused with κ = curvature in Section 2) and σ is another material parameter. Neither will play a role in the analysis and σ will henceforth be set equal to 1. The utility of this particular non-dimensionalization is that it retains the temperature dependence within the parameter μ^2. This parameter is proportional to $T_c(0)-T$ where T is temperature and $T_c(0)$ denotes the critical temperature of the afore-mentioned phase transition in the absence of any applied field. Since we will be studying the energy in the presence of an applied field, it is understood that we are in the regime $T < T_c(0)$, i.e. $\mu^2 > 0$, and that $T_c(\mathbf{B_e}) < T_c(0)$.

The normal state in this description is given by the pair $\psi \equiv 0$, $\tilde{\mathbf{A}} \equiv \mathbf{A_e}$ where $\mathbf{A_e}$ is the applied potential satisfying the condition $\nabla \times \mathbf{A_e} = \mathbf{B_e}$. The phase transition associated with the onset of superconductivity is then characterized by the value of μ^2 (hence, of temperature) at which this normal state loses stability. A routine calculation of the second variation of $G(\psi, \tilde{\mathbf{A}})$ about the normal state then leads one to an eigenvalue problem that determines the value of μ^2, and hence of the critical temperature, at which onset occurs:

$$\mu_c^2(\mathbf{B_e}) = \inf_{\psi} \frac{\int_U |(\nabla - i\mathbf{A_e})\psi|^2 \, dx}{\int_U |\psi|^2 \, dx}. \tag{1.2}$$

Here we make two assumptions that distinguish our investigation from most earlier works and set it in the context of [20]. First, we assume the sample domain U is in fact a sequence of domains $\{U_\varepsilon\}$ consisting of ε-neighborhoods of a limiting simple closed curve in a manner to be made precise in the next section. Second, we assume the given applied field $\mathbf{B_e}$ takes the form

$$\mathbf{B_e} = \frac{\mathbf{B}}{\varepsilon}$$

for a given arbitrary and ε-independent field $\mathbf{B}$ having potential $\mathbf{A}$. This leads us to the object of our study, namely the asymptotic behavior of the following sequence of eigenvalue problems:

$$\lambda_\varepsilon := \inf_{\substack{\psi \in H^1(U_\varepsilon) \\ \psi \not\equiv 0}} E_\varepsilon(\psi), \tag{1.3}$$

where

$$E_\varepsilon(\psi) := \frac{\int_{U_\varepsilon} |(\nabla - i\frac{\mathbf{A}}{\varepsilon})\psi|^2 dx}{\int_{U_\varepsilon} |\psi|^2 dx} \tag{1.4}$$

After breaking the given applied field $\mathbf{B}$ into components B_1, B_2 and B_3 lying along the tangent, normal and bi-normal to the limiting curve respectively, and doing the same for its associated potential $\mathbf{A}$, our main results consist of showing that the eigenvalue λ_ε and corresponding first eigenfunction are drawn asymptotically towards the eigenvalue and first eigenfunction of the 1-dimensional Rayleigh quotient

$$G_{\beta_\varepsilon}(u) := \frac{\int_0^L |\partial_1 u - i\beta_\varepsilon u|^2 + g(y_1)|u|^2 dy_1}{\int_0^L |u|^2 dy_1}. \tag{1.5}$$

Here L is the length of the curve, β_ε is given by (2.13), and the potential $g : [0, L] \to \mathbb{R}$ is given by

$$g(y_1) := \frac{1}{8}(B_1(y_1, 0, 0))^2 + \frac{1}{4}(B_2(y_1, 0, 0))^2 + \frac{1}{4}(B_3(y_1, 0, 0))^2. \tag{1.6}$$

(See Theorems 4.3 and 4.4).

The techniques involved come from dimension reduction and Γ-convergence. One factor complicating the analysis here is that as $\varepsilon \to 0$, the eigenvalue and eigenfunction rapidly oscillate, that is, the Little-Parks oscillations still prevail. This often necessitates passage to zero along convergent subsequences $\{\beta_{\varepsilon_j}\}$.

In the second section we formulate the problem in terms of coordinates given by the Frenet frame of the limiting curve. In the third section we establish some preliminary estimates and then in the final section we present proofs of the main results.

2 Formulation of the Problem

We begin with a precise description of the shrinking domains $\{U_\varepsilon\}$. To this end, let $\mathbf{r} : [0, L] \to \mathbb{R}^3$ be a simple, closed C^2 curve parameterized by arc length and let $\mathbf{t}$ be the unit tangent vector, $\mathbf{n}$ the unit normal vector, and $\mathbf{b}$ the unit binormal vector. The triple $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ forms a Frenet frame for this curve. We define a thin neighborhood $U_\varepsilon \subset \mathbb{R}^3$ of the curve $\mathbf{r}$ as the image of the cylinder

$$\Omega_\varepsilon = \{(z_1, z_2, z_3) : 0 \le z_1 \le L, 0 \le z_2^2 + z_3^3 < \varepsilon^2\} \tag{2.1}$$

under the mapping

$$T(z_1, z_2, z_3) = \mathbf{r}(z_1) + z_2\mathbf{n}(z_1) + z_3\mathbf{b}(z_1). \tag{2.2}$$

That is, $U_\varepsilon \equiv T(\Omega_\varepsilon)$.

We will consider the following Rayleigh quotient

$$E_\varepsilon(\psi) \equiv \frac{\int_{U_\varepsilon} |(\nabla - i\frac{\mathbf{A}}{\varepsilon})\psi|^2 dx}{\int_{U_\varepsilon} |\psi|^2 dx} \tag{2.3}$$

and its corresponding first eigenvalue

$$\lambda_\varepsilon \equiv \inf_{\substack{\psi \in H^1(U_\varepsilon) \\ \psi \neq 0}} E_\varepsilon(\psi). \tag{2.4}$$

We recall from the introduction that $\psi : U_\varepsilon \to \mathbb{C}$ is the order parameter, and $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ is the applied magnetic potential related to the given applied magnetic field $\mathbf{B}$ through $\nabla \times \mathbf{A} = \mathbf{B}$. We take $\mathbf{B} : \mathbb{R}^3 \to \mathbb{R}^3$ to be a C^2 vector field defined.

Using the mapping T given in (2.2) along with the Frenet equations, we can express the gradient of a function, the divergence of a vector field and the curl of a vector field defined on the (z_1, z_2, z_3) coordinates as follows. We will use κ and τ to denote the curvature and the torsion of $\mathbf{r}$, respectively. For any smooth scalar-valued function $\psi = \psi(z_1, z_2, z_3)$ and any smooth vector field $\mathbf{F}(z_1, z_2, z_3) = F_1(z_1, z_2, z_3)\mathbf{t} + F_2(z_1, z_2, z_3)\mathbf{n} + F_3(z_1, z_2, z_3)\mathbf{b}$, we have following the identities :

$$\nabla\psi = \frac{1}{1-\kappa z_2}(\partial_1\psi + \tau z_3 \partial_2\psi - \tau z_2 \partial_3\psi)\mathbf{t} + \partial_2\psi\mathbf{n} + \partial_3\psi\mathbf{b} \tag{2.5}$$

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{1}{1-\kappa z_2}(\partial_1 F_1 - \tau F_2) + \partial_2 F_2 + \partial_3 F_3 \\ &\quad + \frac{\tau z_3}{1-\kappa z_2}\partial_2 F_1 - \frac{\tau z_2}{1-\kappa z_2}\partial_3 F_1 \end{aligned} \tag{2.6}$$

$$\begin{aligned} \nabla \times \mathbf{F} &= (\partial_2 F_3 - \partial_3 F_2)\mathbf{t} \\ &\quad + \Bigg(\partial_3 F_1 - \frac{1}{1-\kappa z_2}\partial_1 F_3 \\ &\quad - \frac{\tau z_3}{1-\kappa z_2}\partial_2 F_3 + \frac{\tau z_2}{1-\kappa z_2}\partial_3 F_3 - \frac{\tau}{1-\kappa z_2}F_2\Bigg)\mathbf{n} \\ &\quad + \Bigg(\frac{1}{1-\kappa z_2}\partial_1 F_2 - \partial_2 F_1 + \frac{\tau z_3}{1-\kappa z_2}\partial_2 F_2 - \frac{\tau z_2}{1-\kappa z_2}\partial_3 F_2 \\ &\quad - \frac{\kappa}{1-\kappa z_2}F_1 - \frac{\tau}{1-\kappa z_2}F_3\Bigg)\mathbf{b} \end{aligned} \tag{2.7}$$

Here we use the notation ∂_1 to denote $\frac{\partial}{\partial z_1}$, etc .

Though $\mathbf{A}$ and $\mathbf{B}$ are defined throughout $\mathbb{R}^3$, in the region Ω_ε we can write $\mathbf{A}$ and $\mathbf{B}$ in the (z_1, z_2, z_3) coordinate system, that is

$$\mathbf{A}(z_1,z_2,z_3) = A_1(z_1,z_2,z_3)\mathbf{t}(z_1) + A_2(z_1,z_2,z_3)\mathbf{n}(z_1) + A_3(z_1,z_2,z_3)\mathbf{b}(z_1),$$
$$\mathbf{B}(z_1,z_2,z_3) = B_1(z_1,z_2,z_3)\mathbf{t}(z_1) + B_2(z_1,z_2,z_3)\mathbf{n}(z_1) + B_3(z_1,z_2,z_3)\mathbf{b}(z_1).$$

From (2.7) and the relation $\nabla \times \mathbf{A} = \mathbf{B}$, we have the following

$$\begin{aligned}
B_1 &= \partial_2 A_3 - \partial_3 A_2 && (2.8)\\
B_2 &= \partial_3 A_1 - \partial_1 A_3 \\
&\quad -\frac{\tau z_3}{1-\kappa z_2}\partial_2 A_3 + \frac{\tau z_2}{1-\kappa z_2}\partial_3 A_3 - \frac{\tau}{1-\kappa z_2} A_2 && (2.9)\\
B_3 &= \frac{1}{1-\kappa z_2}\partial_1 A_2 - \partial_2 A_1 && (2.10)\\
&\quad +\frac{\tau z_3}{1-\kappa z_2}\partial_2 A_2 - \frac{\tau z_2}{1-\kappa z_2}\partial_3 A_2 - \frac{\kappa}{1-\kappa z_2}A_1 - \frac{\tau}{1-\kappa z_2}A_3
\end{aligned}$$

Let us now write the Taylor expansion of $\mathbf{A}$ about the curve $\mathbf{r}$ as

$$\begin{aligned}
&\mathbf{A}(z_1,z_2,z_3) = \\
&\big(A_1(z_1,0,0) + z_2\partial_2 A_1(z_1,0,0) + z_3\partial_3 A_1(z_1,0,0) + R_1\big)\mathbf{t} + \\
&\big(A_2(z_1,0,0) + z_2\partial_2 A_2(z_1,0,0) + z_3\partial_3 A_2(z_1,0,0) + R_2\big)\mathbf{n} + \\
&\big(A_3(z_1,0,0) + z_2\partial_2 A_3(z_1,0,0) + z_3\partial_3 A_3(z_1,0,0) + R_3\big)\mathbf{b}
\end{aligned} \qquad (2.11)$$

where the remainders R_1, R_2, R_3 are of order $\mathcal{O}(z_2^2 + z_3^2)$.

Applying the Taylor expansion (2.11) and (2.5), we write out $E_\varepsilon(\psi)$ in (z_1, z_2, z_3)-coordinates. To compress the notation we will write A_1, A_2, A_3 and $\partial_i A_j$ $(i = 2, 3\,,\ j = 1, 2, 3)$ below to denote $A_1(z_1, 0, 0), A_2(z_1, 0, 0), A_3(z_1, 0, 0)$ and $\partial_i A_j(z_1, 0, 0)$ $(i = 2, 3\,,\ j = 1, 2, 3)$, respectively. We will also use Λ_z to denote the quantity $1-\kappa z_2$.

$$\begin{aligned}
E_\varepsilon(\psi) \;\; = \;\; & \frac{\int_{\Omega_\varepsilon}\Lambda_z\left|\frac{1}{\Lambda_z}\big(\partial_1\psi+\tau z_3\partial_2\psi-\tau z_2\partial_3\psi\big)-\frac{i}{\varepsilon}\big(A_1+z_2\partial_2A_1+z_3\partial_3A_1+R_1\big)\psi\right|^2dz}{\int_{\Omega_\varepsilon}\Lambda_z|\psi|^2dz} \\
& +\frac{\int_{\Omega_\varepsilon}\Lambda_z\left|\partial_2\psi-\frac{i}{\varepsilon}\big(A_2+z_2\partial_2A_2+z_3\partial_3A_2+R_2\big)\psi\right|^2dz}{\int_{\Omega_\varepsilon}\Lambda_z|\psi|^2dz} \\
& +\frac{\int_{\Omega_\varepsilon}\Lambda_z\left|\partial_3\psi-\frac{i}{\varepsilon}\big(A_3+z_2\partial_2A_3+z_3\partial_3A_3+R_3\big)\psi\right|^2dz}{\int_{\Omega_\varepsilon}\Lambda_z|\psi|^2dz}. \qquad (2.12)
\end{aligned}$$

It will be convenient to rephrase the Rayleigh quotient through a further change of variables. First, we choose k_ε be the closest integer to the number $\big(\frac{1}{2\pi}\int_0^L\frac{A_1(t,0,0)}{\varepsilon}dt\big)$. That is,

$$\left|k_\varepsilon-\frac{1}{2\pi}\int_0^L\frac{A_1(t,0,0)}{\varepsilon}dt\right|=\min_{k\in\mathbb{Z}}\left|k-\frac{1}{2\pi}\int_0^L\frac{A_1(t,0,0)}{\varepsilon}dt\right|.$$

Then set

$$\beta_\varepsilon=\left(\frac{1}{L}\int_0^L\frac{A_1(t,0,0)}{\varepsilon}dt\right)-\frac{2\pi}{L}k_\varepsilon \qquad (2.13)$$

For any $\psi\in H^1(\Omega_\varepsilon)$, we set

$$\widetilde{\psi}(z_1,z_2,z_3)=\psi(T(z_1,z_2,z_3))\,e^{-i\phi_\varepsilon} \qquad (2.14)$$

where the phase ϕ_ε is given by

$$\begin{aligned}
\phi_\varepsilon := & \int_0^{z_1}\Big(\frac{A_1(t,0,0)}{\varepsilon}-\beta_\varepsilon\Big)dt \\
& +\frac{1}{\varepsilon}\bigg(z_2A_2(z_1,0,0)+z_3A_3(z_1,0,0)+\frac{z_2^2}{2}\partial_2A_2(z_1,0,0)+\frac{z_3^2}{2}\partial_3A_3(z_1,0,0) \\
& \qquad +\frac{1}{2}z_2z_3\partial_3A_2(z_1,0,0)+\frac{1}{2}z_2z_3\partial_2A_3(z_1,0,0)\bigg)
\end{aligned}$$

Because A_1, A_2, A_3 are periodic functions of z_1, as is ψ, we note from (2.13) that $\widetilde{\psi}$ is also a periodic function of z_1.

Applying (2.14) to (2.12), we obtain an equivalent new functional $\widetilde{E}_\varepsilon : H^1(\Omega_\varepsilon) \to \mathbb{R}$. From here on, we use B_1, B_2, B_3 to denote $B_1(z_1,0,0), B_2(z_1,0,0), B_3(z_1,0,0)$ respectively. Hence we have

$$\begin{aligned}
\widetilde{E}_\varepsilon(\widetilde{\psi}) \;=\; & \frac{\int_{\Omega_\varepsilon} \Lambda_z \Big| \frac{1}{1-\kappa z_2}\big(\partial_1 + \tau z_3 \partial_2 - \tau z_2 \partial_3\big)\widetilde{\psi} - \frac{i}{1-\kappa z_2}\beta_\varepsilon \widetilde{\psi} + \frac{i}{\varepsilon}\big(z_2 B_3 - z_3 B_2 + R\big)\widetilde{\psi}\Big|^2 dz}{\int_{\Omega_\varepsilon} \Lambda_z \big|\widetilde{\psi}\big|^2 dz} \\
& + \frac{\int_{\Omega_\varepsilon} \Lambda_z \Big| \partial_2 \widetilde{\psi} + \frac{i}{\varepsilon}\big(\frac{1}{2} z_3 B_1 - R_2\big)\widetilde{\psi}\Big|^2 dz}{\int_{\Omega_\varepsilon} \Lambda_z \big|\widetilde{\psi}\big|^2 dz} \\
& + \frac{\int_{\Omega_\varepsilon} \Lambda_z \Big| \partial_3 \widetilde{\psi} - \frac{i}{\varepsilon}\big(\frac{1}{2} z_2 B_1 + R_3\big)\widetilde{\psi}\Big|^2 dz}{\int_{\Omega_\varepsilon} \Lambda_z |\widetilde{\psi}|^2 dz}, \qquad (2.15)
\end{aligned}$$

where in the first integral

$$\begin{aligned}
R(z_1,z_2,z_3) \;\equiv\; & \frac{1}{2\Lambda_z}\Big(z_2^2 \partial_1 \partial_2 A_2(z_1,0,0) + z_3^2 \partial_1 \partial_3 A_3(z_1,0,0) \\
& + z_2 z_3 \partial_1 \partial_3 A_2(z_1,0,0) + z_2 z_3 \partial_1 \partial_2 A_3(z_1,0,0)\Big) \\
& - \frac{\tau}{2\Lambda_z}(z_2^2 + z_3^2) B_1(z_1,0,0) \\
& - R_1(z_1,z_2,z_3). \qquad (2.16)
\end{aligned}$$

Note that R is of order $\mathcal{O}(z_2^2 + z_3^2)$.

Finally, it will be convenient to consider the Rayleigh quotient on the fixed domain (i.e. one independent of ε). To this end, set $\Omega = \{(y_1,y_2,y_3) : 0 \le y_1 \le L, 0 \le y_2^2 + y_3^2 < 1\}$ and change the scale in the variables z_2 and z_3 by :

$$y_1 = z_1,\ y_2 = \frac{z_2}{\varepsilon},\ y_3 = \frac{z_3}{\varepsilon}, \qquad (2.17)$$

$$u(y_1,y_2,y_3) \equiv \widetilde{\psi}(z_1,z_2,z_3). \qquad (2.18)$$

This leads to a new equivalent functional $F_\varepsilon : H^1(\Omega) \to \mathbb{R}$:

$$
\begin{aligned}
&F_\varepsilon(u) = \\
&\frac{\int_\Omega \Lambda_y \Big| \frac{1}{\Lambda}\big(\partial_1 + \tau y_3 \partial_2 - \tau y_2 \partial_3\big)u - \frac{i}{\Lambda_y}\beta_\varepsilon u + i\big(y_2 B_3 - y_3 B_2 + \frac{1}{\varepsilon}R(y_1, \varepsilon y_2, \varepsilon y_3)\big)u\Big|^2 dy}{\int_\Omega \Lambda_y \big|u\big|^2 dy} \\
&+\frac{\int_\Omega \Lambda_y \Big| \frac{1}{\varepsilon}\partial_2 u + i\big(\frac{1}{2}y_3 B_1 - \frac{1}{\varepsilon}R_2(y_1, \varepsilon y_2, \varepsilon y_3)\big)u\Big|^2 dy}{\int_\Omega \Lambda_y \big|u\big|^2 dy} \\
&+\frac{\int_\Omega \Lambda_y \Big| \frac{1}{\varepsilon}\partial_3 u - i\big(\frac{1}{2}y_2 B_1 + \frac{1}{\varepsilon}R_3(y_1, \varepsilon y_2, \varepsilon y_3)\big)u\Big|^2 dy}{\int_\Omega \Lambda_y |u|^2 dy}, \qquad (2.19)
\end{aligned}
$$

where $\Lambda_y := 1 - \varepsilon\kappa y_2$.

From here on the notation ∂_1 now denotes $\frac{\partial}{\partial y_1}$, etc. Observe that the remainders $R(y_1, \varepsilon y_2, \varepsilon y_3)$, $R_2(y_1, \varepsilon y_2, \varepsilon y_3)$, and $R_3(y_1, \varepsilon y_2, \varepsilon y_3)$ are of order $\mathcal{O}(\varepsilon^2)$ for any point $(y_1, y_2, y_3) \in \Omega$.

3 Preliminary Estimates

We first write down the Euler-Lagrange equation for F_ε. Let λ_ε be the minimum of F_ε and denote a corresponding minimizer by u^ε. An eigenvalue problem is derived in the following form:

$$
\begin{aligned}
&-\Big(\partial_1 + \tau y_3 \partial_2 - \tau y_2 \partial_3 - i\beta_\varepsilon + i(1 - \varepsilon\kappa y_2)\big(y_2 B_3 - y_3 B_2 + \frac{1}{\varepsilon}R\big)\Big)^2 u^\varepsilon \\
&-\Big(\frac{1 - \varepsilon\kappa y_2}{\varepsilon}\,\partial_2 + i(1 - \varepsilon\kappa y_2)\big(\frac{1}{2}y_3 B_1 - \frac{1}{\varepsilon}R_2\big)\Big)^2 u^\varepsilon \\
&-\Big(\frac{1 - \varepsilon\kappa y_2}{\varepsilon}\,\partial_3 - i(1 - \varepsilon\kappa y_2)\big(\frac{1}{2}y_2 B_1 + \frac{1}{\varepsilon}R_3\big)\Big)^2 u^\varepsilon \\
&= \lambda_\varepsilon (1 - \varepsilon\kappa y_2)^2 u^\varepsilon \qquad \text{in } \Omega \qquad (3.1)
\end{aligned}
$$

The (natural) boundary conditions are

$$
\begin{aligned}
&u(0, y_2, y_3) = u(2\pi, y_2, y_3) \quad \text{on } D, \\
&\partial_1 u(0, y_2, y_3) = \partial_1 u(2\pi, y_2, y_3) \quad \text{on } D, \\
&(y_2\partial_2 + y_3\partial_3)u(y_1, y_2, y_3) = -(y_2 R_2 - y_3 R_3)(y_1, \varepsilon y_2, \varepsilon y_3) \quad \text{on } [0, 2\pi] \times \partial D, \\
&\qquad (3.2)
\end{aligned}
$$

where D denotes the unit disk $\{y_2^2 + y_3^2 \leq 1\}$, R is defined through (2.16) and R_2 and R_3 are defined through (2.11).

Now we derive some crucial estimates concerning a first eigenfunction satisfying (3.1)-(3.2).

Proposition 3.1. *Let u^ε be a minimizer of F_ε. For ε small enough, we have constants C_0, C_1, C_2, C_3 independent of ε such that*

$$\begin{aligned} F_\varepsilon(u^\varepsilon) &\leq C_0, &(3.3)\\ \|\partial_1 u^\varepsilon\|_{L^2(\Omega)} &\leq C_1 \|u^\varepsilon\|_{L^2(\Omega)}, &(3.4)\\ \|\partial_2 u^\varepsilon\|_{L^2(\Omega)} &\leq C_2 \varepsilon \|u^\varepsilon\|_{L^2(\Omega)}, &(3.5)\\ \|\partial_3 u^\varepsilon\|_{L^2(\Omega)} &\leq C_3 \varepsilon \|u^\varepsilon\|_{L^2(\Omega)}. &(3.6)\end{aligned}$$

Proof. For ε is small enough, the bound (3.3) follows immediately by comparing the energy of the minimizer to that of a constant function:

$$\begin{aligned} & F_\varepsilon(u^\varepsilon) \leq F_\varepsilon(1) = \\ & \frac{\int_\Omega (1-\varepsilon\kappa y_2)\Big| - \frac{\beta_\varepsilon}{1-\varepsilon\kappa y_2} + y_3 B_2(y_1,0,0) - y_2 B_3(y_1,0,0) + \frac{1}{\varepsilon} R(y_1, \varepsilon y_2, \varepsilon y_3)\Big|^2 dy}{\int_\Omega (1-\varepsilon\kappa y_2) dy} \\ + \ & \frac{\int_\Omega (1-\varepsilon\kappa y_2)\Big|\frac{1}{2} y_3 B_1(y_1,0,0) - \frac{1}{\varepsilon} R_2(y_1,\varepsilon y_2,\varepsilon y_3)\Big|^2 dy}{\int_\Omega (1-\varepsilon\kappa y_2) dy} \\ + \ & \frac{\Big| - \frac{1}{2} y_2 B_1(y_1,0,0) - \frac{1}{\varepsilon} R_3(y_1,\varepsilon y_2,\varepsilon y_3)\Big|^2\Big) dy}{\int_\Omega (1-\varepsilon\kappa y_2) dy} \\ \leq \ & C_0 \end{aligned} \tag{3.7}$$

since $\beta_\varepsilon, B_1, B_2, B_3$ are bounded.

We next establish the bound (3.5) by the triangle inequality:

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\frac{\|\partial_2 u^\varepsilon\|^2}{\|u^\varepsilon\|^2} \\
\leq\ & C\Bigg(\frac{\int_\Omega (1-\varepsilon\kappa y_2)\Big|\frac{1}{\varepsilon}\partial_2 u^\varepsilon + i\big(\frac{1}{2}y_3 B_1(y_1,0,0) - \frac{1}{\varepsilon}R_2(y_1,\varepsilon y_2,\varepsilon y_3)u^\varepsilon\big)\Big|^2 dy}{\int_\Omega |u^\varepsilon|^2(1-\varepsilon\kappa y_2)dy} \\
& + \frac{\int_\Omega \Big|\big(\frac{1}{2}y_3 B_1(y_1,0,0) - \frac{1}{\varepsilon}R_2(y_1,\varepsilon y_2,\varepsilon y_3)\big)u^\varepsilon\Big|^2 dy)}{\int_\Omega |u^\varepsilon|^2(1-\varepsilon\kappa y_2)dy}\Bigg) \\
\leq\ & C\Bigg(F_\varepsilon(u^\varepsilon) + \frac{\int_\Omega \Big|\big(\frac{1}{2}y_3 B_1(y_1,0,0) - \frac{1}{\varepsilon}R_2(y_1,\varepsilon y_2,\varepsilon y_3)\big)u^\varepsilon\Big|^2 dy}{\int_\Omega |u^\varepsilon|^2(1-\varepsilon\kappa y_2)dy}\Bigg) \\
\leq\ & C
\end{aligned} \tag{3.8}$$

Thus we have $\|\partial_2 u^\varepsilon\|_{L^2(\Omega)} \leq C_2\varepsilon\|u^\varepsilon\|_{L^2(\Omega)}$. Similarly, we will have $\|\partial_3 u^\varepsilon\|_{L^2(\Omega)} \leq C_3\varepsilon\|u^\varepsilon\|_{L^2(\Omega)}$. Combining these inequalities and the triangle inequality, we can return to the definition of $F_\varepsilon(u^\varepsilon)$ in (2.19) obtain $\|\partial_1 u^\varepsilon\|_{L^2(\Omega)} \leq C_1\|u^\varepsilon\|_{L^2(\Omega)}$. □

We now introduce the integral average $\widetilde{u} : [0, 2\pi] \rightarrow R$ of u in order to compare a function of three variables to a function of one variable. It is defined by

$$\widetilde{u}(y_1) := \frac{1}{\pi}\int_{\{y_2^2+y_3^2<1\}} u(y_1,y_2,y_3)dy_2dy_3.$$

We then establish the following simple fact.

Proposition 3.2. *Let u^ε be a minimizer of F_ε and set $v^\varepsilon = u^\varepsilon - \widetilde{u}^\varepsilon$. Then there exists a constant C independent of ε such that for ε small enough*

$$\|v^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon\|u^\varepsilon\|_{L^2(\Omega)} \tag{3.9}$$

Proof. According to the Poincaré inequality and Proposition 1, we have

$$\begin{aligned}
&\|v^\varepsilon\|_{L^2(\Omega)} = \Big(\int_0^{2\pi}\int_D |u^\varepsilon - \widetilde{u}^\varepsilon|^2 dy_2dy_3dy_1\Big)^{1/2} \\
&\leq C\Big(\int_0^{2\pi}\int_D \big(|\partial_2 u^\varepsilon|^2 + |\partial_3 u^\varepsilon|^2\big)dy_2dy_3dy_1\Big)^{1/2} \\
&\leq C\varepsilon\|u^\varepsilon\|_{L^2(\Omega)}.
\end{aligned} \tag{3.10}$$

□

Now we are in a position to improve some of our earlier estimates.

Proposition 3.3. *Let u^ε be a minimizer of F_ε and as before set $v^\varepsilon = u^\varepsilon - \widetilde{u}^\varepsilon$. Then there exist constants C_1, C_2 independent of ε such that for ε small enough*

$$\|\partial_1 v^\varepsilon\|_{L^2(\Omega)} \le \varepsilon^{1/2}(C_1\|u^\varepsilon\|_{L^2(\Omega)} + C_2) \tag{3.11}$$

$$\|\partial_2 v^\varepsilon\|_{L^2(\Omega)} = \|\partial_2 u^\varepsilon\|_{L^2(\Omega)} \le \varepsilon^{3/2}(C_1\|u\|_{L^2(\Omega)} + C_2) \tag{3.12}$$

$$\|\partial_3 v^\varepsilon\|_{L^2(\Omega)} = \|\partial_3 u^\varepsilon\|_{L^2(\Omega)} \le \varepsilon^{3/2}(C_1\|u\|_{L^2(\Omega)} + C_2) \tag{3.13}$$

Note that (3.12) and (3.13) represent an improvement over (3.5) and (3.6).

Proof. First, we rewrite the Euler-Lagrange equation (3.1) in the form

$$\begin{aligned} &-(\partial_1 + \tau y_3 \partial_2 - \tau y_2 \partial_3)^2 u^\varepsilon - \frac{(1-\varepsilon\kappa y_2)^2}{\varepsilon^2}\partial_2^2 u^\varepsilon - \frac{(1-\varepsilon\kappa y_2)^2}{\varepsilon^2}\partial_3^2 u^\varepsilon \\ &= f_0^\varepsilon u^\varepsilon + f_1^\varepsilon \partial_1 u^\varepsilon + \frac{f_2^\varepsilon}{\varepsilon}\partial_2 u^\varepsilon + \frac{f_3^\varepsilon}{\varepsilon}\partial_3 u^\varepsilon \qquad \text{in} \quad \Omega, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} f_0^\varepsilon(y_1,y_2,y_3) \;\equiv\;& \lambda_\varepsilon(1-\varepsilon\kappa y_2)^2 \\ &+i(\partial_1 + \tau y_3\partial_2 - \tau y_2\partial_3)\Big(-\beta_\varepsilon + \big(1-\varepsilon\kappa y_2\big)\big(y_2B_3 - y_3B_2 + \frac{R}{\varepsilon}\big)\Big) \\ &-\Big(-\beta_\varepsilon + \big(1-\varepsilon\kappa y_2\big)\big(y_2B_3 - y_3B_2 + \frac{R}{\varepsilon}\big)\Big)^2 \\ &+\frac{1-\varepsilon\kappa y_2}{\varepsilon}\, i\partial_2\Big((1-\varepsilon\kappa y_2)\big(\frac{y_3B_1}{2} - \frac{R_2}{\varepsilon}\big)\Big) \\ &-(1-\varepsilon\kappa y_2)^2\big(\frac{y_3B_1}{2} - \frac{R_2}{\varepsilon}\big)^2 \\ &-\frac{1-\varepsilon\kappa y_2}{\varepsilon}\, i\partial_3\Big((1-\varepsilon\kappa y_2)\big(\frac{y_2B_1}{2} + \frac{R_3}{\varepsilon}\big)\Big) \\ &-(1-\varepsilon\kappa y_2)^2\big(\frac{y_2B_1}{2} + \frac{R_3}{\varepsilon}\big)^2, \end{aligned}$$

$$\begin{aligned} f_1^\varepsilon(y_1,y_2,y_3) &\equiv 2i\Big(-\beta_\varepsilon+\big(1-\varepsilon\kappa y_2\big)\big(y_2B_3-y_3B_2+\frac{R}{\varepsilon}\big)\Big) \qquad (3.15)\\ f_2^\varepsilon(y_1,y_2,y_3) &\equiv 2i(1-\varepsilon\kappa y_2)^2\big(\frac{y_3B_1}{2}-\frac{R_2}{\varepsilon}\big)\\ &\quad +2i\varepsilon\tau y_3\Big(-\beta_\varepsilon+\big(1-\varepsilon\kappa y_2\big)\big(y_2B_3-y_3B_2+\frac{R}{\varepsilon}\big)\Big) \qquad (3.16)\\ f_3^\varepsilon(y_1,y_2,y_3) &\equiv 2i(1-\varepsilon\kappa y_2)^2\big(\frac{y_2B_1}{2}+\frac{R_3}{\varepsilon}\big)\\ &\quad -2i\varepsilon\tau y_2\Big(-\beta_\varepsilon+\big(1-\varepsilon\kappa y_2\big)\big(y_2B_3-y_3B_2+\frac{R}{\varepsilon}\big)\Big) \qquad (3.17) \end{aligned}$$

Notice $f_0^\varepsilon, f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon$ are uniformly bounded in Ω.

After integrating the equation (3.14) over the unit disk D, dividing by π and subtracting it from the equation (3.14) itself, we get

$$\begin{aligned} &-(\partial_1+\tau y_3\partial_2-\tau y_2\partial_3)^2u^\varepsilon+\frac{1}{\pi}\int_D(\partial_1+\tau y_3\partial_2-\tau y_2\partial_3)^2u^\varepsilon dy_2dy_3\\ &-\frac{(1-\varepsilon\kappa y_2)^2}{\varepsilon^2}(\partial_2^2u^\varepsilon+\partial_3^2u^\varepsilon)+\frac{1}{\varepsilon^2\pi}\int_D(1-\varepsilon\kappa y_2)^2(\partial_2^2u^\varepsilon+\partial_3^2u^\varepsilon)dy_2dy_3\\ &=\Big(f_0^\varepsilon u^\varepsilon-\frac{1}{\pi}\int_D f_0^\varepsilon u^\varepsilon dy_2dy_3\Big)+\Big(f_1^\varepsilon\partial_1u^\varepsilon-\frac{1}{\pi}\int_D f_1^\varepsilon\partial_1u^\varepsilon dy_2dy_3\Big)\\ &+\frac{1}{\varepsilon}\Big(f_2^\varepsilon\partial_2u^\varepsilon-\frac{1}{\pi}\int_D f_2^\varepsilon\partial_2u^\varepsilon dy_2dy_3\Big)+\frac{1}{\varepsilon}\Big(f_3^\varepsilon\partial_3u^\varepsilon-\frac{1}{\pi}\int_D f_3^\varepsilon\partial_3u^\varepsilon dy_2dy_3\Big). \end{aligned} \qquad (3.18)$$

If we multiply the equation (3.18) by $\overline{v}^\varepsilon$ (the complex conjugate of v^ε) and integrate it over Ω, we obtain

$$\begin{aligned} &-\int_\Omega\Big((\partial_1+\tau y_3\partial_2-\tau y_2\partial_3)^2u^\varepsilon\Big)\overline{v}^\varepsilon dy\\ &+\int_\Omega\Big(\frac{1}{\pi}\int_D(\partial_1+\tau y_3\partial_2-\tau y_2\partial_3)^2u^\varepsilon dy_2dy_3\Big)\overline{v}^\varepsilon dy\\ &-\int_\Omega\Big(\frac{(1-\varepsilon\kappa y_2)^2}{\varepsilon^2}(\partial_2^2u^\varepsilon+\partial_3^2u^\varepsilon)\Big)\overline{v}^\varepsilon dy \end{aligned}$$

$$
\begin{aligned}
= \quad & -\int_{\Omega}\Big(\frac{1}{\varepsilon^2\pi}\int_D (1-\varepsilon\kappa y_2)^2(\partial_2^2 u^{\varepsilon}+\partial_3^2 u^{\varepsilon})dy_2dy_3\Big)\overline{v}^{\varepsilon}dy \\
& +\int_{\Omega}\Big(f_0^{\varepsilon}u^{\varepsilon}-\frac{1}{\pi}\int_D f_0^{\varepsilon}u^{\varepsilon}dy_2dy_3\Big)\overline{v}^{\varepsilon}dy \\
& +\int_{\Omega}\Big(f_1^{\varepsilon}\partial_1 u^{\varepsilon}-\frac{1}{\pi}\int_D f_1^{\varepsilon}\partial_1 u^{\varepsilon}dy_2dy_3\Big)\overline{v}^{\varepsilon}dy \\
& +\int_{\Omega}\frac{1}{\varepsilon}\Big(f_2^{\varepsilon}\partial_2 u^{\varepsilon}-\frac{1}{\pi}\int_D f_2^{\varepsilon}\partial_2 u^{\varepsilon}dy_2dy_3\Big)\overline{v}^{\varepsilon}dy \\
& +\int_{\Omega}\frac{1}{\varepsilon}\Big(f_3^{\varepsilon}\partial_3 u^{\varepsilon}-\frac{1}{\pi}\int_D f_1^{\varepsilon}\partial_3 u^{\varepsilon}dy_2dy_3\Big)\overline{v}^{\varepsilon}dy\,. \qquad (3.19)
\end{aligned}
$$

Let us denote the last equation as

$$
I+II+III=IV+V+VI+VII+VIII\,.
$$

We first estimate the term V, VI, VII and VIII. Since $f_0^{\varepsilon}, f_1^{\varepsilon}, f_2^{\varepsilon}, f_3^{\varepsilon}$ are uniformly bounded on Ω, we find that

$$
\begin{aligned}
& \int_{\Omega}\Big|f_0^{\varepsilon}u^{\varepsilon}-\frac{1}{\pi}\int_D f_0^{\varepsilon}u^{\varepsilon}dy_2dy_3\Big|^2 dy \\
& \le \big(\max_{\Omega}|f_0^{\varepsilon}|^2\big)\int_{\Omega}\Big(|u^{\varepsilon}|+\frac{1}{\pi}\int_D|u^{\varepsilon}|dy_2dy_3\Big)^2 dy \\
& \le 2\big(\max_{\Omega}|f_0^{\varepsilon}|^2\big)\int_{\Omega}\Big(|u^{\varepsilon}|^2+\frac{1}{\pi}\int_D|u^{\varepsilon}|^2dy_2dy_3\Big)dy \\
& = 4\big(\max_{\Omega}|f_0^{\varepsilon}|^2\big)\int_{\Omega}|u^{\varepsilon}|^2dy \qquad (3.20)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& |V| = \Big|\int_{\Omega}\Big(f_0^{\varepsilon}u^{\varepsilon}-\frac{1}{\pi}\int_D f_0^{\varepsilon}u^{\varepsilon}dy_2dy_3\Big)\overline{v}^{\varepsilon}dy\Big| \\
& \le \Big\|f_0^{\varepsilon}u^{\varepsilon}-\frac{1}{\pi}\int_D f_0^{\varepsilon}u^{\varepsilon}dy_2dy_3\Big\|_{L^2(\Omega)}\|v^{\varepsilon}\|_{L^2(\Omega)} \\
& \le 2\big(\max_{\Omega}|f_0^{\varepsilon}|\big)\,\|u^{\varepsilon}\|_{L^2(\Omega)}\|v^{\varepsilon}\|_{L^2(\Omega)} \qquad (3.21)
\end{aligned}
$$

Through a similar computation, we also obtain

$$|VI| \leq 2\big(\max_{\Omega}|f_1^\varepsilon|\big)\ \|\partial_1 u^\varepsilon\|_{L^2(\Omega)}\|v^\varepsilon\|_{L^2(\Omega)}\,, \tag{3.22}$$

$$|VII| \leq \frac{2}{\varepsilon}\big(\max_{\Omega}|f_2^\varepsilon|\big)\ \|\partial_2 u^\varepsilon\|_{L^2(\Omega)}\|v^\varepsilon\|_{L^2(\Omega)}\,, \tag{3.23}$$

and

$$|VIII| \leq \frac{2}{\varepsilon}\big(\max_{\Omega}|f_3^\varepsilon|\big)\ \|\partial_3 u^\varepsilon\|_{L^2(\Omega)}\|v^\varepsilon\|_{L^2(\Omega)} \tag{3.24}$$

Turning to the integrals I, II, III and IV, the Divergence Theorem implies that

$$\begin{aligned}&\int_D (\partial_1 + \tau y_3\partial_2 - \tau y_2\partial_3)^2 u^\varepsilon dy_2 dy_3 = \\ &\int_D \partial_1^2 u^\varepsilon + 2(\tau y_3\partial_2 - \tau y_2\partial_3)\partial_1 u^\varepsilon + (\partial_1\tau)(y_3\partial_2 - y_2\partial_3)u^\varepsilon + (\tau y_3\partial_2 - \tau y_2\partial_3)^2 u^\varepsilon dy_2 dy_3 = \\ &\int_D \partial_1^2 u^\varepsilon dy_2 dy_3 = \partial_1^2 \int_D u^\varepsilon dy_2 dy_3 = \partial_1 \widetilde{u}^\varepsilon.\end{aligned} \tag{3.25}$$

Thus,

$$\begin{aligned}&(\partial_1 + \tau y_3\partial_2 - \tau y_2\partial_3)^2 u^\varepsilon - \frac{1}{\pi}\int_D (\partial_1 + \tau y_3\partial_2 - \tau y_2\partial_3)^2 u^\varepsilon dy_2 dy_3 \\ &= (\partial_1 + \tau y_3\partial_2 - \tau y_2\partial_3)^2 v^\varepsilon.\end{aligned} \tag{3.26}$$

Hence, integrating by parts both with respect to y_1 and with respect to y_2 and y_3, and using the periodicity in y_1 (cf. (3.2)), we find

$$\begin{aligned}&I + II \\ &= -\int_\Omega \big((\partial_1 + \tau y_3\partial_2 - \tau y_2\partial_3)^2 v^\varepsilon\big)\overline{v}^\varepsilon dy \\ &= \int_\Omega \big|(\partial_1 + \tau y_3\partial_2 - \tau y_2\partial_3)v^\varepsilon\big|^2 dy\,.\end{aligned} \tag{3.27}$$

Next, we integrate by parts over the cross section D, and again use the boundary condition (3.2) to obtain

$$\begin{aligned}&IV = \\ &-\frac{1}{\varepsilon^2\pi}\int_\Omega \left(\int_{\partial D}(1-\varepsilon\kappa y_2)^2(y_2\partial_2 u^\varepsilon + y_3\partial_3 u^\varepsilon)ds + \int_D \big(\partial_2(1-\varepsilon\kappa y_2)^2\big)\partial_2 u^\varepsilon dy_2 dy_3\right)\overline{v}^\varepsilon dy \\ &= \frac{1}{\pi}\int_\Omega \left(\int_{\partial D}\frac{(1-\varepsilon\kappa y_2)^2}{\varepsilon^2}(y_2 R_2^\varepsilon - y_3 R_3^\varepsilon)ds + \int_D \frac{2(1-\varepsilon\kappa y_2)\kappa}{\varepsilon}\partial_2 u^\varepsilon dy_2 dy_3\right)\overline{v}^\varepsilon dy.\end{aligned}$$

Since $(1-\varepsilon\kappa y_2)^2(y_2 R_2^\varepsilon - y_3 R_3^\varepsilon)$ is of order $\mathcal{O}(\varepsilon^2)$ on ∂D, we can use Proposition 3.1 to obtain

$$\begin{aligned} |IV| &\le C_1\|v^\varepsilon\|_{L^2(\Omega)} + \frac{C_2}{\varepsilon}\|\partial_2 u^\varepsilon\|_{L^2(\Omega)}\|v^\varepsilon\|_{L^2(\Omega)} \\ &\le C_1\|v^\varepsilon\|_{L^2(\Omega)} + C_2\|u^\varepsilon\|_{L^2(\Omega)}\|v^\varepsilon\|_{L^2(\Omega)} \end{aligned} \tag{3.28}$$

Similarly, we use integration by parts over the whole cylinder Ω and apply the boundary conditions to find that

$$\begin{aligned} III &= -\frac{1}{\varepsilon^2}\int_0^L\int_{\partial D}(1-\varepsilon\kappa y_2)^2(y_2\partial_2 u^\varepsilon + y_3\partial_3 u^\varepsilon)\overline{v}^\varepsilon\, ds\, dy_1 \\ &\quad + \frac{1}{\varepsilon^2}\int_\Omega \Big(\partial_2\big((1-\varepsilon\kappa y_2)^2\overline{v}^\varepsilon\big)\partial_2 u^\varepsilon + \partial_3\big((1-\varepsilon\kappa y_2)^2\overline{v}^\varepsilon\big)\partial_3 u^\varepsilon\Big)dy \\ &= -\int_0^L\int_{\partial D}\frac{(1-\varepsilon\kappa y_2)^2}{\varepsilon^2}(y_2 R_2^\varepsilon - y_3 R_3^\varepsilon)\overline{v}^\varepsilon\, ds\, dy_1 - \frac{2}{\varepsilon}\int_\Omega (1-\varepsilon\kappa y_2)\kappa\,\partial_2 u^\varepsilon\,\overline{v}^\varepsilon dy \\ &\quad + \frac{1}{\varepsilon^2}\int_\Omega (1-\varepsilon\kappa y_2)^2\big(\partial_2 u^\varepsilon\partial_2\overline{v}^\varepsilon + \partial_3 u^\varepsilon\partial_3\overline{v}^\varepsilon\big)dy. \end{aligned} \tag{3.29}$$

Applying the Trace Theorem (cf. [7]) on the cross-section D, we can estimate the first term of (3.29) as follows on the right-hand side of

$$\begin{aligned} &\Big|\int_0^L\int_{\partial D}\frac{(1-\varepsilon\kappa y_2)^2}{\varepsilon^2}(y_2 R_2^\varepsilon - y_3 R_3^\varepsilon)\overline{v}^\varepsilon ds\, dy_1\Big| \\ &\le C\Big(\int_0^L\int_{\partial D}|v^\varepsilon|^2 ds\, dy_1\Big)^{1/2} \\ &= C\Big(\int_0^L\int_{\partial D}|v^\varepsilon|^2 ds\, dy_1\Big)^{1/2} \\ &\le C\Big(\int_0^L\int_D\big(|v^\varepsilon|^2 + |\partial_2 v^\varepsilon|^2 + |\partial_3 v^\varepsilon|^2\big)\, dy_2\, dy_3\, dy_1\Big)^{1/2}. \end{aligned} \tag{3.30}$$

Regarding the second to last term in (3.29) the uniform boundedness of $(1-\varepsilon\kappa y_2)\kappa$ and the result of Proposition 3.1 yield

$$\begin{aligned} &\Big|\frac{2}{\varepsilon}\int_\Omega (1-\varepsilon\kappa y_2)\kappa\partial_2 u^\varepsilon\overline{v}^\varepsilon dy\Big| \\ &\le \frac{C}{\varepsilon}\|\partial_2 u^\varepsilon\|_{L^2(\Omega)}\|v^\varepsilon\|_{L^2(\Omega)} \\ &\le C\|u^\varepsilon\|_{L^2(\Omega)}\|v^\varepsilon\|_{L^2(\Omega)}. \end{aligned} \tag{3.31}$$

Turning to the last term in (3.29), and noting that $\partial_2 v^\varepsilon = \partial_2 u^\varepsilon$ and $\partial_3 v^\varepsilon = \partial_3 u^\varepsilon$, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_\Omega (1-\varepsilon\kappa y_2)^2\big(\partial_2 u^\varepsilon \partial_2 \overline{v}^\varepsilon + \partial_3 u^\varepsilon \partial_3 \overline{v}^\varepsilon\big)dy \\
&= \frac{1}{\varepsilon^2}\int_\Omega (1-\varepsilon\kappa y_2)^2\big(|\partial_2 v^\varepsilon|^2 + |\partial_3 v^\varepsilon|^2\big)dy \qquad (3.32)
\end{aligned}$$

Combining Proposition 3.1 and (3.21)-(3.24),(3.27),(3.28)-(3.32), we then conclude that

$$\begin{aligned}
&\int_\Omega \Big(\big|(\partial_1 + \tau y_3 \partial_2 - \tau y_2 \partial_3) v^\varepsilon\big|^2 + \frac{1}{\varepsilon^2}|\partial_2 v^\varepsilon|^2 + \frac{1}{\varepsilon^2}|\partial_3 v^\varepsilon|^2\Big)dy \leq \\
&C\Big(\|u^\varepsilon\|_{L^2(\Omega)}\|v^\varepsilon\|_{L^2(\Omega)} + \|v^\varepsilon\|_{L^2(\Omega)} + \|\partial_2 v^\varepsilon\|_{L^2(\Omega)} + \|\partial_3 v^\varepsilon\|_{L^2(\Omega)}\Big).
\end{aligned} \qquad (3.33)$$

As a consequence of (3.10) and (3.33), we have

$$\Big(\int_\Omega \big(|\partial_2 v^\varepsilon|^2 + |\partial_3 v^\varepsilon|^2\big)dy\Big)^{1/2} \leq \varepsilon^{3/2}\big(C_1\|u^\varepsilon\|_{L^2(\Omega)} + C_2\big). \qquad (3.34)$$

From (3.33), (3.34) and Proposition 3.2, we also have

$$\begin{aligned}
&\int_\Omega |\,\partial_1 v^\varepsilon|^2\, dy \\
\leq\ & 2\Big(\int_\Omega \big|(\partial_1 + \tau y_3 \partial_2 + \tau y_2 \partial_3) v^\varepsilon\big|^2 dy + \int_\Omega \big|(\tau y_3 \partial_2 - \tau y_2 \partial_3) v^\varepsilon\big|^2 dy\Big) \\
\leq\ & C\Big(\|u^\varepsilon\|_{L^2(\Omega)}\|v^\varepsilon\|_{L^2(\Omega)} + \|v^\varepsilon\|_{L^2(\Omega)} + \|\partial_2 v^\varepsilon\|_{L^2(\Omega)} + \|\partial_3 v^\varepsilon\|_{L^2(\Omega)}\Big) \\
\leq\ & \varepsilon\big(C_1\|u^\varepsilon\|_{L^2(\Omega)} + C_2\big). \qquad (3.35)
\end{aligned}$$

This completes the proof of Proposition 3.3. $\square$

4 Main Results

We begin with a compactness result for minimizers of the functionals F_ε. Note that throughout this section, we will identify functions u in $H^1((0,L))$ with elements of $H^1(\Omega)$ by setting $u(y_1, y_2, y_3) = u(y_1)$. Here $H^1_{\text{per}}((0,L))$ refers to periodic functions in $H^1((0,L))$.

Proposition 4.1. *Let* $\{\varepsilon_j\}$ *be any sequence such that* $\{\beta_{\varepsilon_j}\}$ *given by (2.13) converges to a limit* $\beta_0 \in [-\frac{\pi}{L}, \frac{\pi}{L}]$ *and let* u^{ε_j} *denote a minimizer of* F_{ε_j} *normalized so that* $\|u^{\varepsilon_j}\|_{L^2(\Omega)} = 1$. *Then passing to a subsequence, still denoted by* $\{\varepsilon_j\}$, *one has*

$$u^{\varepsilon_j} \rightharpoonup u_0 \quad \text{weakly in } H^1(\Omega), \tag{4.1}$$

$$u^{\varepsilon_j} \to u_0 \quad \text{strongly in } L^q(\Omega), 1 \leq q < 6 \tag{4.2}$$

for some function $u_0 = u_0(y_1)$ *in* $H^1_{\text{per}}((0, L))$.

Proof. Proposition 3.1 implies that the sequence $\{u^\varepsilon\}$ is uniformly bounded in $H^1(\Omega)$. By weak compactness and the Rellich-Kondrachov compactness theorem, there exists $u_0 \in H^1(\Omega)$ such that a subsequence $\{u^{\varepsilon_j}\}$ converges weakly to u_0 in $H^1(\Omega)$, and strongly to u_0 in $L^q(\Omega)$ for $1 \leq q < 6$.

By Proposition 3.1 and weak lower-semicontinuity, we know $\|\partial_2 u_0\|_{L^2(\Omega)} = \|\partial_3 u_0\|_{L^2(\Omega)} = 0$. This implies u_0 is a function of y_1 only. □

Next we identify a limiting energy minimized by u_0. To this end, for each $\beta \in \mathbb{R}$, we define a new functional $G_\beta : H^1_{\text{per}}((0, L)) \to \mathbb{R}$ by

$$G_\beta(u) := \frac{\int_0^L |\,\partial_1 u - i\beta u|^2 + g(y_1)|u|^2 dy_1}{\int_0^L |u|^2 dy_1} \tag{4.3}$$

where

$$g(y_1) := \frac{1}{8}(B_1(y_1, 0, 0))^2 + \frac{1}{4}(B_2(y_1, 0, 0))^2 + \frac{1}{4}(B_3(y_1, 0, 0))^2. \tag{4.4}$$

Theorem 4.2. *The function* u_0 *provided by Proposition 4 minimizes* G_{β_0}. *Moreover,*

$$\liminf_{\varepsilon_j \to 0} F_{\varepsilon_j}(u^{\varepsilon_j}) = G_{\beta_0}(u_0). \tag{4.5}$$

Proof. The proof hinges on showing that

$$\liminf_{\varepsilon_j \to 0} F_{\varepsilon_j}(u^{\varepsilon_j}) \geq G_{\beta_0}(u_0) \tag{4.6}$$

when $\beta_{\varepsilon_j} \to \beta_0$. Then it will follow from the minimizing property of $\{u^\varepsilon\}$ that for any $v \in H^1((0, L))$ we have

$$G_{\beta_0}(u_0) \leq \liminf_{\varepsilon_j \to 0} F_{\varepsilon_j}(u^{\varepsilon_j}) \leq \liminf_{\varepsilon_j \to 0} F_{\varepsilon_j}(v) = G_{\beta_0}(v) \tag{4.7}$$

so that the function u_0 is indeed a minimizer of G_{β_0}. Of course, from this we will also conclude that $\liminf_{\varepsilon_j \to 0} F_{\varepsilon_j}(u^{\varepsilon_j}) = G_{\beta_0}(u_0)$.

Turning to the proof of claim (4.6), we decompose F_{ε_j} as follows :

$$
\begin{aligned}
&F_{\varepsilon_j}(u^{\varepsilon_j}) = \\
&\frac{\int_\Omega (1-\varepsilon_j\kappa y_2)\Big|\frac{1}{1-\varepsilon_j\kappa y_2}\big(\partial_1 + \tau y_3\partial_2 - \tau y_2\partial_3\big)u^{\varepsilon_j} + iQ_1^{\varepsilon_j}u^{\varepsilon_j}\Big|^2 dy}{\int_\Omega (1-\varepsilon_j\kappa y_2)\big|u^{\varepsilon_j}\big|^2 dy} + \\
&\frac{\int_\Omega (1-\varepsilon_j\kappa y_2)\Big|\frac{1}{\varepsilon_j}\partial_2 u^{\varepsilon_j} + iQ_2^{\varepsilon_j}u^{\varepsilon_j}\Big|^2 dy}{\int_\Omega (1-\varepsilon_j\kappa y_2)\big|u^{\varepsilon_j}\big|^2 dy} + \\
&\frac{\int_\Omega (1-\varepsilon_j\kappa y_2)\Big|\frac{1}{\varepsilon_j}\partial_3 u^{\varepsilon_j} - iQ_3^{\varepsilon_j}u^{\varepsilon_j}\Big|^2 dy}{\int_\Omega (1-\varepsilon_j\kappa y_2)|u^{\varepsilon_j}|^2 dy} = \\
&\frac{\int_\Omega \frac{1}{1-\varepsilon_j\kappa y_2}\big|\partial_1 u^{\varepsilon_j}\big|^2 + (1-\varepsilon_j\kappa y_2)\big|Q_1^{\varepsilon_j}\big|^2\big|u^{\varepsilon_j}\big|^2 + iQ_1^{\varepsilon_j}\big(\partial_1\overline{u}^{\varepsilon_j}u^{\varepsilon_j} - \partial_1 u^{\varepsilon_j}\overline{u}\big)dy}{\int_\Omega (1-\varepsilon_j\kappa y_2)\big|u^{\varepsilon_j}\big|^2 dy} + \\
&\frac{\int_\Omega \frac{1}{1-\varepsilon_j\kappa y_2}\Big|\big(\tau y_3\partial_2 - \tau y_2\partial_3\big)u^{\varepsilon_j}\Big|^2 dy}{\int_\Omega (1-\varepsilon_j\kappa y_2)\big|u^{\varepsilon_j}\big|^2 dy} + \\
&\frac{\int_\Omega \frac{i}{1-\varepsilon_j\kappa y_2}\Big(\partial_1 u^{\varepsilon_j}\big(\tau y_3\partial_2 - \tau y_2\partial_3\big)\overline{u}^{\varepsilon_j} + \partial_1\overline{u}^{\varepsilon_j}\big(\tau y_3\partial_2 - \tau y_2\partial_3\big)u^{\varepsilon_j}\Big)dy}{\int_\Omega (1-\varepsilon_j\kappa y_2)\big|u^{\varepsilon_j}\big|^2 dy} + \\
&\frac{\int_\Omega iQ_1^{\varepsilon_j}\Big(u^{\varepsilon_j}\big(\tau y_3\partial_2 - \tau y_2\partial_3\big)\overline{u}^{\varepsilon_j} - \overline{u}^{\varepsilon_j}\big(\tau y_3\partial_2 - \tau y_2\partial_3\big)u^{\varepsilon_j}\Big)dy}{\int_\Omega (1-\varepsilon_j\kappa y_2)\big|u^{\varepsilon_j}\big|^2 dy} + \\
&\frac{\int_\Omega (1-\varepsilon_j\kappa y_2)\Big(\frac{1}{{\varepsilon_j}^2}|\partial_2 u^{\varepsilon_j}|^2 + |Q_2^{\varepsilon_j}|^2|u^{\varepsilon_j}|^2 + i\frac{Q_2^{\varepsilon_j}}{\varepsilon_j}(\partial_2\overline{u}^{\varepsilon_j}u^{\varepsilon_j} - \partial_2 u^{\varepsilon_j}\overline{u}^{\varepsilon_j})\Big)dy}{\int_\Omega (1-\varepsilon_j\kappa y_2)|u^{\varepsilon_j}|^2 dy} + \\
&\frac{\int_\Omega (1-\varepsilon_j\kappa y_2)\Big(\frac{1}{{\varepsilon_j}^2}|\partial_3 u^{\varepsilon_j}|^2 + |Q_3^{\varepsilon_j}|^2|u^{\varepsilon_j}|^2 - i\frac{Q_3^{\varepsilon_j}}{\varepsilon_j}(\partial_3\overline{u}^{\varepsilon_j}u^{\varepsilon_j} - \partial_3 u^{\varepsilon_j}\overline{u}^{\varepsilon_j})\Big)dy}{\int_\Omega (1-\varepsilon_j\kappa y_2)|u^{\varepsilon_j}|^2 dy}
\end{aligned}
\tag{4.8}
$$

where

$$\begin{aligned}
Q_1^\varepsilon(y_1,y_2,y_3) &= -\frac{1}{1-\varepsilon\kappa y_2}\beta_\varepsilon + y_2B_3 - y_3B_2 + \frac{1}{\varepsilon}R(y_1,\varepsilon y_2,\varepsilon y_3)\\
Q_2^\varepsilon(y_1,y_2,y_3) &= \frac{1}{2}y_3B_1 - \frac{1}{\varepsilon}R_2(y_1,\varepsilon y_2,\varepsilon y_3)\\
Q_3^\varepsilon(y_1,y_2,y_3) &= \frac{1}{2}y_2B_1 + \frac{1}{\varepsilon}R_3(y_1,\varepsilon y_2,\varepsilon y_3). \qquad (4.9)
\end{aligned}$$

We will analyze each term separately, but first we observe that

$$Q_1^{\varepsilon_j}(y_1,y_2,y_3) \to -\beta_0 + y_2B_3(y_1,0,0) - y_3B_2(y_1,0,0) \qquad (4.10)$$

$$Q_2^{\varepsilon_j}(y_1,y_2,y_3) \to \frac{y_3}{2}B_1(y_1,0,0) \qquad (4.11)$$

$$Q_3^{\varepsilon_j}(y_1,y_2,y_3) \to \frac{y_2}{2}B_1(y_1,0,0) \qquad (4.12)$$

uniformly on Ω as ε_j tends to 0. Now, one uses (4.2) to calculate the limit of the denominator of (4.8):

$$\lim_{j\to\infty}\int_\Omega |u^{\varepsilon_j}|^2(1-\varepsilon_j\kappa y_2)dy = \int_\Omega |u_0|^2dy = \pi\int_0^L |u_0|^2dy_1. \qquad (4.13)$$

Next we analyze each term of the numerator. Because of (4.1) and lower semi-continuity of the H^1-norm, we have

$$\liminf_{j\to\infty}\int_\Omega \frac{1}{1-\varepsilon_j\kappa y_2}|\partial_1 u^{\varepsilon_j}|^2dy \geq \int_\Omega |\partial_1 u_0|^2dy = \pi\int_0^L |\partial_1 u_0|^2dy_1. \qquad (4.14)$$

From (4.2) and (4.10) and the symmetry of the cross-section of Ω (which we have taken to be a disc), we have that

$$\begin{aligned}
&\lim_{j\to\infty}\int_\Omega (1-\varepsilon_j\kappa y_2)|\,Q_1^{\varepsilon_j}|^2|\,u^{\varepsilon_j}|^2dy\\
&= \int_\Omega \big|-\beta_0 + y_2B_3(y_1,0,0) - y_3B_2(y_1,0,0)\big|^2|u_0|^2dy\\
&= \pi\int_0^L \Big(\beta_0 + \frac{1}{4}B_2(y_1,0,0)^2 + \frac{1}{4}B_3(y_1,0,0)^2\Big)|\,u_0|^2\,dy_1 \qquad (4.15)
\end{aligned}$$

In light of (4.1), (4.2) and (4.10), we have that

$$\begin{aligned}
&\lim_{j\to\infty}\int_\Omega Q_1^{\varepsilon_j}\big(\partial_1\overline{u}^{\varepsilon_j}\,u^{\varepsilon_j}-\partial_1 u^{\varepsilon_j}\,\overline{u}^{\varepsilon_j}\big)\,dy\\
&=\int_\Omega\Big(-\beta_0+y_2B_3(y_1,0,0)-y_3B_2(y_1,0,0)\Big)\big(\partial_1\overline{u}_0\,u_0-\partial_1u_0\,\overline{u}_0\big)\,dy\\
&=\pi\int_0^L-\beta_0\big(\partial_1\overline{u}_0\,u_0-\partial_1u_0\,\overline{u}_0\big)\,dy_1\,. \qquad (4.16)
\end{aligned}$$

Then by (3.12) and (3.13), we get

$$\begin{aligned}
\lim_{j\to\infty}\int_\Omega\frac{1}{1-\varepsilon_j\kappa y_2}\big|\big(\tau y_3\partial_2-\tau y_2\partial_3\big)u^{\varepsilon_j}\big|^2dy &\le \lim_{j\to\infty}C\big(\|\,\partial_2u^{\varepsilon_j}\|^2+\|\,\partial_3u^{\varepsilon_j}\|^2\big)\\
&\le \lim_{j\to\infty}C{\varepsilon_j}^3=0\,. \qquad (4.17)
\end{aligned}$$

Similarly, we find that

$$\int_\Omega\frac{1-\varepsilon_j\kappa y_2}{{\varepsilon_j}^2}|\,\partial_2u^{\varepsilon_j}|^2\,dy\le C\varepsilon_j\to0 \qquad (4.18)$$

$$\int_\Omega\frac{1-\varepsilon_j\kappa y_2}{{\varepsilon_j}^2}|\,\partial_3u^{\varepsilon_j}|^2\,dy\le C\varepsilon_j\to0. \qquad (4.19)$$

Next, we use (3.4), (3.12) and (3.13) to calculate

$$\begin{aligned}
&\lim_{j\to\infty}\Big|\int_\Omega\frac{i}{1-\varepsilon_j\kappa y_2}\Big(\partial_1u^{\varepsilon_j}\big(\tau y_3\partial_2-\tau y_2\partial_3\big)\overline{u}^{\varepsilon_j}+\partial_1\overline{u}^{\varepsilon_j}\big(\tau y_3\partial_2-\tau y_2\partial_3\big)u^{\varepsilon_j}\Big)dy\Big|\\
&\le\lim_{j\to\infty}C\,\|\,\partial_1u^{\varepsilon_j}\|_{L^2(\Omega)}\,\big(\|\,\partial_2u^{\varepsilon_j}\|_{L^2(\Omega)}+\|\,\partial_3u^{\varepsilon_j}\|_{L^2(\Omega)}\big)\\
&=\lim_{j\to\infty}C{\varepsilon_j}^{3/2}=0\,. \qquad (4.20)
\end{aligned}$$

The third to last line of (4.8) likewise disappears in the limit since

$$\begin{aligned}
&\lim_{j\to\infty}\Big|\int_\Omega iQ_1^{\varepsilon_j}\Big(u^{\varepsilon_j}\big(\tau y_3\partial_2-\tau y_2\partial_3\big)\overline{u}^{\varepsilon_j}-\overline{u}^{\varepsilon_j}\big(\tau y_3\partial_2-\tau y_2\partial_3\big)u^{\varepsilon_j}\Big)dy\Big|\\
&\le\lim_{j\to\infty}C\,\|u^{\varepsilon_j}\|_{L^2(\Omega)}\,\big(\|\partial_2u^{\varepsilon_j}\|_{L^2(\Omega)}+\|\partial_3u^{\varepsilon_j}\|_{L^2(\Omega)}\big)\\
&=\lim_{j\to\infty}C{\varepsilon_j}^{3/2}=0\,. \qquad (4.21)
\end{aligned}$$

As the consequence of (4.2) and (4.11), we see that

$$\begin{aligned}
&\lim_{j\to\infty}\int_\Omega |Q_2^{\varepsilon_j}|^2|u^{\varepsilon_j}|^2dy\\
&=\int_\Omega \big|\frac{y_3}{2}B_1(y_1,0,0)\big|^2|u_0|^2dy\\
&=\frac{\pi}{16}\int_0^L B_1^2|u_0|^2dy_1.
\end{aligned}\tag{4.22}$$

Then by (3.12) and (4.11) we obtain that

$$\left|\int_\Omega \frac{Q_2^{\varepsilon_j}}{\varepsilon_j}\big(\partial_2\overline{u}^{\varepsilon_j}u^{\varepsilon_j}-\partial_2 u^{\varepsilon_j}\overline{u}^{\varepsilon_j}\big)dy\right|\le\frac{C}{\varepsilon_j}\|\,\partial_2 u^{\varepsilon_j}\|_{L^2(\Omega)}\|\,u^{\varepsilon_j}\|_{L^2(\Omega)}\le C{\varepsilon_j}^{1/2}\to 0\tag{4.23}$$

and similarly

$$\lim_{j\to\infty}\int_\Omega \frac{Q_3^{\varepsilon_j}}{\varepsilon_j}\big(\partial_3\overline{u}^{\varepsilon_j}u^{\varepsilon_j}-\partial_3 u^{\varepsilon_j}\overline{u}^{\varepsilon_j}\big)dy=0\,.\tag{4.24}$$

Finally, we have from (4.12) that

$$\lim_{j\to\infty}\int_\Omega |\,Q_3^{\varepsilon_j}|^2|\,u^{\varepsilon_j}|^2dy=\frac{\pi}{16}\int_0^L B_1(y_1,0,0)^2|\,u_0|^2\,dy_1\,.\tag{4.25}$$

Applying (4.13)-(4.25) to (4.8), we conclude that

$$\liminf_{\varepsilon_j\to 0} F_{\varepsilon_j}(u^{\varepsilon_j})\ge G_{\beta_0}(u_0).$$

This completes the proof of Theorem 4.2. □

Invoking some techniques from the proof of Theorem 4.2, we can now improve the compactness result from Proposition 4.1.

Theorem 4.3. *Suppose $\{\beta_{\varepsilon_j}\}$ given by (2.13) converges to a limit $\beta_0\in[-\frac{\pi}{L},\frac{\pi}{L}]$ as $\varepsilon_j\to 0$. Let u^ε denote a minimizer of F_ε with $\|u^\varepsilon\|_{L^2(\Omega)}=1$. Then passing to a subsequence, still denoted by $\{\varepsilon_j\}$, one has*

$$u^{\varepsilon_j}\quad\to\quad u_0\qquad\textit{strongly in } H^1(\Omega)\tag{4.26}$$

where the function $u_0=u_0(y_1)$ minimizes G_{β_0} in $H^1_{\mathrm{per}}((0,L))$.

Proof. By (4.5), we can pass to a further subsequence, still denoted by u^{ε_j}, such that

$$\lim_{\varepsilon_j \to 0} F_{\varepsilon_j}(u^{\varepsilon_j}) = G_{\beta_0}(u_0)\,.$$

Then in light of the convergence established in (4.13)-(4.25), the inequality in (4.14) must be an equality. Hence

$$\lim_{\varepsilon_j \to 0} \|\, \partial_1 u^{\varepsilon_j} \|_{L^2(\Omega)} = \|\, \partial_1 u_0 \|_{L^2(\Omega)}\,. \tag{4.27}$$

We also know

$$\lim_{\varepsilon_j \to 0} \|\, \partial_2 u^{\varepsilon_j} \|_{L^2(\Omega)} = 0 = \|\, \partial_2 u_0 \|_{L^2(\Omega)}$$

and

$$\lim_{\varepsilon_j \to 0} \|\, \partial_3 u^{\varepsilon_j} \|_{L^2(\Omega)} = 0 = \|\, \partial_3 u_0 \|_{L^2(\Omega)}.$$

Therefore, by (4.2), we obtain

$$\lim_{\varepsilon_j \to 0} \|\, u^{\varepsilon_j} \|_{H^1(\Omega)} = \|\, u_0 \|_{H^1(\Omega)}\,.$$

Combining this fact with the weak convergence already established in (4.1), we conclude that $\{u^{\varepsilon_j}\}$ converges to u_0 strongly in $H^1(\Omega)$. Of course, by Theorem 4.2 we already established that u_0 minimizes G_{β_0}. □

The following theorem shows the minimum of the limiting Rayleigh quotient G_{β_ε} characterizes the asymptotic behavior of the minimum of the Rayleigh quotient F_ε as $\varepsilon \to 0$ by analyzing the difference between the first eigenvalues.

Theorem 4.4. *Let λ_ε be the minimum of F_ε and let μ_ε be the minimum of G_{β_ε}. Then*

$$\lim_{\varepsilon \to 0} (\lambda_\varepsilon - \mu_\varepsilon) = 0 \tag{4.28}$$

Proof. First, we will show that

$$\limsup_{\varepsilon \to 0}\, (\lambda_\varepsilon - \mu_\varepsilon) \le 0. \tag{4.29}$$

Next, we show that for any sequence $\{\varepsilon_j\} \to 0$ there exists a subsequence $\{\varepsilon_{j_k}\}$ such that

$$\liminf_{k \to \infty}\, (\lambda_{\varepsilon_{j_k}} - \mu_{\varepsilon_{j_k}}) \ge 0\,. \tag{4.30}$$

This will imply that

$$\lim_{\varepsilon\to 0}(\lambda_{\varepsilon_{j_k}} - \mu_{\varepsilon_{j_k}}) = 0.$$

Since the limit is independent of the sequence $\{\varepsilon_j\}$ we choose, we will conclude that

$$\lim_{\varepsilon\to 0}(\lambda_\varepsilon - \mu_\varepsilon) = 0.$$

To establish (4.29), suppose $\{u^\varepsilon\}$ are minimizers of $\{F_\varepsilon\}$ and $\{v^\varepsilon\}$ are minimizers of $\{G_{\beta_\varepsilon}\}$. Then we calculate

$$\begin{aligned}
&\lambda_\varepsilon - \mu_\varepsilon = F_\varepsilon(u^\varepsilon) - G_\varepsilon(v^\varepsilon)\\
&\le F_\varepsilon(v^\varepsilon) - G_\varepsilon(v^\varepsilon) =\\
&\frac{\int_\Omega \frac{1}{1-\varepsilon\kappa y_2}\left|\partial_1 v^\varepsilon + i(1-\varepsilon\kappa y_2)Q_1^\varepsilon v^\varepsilon\right|^2 dy}{\int_\Omega |v^\varepsilon|^2(1-\varepsilon\kappa y_2)dy}\\
&+\frac{\int_\Omega (1-\varepsilon\kappa y_2)\left(\left|Q_2^\varepsilon v^\varepsilon\right|^2 + \left|Q_3^\varepsilon v^\varepsilon\right|^2\right)dy}{\int_\Omega |v^\varepsilon|^2(1-\varepsilon\kappa y_2)dy}\\
&-\frac{\int_0^{2\pi} |\partial_1 v^\varepsilon - i\beta_\varepsilon v^\varepsilon|^2 + g(y_1)|v^\varepsilon|^2 dy_1}{\int_\Omega |v^\varepsilon|^2 dy_1}.
\end{aligned} \tag{4.31}$$

Taking the lim sup as $\varepsilon \to 0$ and applying the estimates analogous to (4.15), (4.16), (4.22) and (4.25) for the proof of Theorem 4.2, we obtain (4.29).

To establish claim (4.30), we invoke Proposition 4.1 to obtain a subsequence $\{\varepsilon_{j_k}\}$, a real number $\beta_0 \in [-\frac{\pi}{L}, \frac{\pi}{L}]$, and a function $u_0 \in H^1_{\mathrm{per}}((0, L))$ satisfying (4.1) and (4.2). From (4.5), we obtain that

$$\begin{aligned}
\liminf_{k\to\infty}(\lambda_{\varepsilon_{j_k}} - \mu_{\varepsilon_{j_k}}) &= \liminf_{k\to\infty}\left(F_{\varepsilon_{j_k}}(u^{\varepsilon_{j_k}}) - G_{\beta_{\varepsilon_{j_k}}}(v^{\varepsilon_{j_k}})\right)\\
&= G_{\beta_0}(u_0) - \limsup_{k\to\infty} G_{\beta_{\varepsilon_{j_k}}}(v^{\varepsilon_{j_k}})\\
&\ge G_{\beta_0}(u_0) - \limsup_{k\to\infty} G_{\beta_{\varepsilon_{j_k}}}(u_0)\\
&= G_{\beta_0}(u_0) - G_{\beta_0}(u_0)\\
&= 0\,.
\end{aligned}$$

□

We state the asymptotic relationship between the first eigenspace of the functionals F_ε and G_ε in the following theorem.

Theorem 4.5. *Let $\varepsilon_j \to 0$ be any sequence such that*

$$-\frac{\pi}{L} < \liminf_{j\to\infty} \beta_{\varepsilon_j} \leq \limsup_{j\to\infty} \beta_{\varepsilon_j} < \frac{\pi}{L}\,. \tag{4.32}$$

and let u^{ε_j} be a minimizer of F_{ε_j} in $H^1(\Omega)$ with $\|u^{\varepsilon_j}\|_{L^2(\Omega)} = 1$. Then there exists a sequence $\{v^{\varepsilon_j}\}$ minimizing $\{G_{\beta_{\varepsilon_j}}\}$ in $H^1_{\mathrm{per}}((0,L))$ with $\|v^{\varepsilon_j}\|_{L^2(\Omega)} = 1$ such that

$$v^{\varepsilon_j} - u^{\varepsilon_j} \quad \to \quad 0 \quad \textit{strongly in } H^1(\Omega)\,. \tag{4.33}$$

Proof. By Theorem 4.3 and (4.32), we know there exists a subsequence $\beta_{\varepsilon_{j_k}} \to \beta_0 \in (-\frac{\pi}{L}, \frac{\pi}{L})$ and a minimizer $u_0 \in H^1_{\mathrm{per}}((0,L))$ of G_{β_0} with $\|u_0\|_{L^2(\Omega)} = 1$ such that $\{u^{\varepsilon_{j_k}}\}$ converges to u_0 strongly in $H^1(\Omega)$. We claim that after perhaps passing to a further subsequence, still denoted by $\{\varepsilon_{j_k}\}$, one has:

$$\begin{aligned}&\text{There exists a sequence of minimizers } \{v^{\varepsilon_{j_k}}\} \subset H^1_{\mathrm{per}}((0,L)) \text{ of } G_{\beta_{\varepsilon_{j_k}}}\\ &\text{such that } v^{\varepsilon_{j_k}} \to u_0 \quad \text{strongly in } H^1(\Omega).\end{aligned} \tag{4.34}$$

To prove claim (4.34), first note that the existence of the minimizer of G_β is given by a standard application of the direct method in the calculus of variations. Note also that there is an intrinsic degeneracy to the problem related to rotation invariance. That is, if u_β is a first eigenfunction of the Rayleigh quotient G_β, then $e^{i\theta}u_\beta$ is also a first eigenfunction for any $\theta \in [0, 2\pi)$. We address this degeneracy by letting v^{ε_j} be a minimizer of $G_{\beta_{\varepsilon_j}}$ with $\|v^{\varepsilon_j}\|_{L^2(\Omega)} = 1$ such that

$$\|v^{\varepsilon_j} - u_0\|_{L^2((0,L))} = \min_{\theta\in[0,2\pi)} \|e^{i\theta}v^{\varepsilon_j} - u_0\|_{L^2((0,L))}\,. \tag{4.35}$$

The minimizing property of v^{ε_j} and the condition $\|v^{\varepsilon_j}\|_{L^2(\Omega)} = 1$ yield a uniform $H^1((0,L))$ bound. By weak compactness and the Rellich-Kondrachov theorem, there exists a subsequence still denoted by $\{\beta_{\varepsilon_{j_k}}\}$ and a function $v_0 \in H^1_{\mathrm{per}}((0,L))$ such that

$$v^{\varepsilon_{j_k}} \quad \rightharpoonup \quad v_0 \quad \text{weakly in } H^1((0,L))\,, \tag{4.36}$$

$$v^{\varepsilon_{j_k}} \quad \to \quad v_0 \quad \text{strongly in } L^2((0,L))\,. \tag{4.37}$$

Because of (4.36), (4.37) and the minimizing property of $\{v^{\varepsilon_{j_k}}\}$, we obtain

$$G_{\beta_0}(v_0) \leq \liminf_{k\to\infty} G_{\beta_{\varepsilon_{j_k}}}(v^{\varepsilon_{j_k}}) \leq \liminf_{k\to\infty} G_{\beta_{\varepsilon_{j_k}}}(v) = G_{\beta_0}(v) \tag{4.38}$$

for any $v \in H^1_{\mathrm{per}}((0,L))$. This leads us to conclude that v_0 minimizes G_{β_0} and $\liminf_{k\to\infty} G_{\beta_{\varepsilon_{j_k}}}(v^{\varepsilon_{j_k}}) = G_{\beta_0}(v_0)$. Passing to a further subsequence, still denoted by $\{v^{\varepsilon_{j_k}}\}$, we obtain $\lim_{k\to\infty} G_{\beta_{\varepsilon_{j_k}}}(v^{\varepsilon_{j_k}}) = G_{\beta_0}(v_0)$. Therefore, we obtain $\lim_{k\to\infty} \|\partial_1 v^{\varepsilon_{j_k}}\|_{L^2(\Omega)} = \|\partial_1 v_0\|_{L^2(\Omega)}$. Since $\{v^{\varepsilon_{j_k}}\}$ converges weakly to v_0 in $H^1_{\mathrm{per}}((0,L))$ and

$$\lim_{k\to\infty} \|v^{\varepsilon_{j_k}}\|_{H^1((0,L))} = \|v_0\|_{H^1(0,L)}\,, \tag{4.39}$$

we obtain that $\{v^{\varepsilon_{j_k}}\}$ converges strongly to v_0 in $H^1_{\mathrm{per}}((0,L))$. By Theorem XIII.89 in [19], we know the first eigenvalue of G_β is simple for $\beta \in (-\frac{\pi}{L}, \frac{\pi}{L})$. By (4.35) and the simplicity of the eigenspace of G_{β_0}, v_0 must equal u_0. This completes the verification of (4.34).

Since every subsequence of $\{\beta_{\varepsilon_j}\}$ satisfying (4.32) possesses a further subsequence $\{\beta_{\varepsilon_{j_k}}\}$ such that

$$v^{\varepsilon_{j_k}} - u^{\varepsilon_{j_k}} \quad \to \quad 0 \quad \text{strongly in } H^1(\Omega)\,, \tag{4.40}$$

we can conclude that (4.33) holds. $\square$

Remark 4.6. The Euler-Lagrange equation associated with critical points of the Rayleigh quotient G_β takes the form of Hill's equation and has been studied extensively. From the general theory of the Hill's equation (see [6]), the first eigenvalue is also simple for $\beta = \pm\frac{\pi}{L}$ if the function g given by (4.4) is C^∞ and the condition $\int_0^L e^{\frac{2\pi i s}{L}} g(s)\,ds \neq 0$ is satisfied. In this case, we can also conclude (4.33) as $\varepsilon \to 0$ without any exception.

Remark 4.7. As explained in the introduction, the minimum value of the Rayleigh quotient is proportional to the difference between the critical temperature below which a material first exhibits superconducivity in the absence of any applied field and the critical temperature in the presence of an applied field. The problem of finding the critical temperature in the case of a thin superconducting ring subjected to an order $\mathcal{O}(1)$ applied field has been particularly well-studied (see e.g. [21]), especially in the situation where the field is constant and directed orthogonally to the plane of the ring . In this case, the problem reduces to minimizing the functional G_β (cf. (4.3)) with $g \equiv 0$ and β given by (2.13) with $\varepsilon = 1$. This is a simple eigenvalue problem that can be solved explicitly to reveal an oscillatory dependence of the critical temperature on the applied field known as the Little-Parks effect, [15]. (See e.g. [12] for a simple mathematical treatment.) What the results developed formally in [20] and rigorously here reveal in this context is that the effect of a constant $\mathcal{O}(1/\varepsilon)$ applied field $= (0, 0, B_3/\varepsilon)$ is to raise the first eigenvalue–thus, lowering the critical temperature, by the value $\frac{B_3^2}{4}$ (modulo a factor depending on material parameters coming from (1.1)). Of course, in the case of a non-constant applied field, the form of G_β still reveals an upward shift in the first eigenvalue but this time the appearance of the positive but no longer constant potential g generally precludes an explicit solution of the problem.

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