Ginzburg-Landau model in thin loops with narrow constrictions

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Abstract

We consider the Ginzburg-Landau model for a superconducting thin ring in the presence of an applied field. The ring is constricted and we derive an asymptotic form for the energy as the ring thickness tends to zero. The constriction leads in the limit to a jump condition for the order parameter, yielding a transmission condition across the weak link of the type postulated by deGennes for superconducting/normal/superconducting junctions.

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1 Introduction

The Josephson effect models the peculiar flow of currents through a normal thin layer, called a junction, separating two bulk superconducting samples. The fundamental feature of this effect is an expression for the current as a function of the phase difference across the junction:

$$J = J_M \sin[\phi]. \tag{1.1}$$

The parameter J_M is the maximal current that the junction can transmit, and $[\phi]$ denotes the difference between the phase of the superconducting wave function on the two ends of the junction.

Josephson predicted the relation (1.1) from the microscopic BCS theory, [13] . Soon thereafter, it was realized that a similar expression, and some other features of the junction can also be derived by ad-hoc models that are coupled to the macroscopic Ginzburg-Landau (GL) model of superconductivity. In particular deGennes [7] modeled the junction through a set of linear conditions relating the wave function and its derivatives on the two sides of the thin normal layer. Similar equations were written also in [1]. Alternatively, other authors, and in fact, most of the physics literature on the subject (e.g. [19], [2]) use equation (1.1) as a basic paradigm and supplement it with equations and arguments based on the GL theory and classical electromagnetism.

It is therefore desirable to develop a theory for Josephson junctions that is built up coherently and directly upon the GL model. One way to do so is to model the normal layer into the GL energy functional. This was done in [5], [10], [11] and [18]. In particular it was shown in [18] that a large variety of junctions can be modeled in this way, leading to different types of current flow patterns.

The purpose of the current paper is to construct, directly from the GL equations, a 'geometrical' Josephson junction. Such junctions, called in the literature 'weak links', are characterized by a sharp constriction in the the thickness of the sample, [14]. We shall show below that under appropriate selection of the sample geometry and its scaling, the GL model converges to a new model that provides in a natural way the linear relations postulated by deGennes. The convergence is established rigorously in Section 2, with the proof inspired in part by an analogous convergence result in the field of elasticity, [6]. In Section 3 we discuss some implications of the convergence result.

2 Formulation

We begin with a description of the geometry of the region Ω_ε to be occupied by the sample. To this end, we introduce a continuous, piecewise linear function $g_\varepsilon : [-\pi, \pi] \to \mathbb{R}^1$ that will govern the thickness of the ring. Fixing any positive number $p < 1$, we define g_ε via:

$$g_\varepsilon(y_1) = (\varepsilon^{1-p} - 2\varepsilon)\,|y_1| + 2\varepsilon^{1+p} \quad \text{for} \quad |y_1| \le \varepsilon^p, \tag{2.1}$$

$$g_\varepsilon(y_1) = \varepsilon \quad \text{for} \quad \varepsilon^p \le |y_1| \le \pi, \tag{2.2}$$

We then define a ring-shaped region $\Omega_\varepsilon \subset \mathbb{R}^3$ of thickness g_ε as the image of the cylinder

$$\mathcal{C} = \{(y_1, y_2, y_3) : -\pi \le y_1 \le \pi,\, 0 \le y_2^2 + y_3^2 < 1\} \tag{2.3}$$

under the mapping $T_\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$x = T_\varepsilon(y_1, y_2, y_3) = \big((1+g_\varepsilon(y_1)y_2)\cos y_1, (1+g_\varepsilon(y_1)y_2)\sin y_1, g_\varepsilon(y_1)y_3\big) \tag{2.4}$$

That is, $\Omega_\varepsilon \equiv T_\varepsilon(\mathcal{C})$. Note, in particular, that in $(x_1, x_2, x_3)-$space, the variable y_1 corresponds to the polar angle in the x_1x_2-plane and that the ring Ω_ε has uniform thickness ε except near the constriction at $y_1 = 0$. See Figure 1.

We will use the following non-dimensional version of the Ginzburg-Landau energy functional

$$G_\epsilon(u, \mathbf{A}) = \frac{1}{\varepsilon^2}\int_{\Omega_\epsilon}\left(|(i\nabla + \mathbf{A})u|^2 + \frac{\nu^2}{2}\left(|u|^2 - \mu^2\right)^2\right) dx + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^3} |\nabla \times \mathbf{A} - \mathbf{H}^e|^2\, dx. \tag{2.5}$$

Here $u : \Omega_\varepsilon \to \mathbb{C}$ is the order parameter, $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ is the magnetic potential associated with the magnetic field $\mathbf{H}$ through $\nabla \times \mathbf{A} = \mathbf{H}$, $\mathbf{H}^e$ is a given, smoothly varying, applied magnetic field directed along the x_3-axis and taken to be independent of the coordinate x_3. The quantities ν and μ are material parameters with μ^2 proportional to the difference between the critical temperature T_c and the temperature of the sample, [18]. We assume we are in the superconducting temperature regime where this difference is positive. The energy G_ε has been scaled so that the minimum energy remains uniformly bounded away from both zero and infinity for small ε.

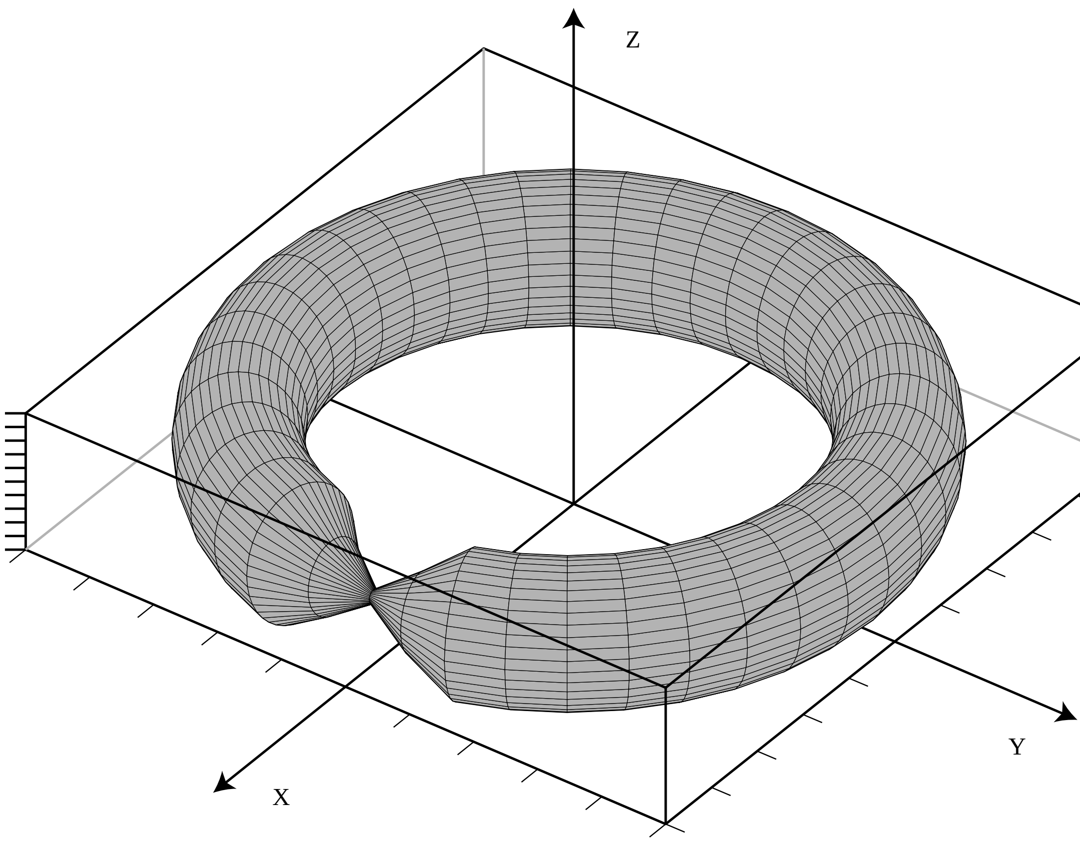

Figure 1: The constricted ring Ω_ε.

We would like to investigate the asymptotic behavior of minimizers to (2.5) and this will require a precise description of function spaces over which the minimization is to take place. For the order parameter u we shall take competitors in the standard Sobolev space $W^{1,2}(\Omega_\varepsilon;\mathbb{C})$ consisting of square-integrable functions with square-integrable first derivatives. For the magnetic potential $\mathbf{A}$ we introduce the space $\mathcal{H}$ as the completion of the set

$$\{\phi \in C^\infty(\mathbb{R}^3;\mathbb{R}^3) : \phi \text{ compactly supported}\}$$

with respect to the norm $\|\nabla\phi\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla\phi|^2\, dx\right)^{1/2}$. Then we define $\mathcal{H}_0$ to be

$$\mathcal{H}_0 = \{\phi \in \mathcal{H} : \operatorname{div}\phi = 0\},$$

and consider competitors $\mathbf{A}$ satisfying $\mathbf{A} - \mathbf{A}^e \in \mathcal{H}_0$, where $\mathbf{A}^e = \mathbf{A}^e(x_1, x_2)$ is the applied magnetic potential satisfying

$$\nabla \times \mathbf{A}^e = \mathbf{H}^e \quad \text{and } \operatorname{div}\mathbf{A}^e = 0 \quad \text{in } \mathbb{R}^3, \tag{2.6}$$

$$\mathbf{A}^e \cdot (0,0,1) = 0 \quad \text{in } \mathbb{R}^3, \tag{2.7}$$

Condition (2.7) holds by our assumptions on $\mathbf{H}^e$, while we can arrange for the zero divergence condition by a suitable choice of gauge.

Through a rather standard application of the Direct Method in the calculus of variations, along with standard elliptic regularity theory, one obtains the following:

2.1 Theorem. *For all positive $\varepsilon < 1$, there exists a pair $(u^\varepsilon, \mathbf{A}^\varepsilon)$ solving the variational problem*

$$\inf_{\{u \in W^{1,2}(\Omega_\varepsilon;\mathbb{C}), \mathbf{A}-\mathbf{A}^e \in \mathcal{H}_0\}} G_\varepsilon(u, \mathbf{A}). \tag{2.8}$$

The function u^ε is smooth in Ω_ε while the function $\mathbf{A}^\varepsilon$ is smooth in $\mathbb{R}^3 \setminus \partial\Omega_\varepsilon$ and continuously differentiable across $\partial\Omega$. Furthermore, the minimizers satisfy the Ginzburg-Landau system

$$(i\nabla + \mathbf{A}^\varepsilon)^2 u^\varepsilon = \nu^2(|u^\varepsilon|^2 - \mu^2)u^\varepsilon \quad in \quad \Omega_\varepsilon, \tag{2.9}$$

$$\nabla \times \nabla \times (\mathbf{A}^\varepsilon - \mathbf{A}^e) = -\Delta(\mathbf{A}^\varepsilon - \mathbf{A}^e) =$$
$$\begin{cases} \frac{i}{2}(\overline{u^\varepsilon}\nabla u^\varepsilon - u^\varepsilon \nabla \overline{u^\varepsilon}) - |u^\varepsilon|^2 \mathbf{A}^\varepsilon & for\ x \in \Omega_\varepsilon \\ 0 & for\ x \in \mathbb{R}^3 \setminus \overline{\Omega_\varepsilon} \end{cases} \tag{2.10}$$

and the boundary condition

$$(i\nabla + \mathbf{A}^\varepsilon)u^\varepsilon \cdot \mathbf{n} = 0 \quad on \quad \partial\Omega_\varepsilon. \tag{2.11}$$

Here $\bar{\cdot}$ denotes complex conjugation and $\mathbf{n}$ denotes the outer unit normal along $\partial\Omega_\varepsilon$.

Finally, the order parameter u^ε satisfies the condition

$$|u^\varepsilon| \leq \mu \quad in \quad \overline{\Omega_\varepsilon}. \tag{2.12}$$

Application of the Direct Method to establishing existence of minimizers to the Ginzburg-Landau energy can be found for instance in [8] or [17]. The regularity theory in this context can be found for instance in [12]. Inequality (2.12) is an easy consequence of the maximum principle, see e.g. [8].

2.2 Proposition. *There exist positive constants C_1 and C_2 independent of ε such that*

$$G_\varepsilon(u^\varepsilon, \mathbf{A}^\varepsilon) \leq C_1 \quad and \tag{2.13}$$

$$\int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^2 \, dx \leq C_2 \varepsilon^2. \tag{2.14}$$

Furthermore, one has the uniform convergence

$$\|\mathbf{A}^\varepsilon - \mathbf{A}^e\|_{L^\infty(B_R(0);\mathbb{R}^3)} \to 0 \quad \textit{as} \quad \varepsilon \to 0 \quad \textit{for every } R > 0, \tag{2.15}$$

where $B_R(0) = \{x \in \mathbb{R}^3 : |x| < R\}$. *Condition* (2.15) *in particular implies that*

$$\sup_{y\in\mathcal{C}} |\mathbf{A}^\varepsilon(T_\varepsilon(y)) - \mathbf{A}^e(\cos y_1, \sin y_1, 0)| \to 0 \quad \textit{as} \quad \varepsilon \to 0. \tag{2.16}$$

Proof. The bound (2.13) follows immediately by comparing the energy of the minimizer to that of the pair $(\mu, \mathbf{A}^e)$:

$$G_\varepsilon(u^\varepsilon, \mathbf{A}^\varepsilon) \le G_\varepsilon(\mu, \mathbf{A}^e) = \frac{\mu^2}{\varepsilon^2}\int_{\Omega_\varepsilon} |\mathbf{A}^e|^2 \, dx \le \frac{\mu^2 \mathrm{vol}(\Omega_\varepsilon)}{\varepsilon^2} \|\mathbf{A}^e\|^2_{L^\infty(\Omega_\varepsilon)} \le C_1$$

since $\mathrm{vol}(\Omega_\varepsilon) = \mathcal{O}(\varepsilon^2)$.

We next establish the convergence (2.15) and the bound (2.14) using (2.13) by decomposing G_ε as

$$\begin{aligned}\varepsilon^2 G_\varepsilon(u^\varepsilon, \mathbf{A}^\varepsilon) &= \int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^2 \, dx + i\int_{\Omega_\varepsilon} (\overline{u^\varepsilon}\nabla u^\varepsilon - u^\varepsilon \nabla\overline{u^\varepsilon}) \cdot \mathbf{A}^\varepsilon \, dx \\ &+ \int_{\Omega_\varepsilon} |u^\varepsilon|^2 |\mathbf{A}^\varepsilon|^2 + \frac{\nu^2}{2}(|u^\varepsilon|^2 - \mu^2)^2 \, dx + \int_{\mathbb{R}^3} |\nabla \times (\mathbf{A}^\varepsilon - \mathbf{A}^e)|^2 \, dx.\end{aligned} \tag{2.17}$$

Applying (2.12), we find that

$$\begin{aligned}&\left| i\int_{\Omega_\varepsilon} (\overline{u^\varepsilon}\nabla u^\varepsilon - u^\varepsilon \nabla\overline{u^\varepsilon}) \cdot \mathbf{A}^\varepsilon \, dx \right| \le 2\mu \int_{\Omega_\varepsilon} |\nabla u^\varepsilon| \, |\mathbf{A}^\varepsilon| \, dx \\ &\le \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^2 \, dx + C(\mu)\int_{\Omega_\varepsilon} |\mathbf{A}^\varepsilon|^2 \, dx.\end{aligned}$$

Hence, (2.13) and (2.17) yield the bound

$$\begin{aligned}&\int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^2 \, dx \le 2C_1\varepsilon^2 + C(\mu)\int_{\Omega\varepsilon} |\mathbf{A}^\varepsilon|^2 \, dx \\ &\le 2C_1\varepsilon^2 + C(\mu)\left(\|\mathbf{A}^\varepsilon - \mathbf{A}^e\|^2_{L^\infty(\Omega_\varepsilon;\mathbb{R}^3)} + \|\mathbf{A}^e\|^2_{L^\infty(\Omega_\varepsilon;\mathbb{R}^3)}\right) \mathrm{vol}\,(\Omega_\varepsilon).\end{aligned} \tag{2.18}$$

Now we turn to the equation (2.10) satisfied by the difference $\mathbf{A}^\varepsilon - \mathbf{A}^e$ and utilize the fact that this difference lies in $\mathcal{H}_0$. This decay at infinity allows us to express it via the fundamental solution to the Laplacian in $\mathbb{R}^3$:

$$\mathbf{A}^\varepsilon - \mathbf{A}^e = \int_{\Omega_\varepsilon} \Gamma(x - z) f^\varepsilon(z) \, dz \tag{2.19}$$

where $\Gamma(x) \equiv \frac{1}{4\pi|x|}$ and $f^\varepsilon(z) \equiv \frac{i}{2}(\overline{u^\varepsilon}\nabla u^\varepsilon - u^\varepsilon\nabla\overline{u^\varepsilon}) - |u^\varepsilon|^2 \mathbf{A}^\varepsilon$.

Given any $R > 0$, one readily checks that

$$\int_{\Omega_\varepsilon} |\Gamma(x-z)|^2 \, dz \leq C(R)$$

provided $|x| \leq R$, so that by Hölder's inequality, we obtain

$$\|\mathbf{A}^\varepsilon - \mathbf{A}^e\|_{L^\infty(B_R(0);\mathbb{R}^3)} \leq \sqrt{C(R)} \, \|f^\varepsilon\|_{L^2(\Omega_\varepsilon)} \,. \tag{2.20}$$

Then writing

$$f^\varepsilon = \frac{i}{2}(\overline{u^\varepsilon}\nabla u^\varepsilon - u^\varepsilon\nabla\overline{u^\varepsilon}) - |u^\varepsilon|^2 (\mathbf{A}^\varepsilon - \mathbf{A}^e) - |u^\varepsilon|^2 \mathbf{A}^e,$$

we can combine inequalities (2.12), (2.18) and (2.20) to conclude that

$$\begin{aligned} &\|\mathbf{A}^\varepsilon - \mathbf{A}^e\|_{L^\infty(B_R(0);\mathbb{R}^3)} \leq \\ &C\left(\|\nabla u^\varepsilon\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} + \|\mathbf{A}^\varepsilon - \mathbf{A}^e\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} + \|\mathbf{A}^e\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)}\right) \\ &\leq C\left(\varepsilon + \|\mathbf{A}^\varepsilon - \mathbf{A}^e\|_{L^\infty(\Omega_\varepsilon;\mathbb{R}^3)} \sqrt{\operatorname{vol}\Omega_\varepsilon} + \|\mathbf{A}^e\|_{L^\infty(\Omega_\varepsilon;\mathbb{R}^3)} \sqrt{\operatorname{vol}\Omega_\varepsilon}\right), \end{aligned} \tag{2.21}$$

where again C depends on R. Hence we obtain (2.15) in that

$$\|\mathbf{A}^\varepsilon - \mathbf{A}^e\|_{L^\infty(B_R(0);\mathbb{R}^3)} \leq C\varepsilon \tag{2.22}$$

for some constant C independent of ε but depending on R, μ, C_1 and $\mathbf{A}^e$. Then (2.14) follows from (2.15) and (2.18). □

Our characterization of the asymptotic behavior of the sequence of minimizers $\{u^\varepsilon\}$ is most conveniently carried out using the variables (y_1, y_2, y_3) defined in (2.4). Thus, we introduce the notation

$$U^\varepsilon(y_1, y_2, y_3) := u^\varepsilon(T_\varepsilon(y_1, y_2, y_3)). \tag{2.23}$$

2.3 Proposition. *There exists a subsequence $\{\varepsilon_j\} \to 0$ and a function $U^0 \in BV(\mathcal{C};\mathbb{C})$ such that $U^{\varepsilon_j} \to U^0$ in $L^1(\mathcal{C};\mathbb{C})$. Furthermore, U^0 is a function of y_1 only.*

Here $BV(\mathcal{C};\mathbb{C})$ denotes the space of complex-valued functions of bounded variation. The compactness assertion is based on the fact that sequences uniformly bounded in $W^{1,1}(\mathcal{C};\mathbb{C})$ (hence in BV) contain L^1-convergent subsequences, cf. e.g. [20], section 5.3.

Proof. To reiterate, the goal is a uniform $W^{1,1}(\mathcal{C};\mathbb{C})$-bound on the sequence $\{U^\varepsilon\}$. Such a bound will come from (2.14) once we express $\int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^2\,dx$ in terms of U^ε.

To this end, one carries out a lengthy but routine calculation to obtain

$$\int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^2\,dx = \frac{1}{2}\int_{\mathcal{C}} a_{ik}\big(U^\varepsilon_{y_i}\overline{U^\varepsilon}_{y_k} + \overline{U^\varepsilon}_{y_i}U^\varepsilon_{y_k}\big)\,dy \tag{2.24}$$

where the 3×3 matrix A with entries $a_{ik} = a_{ik}(y_1,y_2,y_3)$ can be written in the form

$$A = \frac{1}{(1+g_\varepsilon(y_1)y_2)}\big(B+D\big) \tag{2.25}$$

with

$$B = \begin{pmatrix} g_\varepsilon(y_1)^2 & -g_\varepsilon(y_1)g'_\varepsilon(y_1)y_2 & -g_\varepsilon(y_1)g'_\varepsilon(y_1)y_3 \\ -g_\varepsilon(y_1)g'_\varepsilon(y_1)y_2 & 1+g'_\varepsilon(y_1)^2y_2^2 & g'_\varepsilon(y_1)^2y_2y_3 \\ -g_\varepsilon(y_1)g'_\varepsilon(y_1)y_3 & g'_\varepsilon(y_1)^2y_2y_3 & 1+g'_\varepsilon(y_1)^2y_3^2 \end{pmatrix} \tag{2.26}$$

and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2g_\varepsilon(y_1)y_2 + g_\varepsilon(y_1)^2y_2^2 \end{pmatrix}. \tag{2.27}$$

At this point, we appeal to [6], where the eigenvalues of the matrix B are explicitly calculated and found to be given by the formulas

$$\lambda_1 = \frac{1+g_\varepsilon(y_1)^2+g'_\varepsilon(y_1)^2(y_2^2+y_3^2)}{2} - \frac{\sqrt{[1+g_\varepsilon(y_1)^2+g'_\varepsilon(y_1)^2(y_2^2+y_3^2)]^2-4g_\varepsilon(y_1)^2}}{2}, \tag{2.28}$$

$$\lambda_2 = \frac{1+g_\varepsilon(y_1)^2+g'_\varepsilon(y_1)^2(y_2^2+y_3^2)}{2} + \frac{\sqrt{[1+g_\varepsilon(y_1)^2+g'_\varepsilon(y_1)^2(y_2^2+y_3^2)]^2-4g_\varepsilon(y_1)^2}}{2}, \tag{2.29}$$

$$\lambda_3 = 1.$$

Note, in particular, that $\lambda_2,\lambda_3 \geq 1$ while expansion of (2.28) reveals that

$$\frac{g_\varepsilon^2}{1+g_\varepsilon^2+(g'_\varepsilon)^2} \leq \lambda_1 \leq g_\varepsilon^2 \quad \text{for small } \varepsilon. \tag{2.30}$$

Let us now denote the eigenvalues of A by μ_1,μ_2 and μ_3. Since A is a regular perturbation of B, it follows easily that

$$\lim_{\varepsilon\to 0}|\mu_2-\lambda_2| = 0, \quad \text{and} \quad \lim_{\varepsilon\to 0}|\mu_3-1| = 0. \tag{2.31}$$

Also, expanding $\det A$ along its last row, one readily checks that

$$(1+g_\varepsilon(y_1)y_2)^3 \det A = \det B + \mathcal{O}(g_\varepsilon^3) \quad \text{as } \varepsilon \to 0. \tag{2.32}$$

Then phrasing (2.32) in terms of λ_i and μ_i and using (2.30) and (2.31), it is not hard to verify that

$$|\mu_1 - \lambda_1| = o(g_\varepsilon^2). \tag{2.33}$$

Hence,

$$\frac{g_\varepsilon^2}{1+g_\varepsilon^2+(g_\varepsilon')^2} \le \mu_1 \le g_\varepsilon^2. \tag{2.34}$$

If we then introduce the quantity

$$a_\varepsilon(y_1) := \frac{g_\varepsilon(y_1)^2}{\varepsilon^2\big(1+g_\varepsilon(y_1)^2+(g_\varepsilon'(y_1))^2\big)}, \tag{2.35}$$

we can combine (2.14), (2.24), (2.31) and (2.34) to conclude that

$$\begin{aligned}\int_{\mathcal{C}} a_\varepsilon \left|U^\varepsilon_{y_1}\right|^2 dy &\le \frac{1}{2\varepsilon^2}\int_{\mathcal{C}} a_{ik}\big(U^\varepsilon_{y_i}\overline{U^\varepsilon}_{y_k} + \overline{U^\varepsilon}_{y_i}U^\varepsilon_{y_k}\big)\, dy \\ &\le \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^2 \, dx \le C_2 \end{aligned} \tag{2.36}$$

Hence,

$$\int_{\mathcal{C}} \left\{ g_\varepsilon(y_1)^2 \left|U^\varepsilon_{y_1}\right|^2 + \left|U^\varepsilon_{y_2}\right|^2 + \left|U^\varepsilon_{y_3}\right|^2 \right\} dy \le C\varepsilon^2 \tag{2.37}$$

for some constant C independent of ε. In particular, it follows that

$$\int_{\mathcal{C}} \left|U^\varepsilon_{y_2}\right|^2 + \left|U^\varepsilon_{y_3}\right|^2 dy \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.38}$$

Arguing as in [6], Theorem 6.1, this leads to control of $\{\|\nabla U^\varepsilon\|_{L^1(\mathcal{C})}\}$ via (2.36) as follows:

$$\begin{aligned}\int_{\mathcal{C}} |\nabla U^\varepsilon| \, dy &= \int_{\mathcal{C}} \frac{1}{\sqrt{a_\varepsilon}}\sqrt{a_\varepsilon}\, |\nabla U^\varepsilon| \, dy \\ &\le \left(\int_{\mathcal{C}} \frac{1}{a_\varepsilon}\, dy\right)^{1/2} \left(\int_{\mathcal{C}} a_\varepsilon |\nabla U^\varepsilon|^2 \, dy\right)^{1/2} \\ &\le C_2^{1/2}\left(\int_{\mathcal{C}} \frac{1}{a_\varepsilon}\, dy\right)^{1/2}. \end{aligned} \tag{2.39}$$

Referring back to assumptions (2.1)–(2.2), we note that

$$\int_{-\pi}^{\pi} \frac{1}{a_\varepsilon(y_1)}\, dy_1 \to 2\pi + 1 \quad \text{as } \varepsilon \to 0. \tag{2.40}$$

In light of the bound (2.12), we see that

$$\|U^\varepsilon\|_{W^{1,1}(\mathcal{C})} < C$$

and the L^1-convergence of a subsequence $\{U^{\varepsilon_j}\}$ to a $BV(\mathcal{C};\mathbb{C})$ function U^0 follows. In view of (2.38), one sees that U^0 is independent of y_2 and y_3. $\square$

Next we wish to identify a limiting energy for G_ε. To this end, we introduce notation for the tangential component of the applied potential restricted to the unit circle:

$$A_1^e(y_1) \equiv \mathbf{A}^e(\cos y_1, \sin y_1, 0) \cdot (-\sin y_1, \cos y_1, 0). \tag{2.41}$$

We also introduce the function λ_ε via the formula

$$\frac{1}{a_\varepsilon(y_1)} = 1 + \lambda_\varepsilon(y_1). \tag{2.42}$$

One can easily check that the $\lambda_\varepsilon \, dy_1 \overset{*}{\rightharpoonup} \delta_0$ weakly as measures so that

$$\frac{1}{a_\varepsilon} \, dy_1 \overset{*}{\rightharpoonup} 1 \, dy_1 + \delta_0. \tag{2.43}$$

This allows us to establish a generalization of Theorem 6.2 of [6]. To state the result, we define the functional G_0 acting on functions in $L^1((-\pi,\pi);\mathbb{C})$ by the formula

$$G_0(U) \quad = \begin{cases} \int_{(-\pi,\pi)\setminus\{0\}} \left(\left|(i\frac{d}{dy_1} + A_1^e)U\right|^2 + \frac{\nu^2}{2}(|U|^2 - \mu^2)^2 \right) dy_1 + |U^+ - U^-|^2 \\ \qquad \text{if } U \in W^{1,2}((-\pi,\pi)\setminus\{0\};\mathbb{C}), \quad U(-\pi) = U(\pi) \\ \qquad\qquad +\infty \quad \text{otherwise,} \end{cases} \tag{2.44}$$

where

$$U^+ = \lim_{y_1 \to 0^+} U(y_1) \quad \text{and} \quad U^- = \lim_{y_1 \to 0^-} U(y_1).$$

We point out that the condition $U \in W^{1,2}((-\pi,\pi)\setminus\{0\};\mathbb{C})$ in particular implies that U can be continuously defined on $[-\pi, 0]$ and on $[0, \pi]$, see e.g. [20].

We then will prove:

2.4 Theorem. *The function U^0 provided by Proposition 2.3 lies in the space $W^{1,2}((-\pi,\pi)\setminus\{0\};\mathbb{C})$ and minimizes G_0.*

Proof. The identification of U^0 as a minimizer of G_0 will be achieved through two claims. First we will show that

$$\liminf_{\varepsilon_j \to 0} G_{\varepsilon_j}(u^{\varepsilon_j}, \mathbf{A}^{\varepsilon_j}) \geq \pi G_0(U^0). \tag{2.45}$$

Then we will show that for any $V \in L^1((-\pi,\pi);\mathbb{C})$, there exists a sequence $\{v^\varepsilon\} \subset W^{1,2}(\Omega_\varepsilon;\mathbb{C})$ such that

$$\lim_{\varepsilon \to 0} G_\varepsilon(v^\varepsilon, \mathbf{A}^e) = \pi G_0(V). \tag{2.46}$$

Using the minimizing property of $\{(u^\varepsilon, \mathbf{A}^\varepsilon)\}$, we can then combine (2.45) and (2.46) to obtain $G_0(U^0) \leq G_0(V)$ as asserted. Of course, it will then also follow that $U^0 \in W^{1,2}((-\pi,\pi)\setminus\{0\};\mathbb{C})$.

Proof of Claim (2.45).

For this argument it will be convenient to work in a different gauge. Specifically, we take an applied magnetic potential to still satisfy the condition (2.7) but now also to satisfy

$$\mathbf{A}^e(x_1,x_2)\cdot(x_1,x_2,0) = 0 \quad \text{for } x_1^2 + x_2^2 = 1. \tag{2.47}$$

We can achieve this if we drop the divergence-free requirement and only insist that $\operatorname{div} \mathbf{A}^e = 0$ in $\{x : x_1^2 + x_2^2 < 1\}$ by replacing $\mathbf{A}^e$ with $\mathbf{A}^e - \nabla\phi$ where ϕ is any smooth extension to $\mathbb{R}^3$ of the solution to

$$\begin{aligned} \Delta\phi &= 0 \quad \text{in } x_1^2 + x_2^2 < 1, \\ \nabla\phi\cdot(x_1,x_2,0) &= \mathbf{A}^e\cdot(x_1,x_2,0) \quad \text{on} \quad x_1^2+x_2^2 = 1. \end{aligned}$$

Note that a solution ϕ exists in light of the divergence-free condition on $\mathbf{A}^e$ inside the disc, and the solution is independent of x_3 since the original $\mathbf{A}^e$ was as well. Consequently, $\mathbf{A}^e - \nabla\phi$ will, in particular, still satisfy (2.7).

We observe that as a consequence of (2.7) and (2.47), we have that

$$|\mathbf{A}^e(\cos y_1, \sin y_1, 0)| = |A_1^e(y_1)| \quad \text{for } -\pi \leq y_1 \leq \pi \tag{2.48}$$

(cf. (2.41)).

For the remainder of the proof, we then replace the original $\mathbf{A}^e$ by $\mathbf{A}^e - \nabla\phi$, $\mathbf{A}^\varepsilon$ by $\mathbf{A}^\varepsilon - \nabla\phi$ and u^ε by $u^\varepsilon e^{-i\phi}$. Of course, through gauge-invariance, we have that $G_\varepsilon(u^\varepsilon, \mathbf{A}^\varepsilon) = G_\varepsilon(u^\varepsilon e^{-i\phi}, \mathbf{A}^\varepsilon - \nabla\phi)$. We will not introduce new notation but through an abuse of notation, still denote these three quantities using their original designations. We should also remark that this change does not affect the estimates (2.15) and (2.16) measuring the L^∞-norm of $\mathbf{A}^\varepsilon - \mathbf{A}^e$ since this difference is unchanged by the gauge transformation.

Discarding the non-negative term $\int_{\mathbb{R}^3} |\nabla \times \mathbf{A}^{\varepsilon_j} - \mathbf{H}^e|^2 \, dx$, we begin with the decomposition of G_{ε_j} as

$$G_{\varepsilon_j}(u^{\varepsilon_j}, \mathbf{A}^{\varepsilon_j}) \geq \frac{1}{\varepsilon_j^2} \int_{\Omega_{\varepsilon_j}} |\nabla u^{\varepsilon_j}|^2 \, dx + \frac{1}{\varepsilon_j^2} \int_{\Omega_{\varepsilon_j}} i(\overline{u^{\varepsilon_j}} \nabla u^{\varepsilon_j} - u^{\varepsilon_j} \nabla \overline{u^{\varepsilon_j}}) \cdot \mathbf{A}^{\varepsilon_j} \, dx$$

$$+ \frac{1}{\varepsilon_j^2} \int_{\Omega_{\varepsilon_j}} |u^{\varepsilon_j}|^2 \, |\mathbf{A}^\varepsilon|^2 \, dx + \frac{1}{\varepsilon_j^2} \int_{\Omega_{\varepsilon_j}} \frac{\nu^2}{2} (|u^{\varepsilon_j}|^2 - \mu^2)^2 \, dx. \tag{2.49}$$

We will analyze the limit of each of the four terms above separately. Most crucial is the first term, where in light of (2.36) we have

$$\begin{aligned}
\liminf_{\varepsilon_j \to 0} \frac{1}{\varepsilon_j^2} \int_{\Omega_{\varepsilon_j}} |\nabla u^{\varepsilon_j}|^2 \, dx &\geq \liminf_{\varepsilon_j \to 0} \int_{\mathcal{C}} a_{\varepsilon_j}(y_1) \left| U^{\varepsilon_j}_{y_1} \right|^2 \, dy \\
&= \liminf_{\varepsilon_j \to 0} \int_{\mathcal{C}} \left| a_\varepsilon(y_1) U^{\varepsilon_j}_{y_1} \right|^2 \frac{1}{a_{\varepsilon_j}(y_1)} \, dy.
\end{aligned}$$

Then the conditions (2.40) and (2.43) allow for an appeal to [6], Theorem 6.2 to conclude that

$$\liminf_{\varepsilon_j \to 0} \frac{1}{\varepsilon_j^2} \int_{\Omega_{\varepsilon_j}} |\nabla u^{\varepsilon_j}|^2 \, dx \geq \pi \int_{(-\pi,\pi)\setminus\{0\}} \left| \frac{d\,U^0}{d\,y_1} \right|^2 \, dy_1 + \pi \left| (U^0)^+ - (U^0)^- \right|^2. \tag{2.50}$$

(See also [3] and [4].)

It remains to determine the limits of the last three integrals on the right-hand side of (2.49). To this end, we fix a positive number δ, and denote by $\mathcal{C}_\delta$ the set $\{y \in \mathcal{C} : |y_1| > \delta\}$. It then follows from (2.37) that $\{U^\varepsilon\}$ is bounded uniformly in $W^{1,2}(\mathcal{C}_\delta; \mathbb{C})$. Hence, we conclude from (2.38) and the Sobolev embedding theorem (see e.g. [20]) that

$$U^{\varepsilon_{j_k}} \rightharpoonup V^0 \quad \text{in} \quad W^{1,2}(\mathcal{C}_\delta; \mathbb{C}), \quad \text{and} \tag{2.51}$$

$$U^{\varepsilon_{j_k}} \to V^0 \quad \text{in} \quad L^q(\mathcal{C}_\delta; \mathbb{C}) \quad \text{for all} \quad q < 6 \tag{2.52}$$

for some $W^{1,2}(\mathcal{C}_\delta; \mathbb{C})$ function V^0 that is independent of y_2 and y_3. In light of Proposition 2.3, we may then identify $V^0 = U^0$ and observe that the convergences above must hold along the full sequence $\{\varepsilon_j\}$. We caution, however, that the $W^{1,2}(\mathcal{C}_\delta; \mathbb{C})$-norm of U^0 depends on δ so we should not in general expect that $U^0 \in W^{1,2}(\mathcal{C}; \mathbb{C})$.

Since the Jacobian of the mapping T_ε given by (2.4) is found to be $g_\varepsilon(y_1)^2(1 + g_\varepsilon(y_1) y_2)$, one uses (2.1)–(2.2), (2.12) and (2.52) to calculate the limit of the last term of (2.49) as follows:

$$
\begin{aligned}
&\lim_{\varepsilon_j\to 0}\int_{\Omega_{\varepsilon_j}}\frac{\nu^2}{2\varepsilon_j^2}(|u^{\varepsilon_j}|^2-\mu^2)^2\,dx=\\
&\lim_{\varepsilon_j\to 0}\int_{\mathcal{C}}\frac{\nu^2}{2\varepsilon_j^2}(|U^{\varepsilon_j}|^2-\mu^2)^2 g_{\varepsilon_j}(y_1)^2(1+g_{\varepsilon}(y_1)y_2)\,dy=\\
&\lim_{\varepsilon_j\to 0}\int_{\mathcal{C}_\delta}\frac{\nu^2}{2}(|U^{\varepsilon_j}|^2-\mu^2)^2\,dy+\mathcal{O}(\delta)=\\
&\int_{\mathcal{C}_\delta}\frac{\nu^2}{2}(\left|U^0\right|^2-\mu^2)^2\,dy+\mathcal{O}(\delta)=\\
&\pi\int_{\{\delta<|y_1|<\pi\}}\frac{\nu^2}{2}(\left|U^0\right|^2-\mu^2)^2\,dy_1+\mathcal{O}(\delta).
\end{aligned}
\tag{2.53}
$$

Similarly, in light of (2.48), (2.16) and (2.52), we have for the second to last integral in (2.49) that

$$
\begin{aligned}
&\lim_{\varepsilon_j\to 0}\frac{1}{\varepsilon_j^2}\int_{\Omega_{\varepsilon_j}}|u^{\varepsilon_j}|^2\left|\mathbf{A}^{\varepsilon_j}\right|^2\,dx=\\
&\lim_{\varepsilon_j\to 0}\frac{1}{\varepsilon_j^2}\int_{\mathcal{C}}|U^{\varepsilon_j}(y)|^2\left|\mathbf{A}^{\varepsilon_j}(y)\right|^2 g_{\varepsilon_j}(y_1)^2(1+g_{\varepsilon_j}(y_1)y_2)\,dy=\\
&\lim_{\varepsilon_j\to 0}\frac{1}{\varepsilon_j^2}\int_{\mathcal{C}}|U^{\varepsilon_j}(y)|^2\left|\mathbf{A}^{e}(\cos y_1,\sin y_1,0)\right|^2 g_{\varepsilon_j}(y_1)^2(1+g_{\varepsilon_j}(y_1)y_2)\,dy=\\
&\lim_{\varepsilon_j\to 0}\frac{1}{\varepsilon_j^2}\int_{\mathcal{C}}|U^{\varepsilon_j}(y)|^2\left|A^{e}_1(y_1)\right|^2 g_{\varepsilon_j}(y_1)^2(1+g_{\varepsilon_j}(y_1)y_2)\,dy=\\
&\lim_{\varepsilon_j\to 0}\int_{\mathcal{C}_\delta}|U^{\varepsilon_j}|^2\left|A^e_1\right|^2\,dy+\mathcal{O}(\delta)=\\
&\int_{\mathcal{C}_\delta}\left|U^0\right|^2\left|A^e_1\right|^2\,dy_1+\mathcal{O}(\delta)=\\
&\pi\int_{\{\delta<|y_1|<\pi\}}\left|U^0\right|^2\left|A^e_1\right|^2\,dy_1+\mathcal{O}(\delta).
\end{aligned}
\tag{2.54}
$$

Finally, we turn to the limit of the remaining integral in (2.49), namely,

$$
\liminf_{\varepsilon_j\to 0}\frac{1}{\varepsilon_j^2}\int_{\Omega_{\varepsilon_j}} i(\overline{u^{\varepsilon_j}}\nabla u^{\varepsilon_j}-u^{\varepsilon_j}\nabla\overline{u^{\varepsilon_j}})\cdot\mathbf{A}^{\varepsilon_j}\,dx. \tag{2.55}
$$

Another straight-forward but laborious calculation based on the change of variables $x = T_\varepsilon(y)$ leads to the fact that

$$
\begin{aligned}
&\int_{\Omega_{\varepsilon_j}} i(\overline{u^{\varepsilon_j}}\nabla u^{\varepsilon_j} - u^{\varepsilon_j}\nabla\overline{u^{\varepsilon_j}})\cdot \mathbf{A}^{\varepsilon_j}\,dx = \\
&\int_{\mathcal{C}} i\left(\overline{U^{\varepsilon_j}}U^{\varepsilon_j}_{y_1} - U^{\varepsilon_j}\overline{U^{\varepsilon_j}}_{y_1}\right)\left[(-\sin y_1, \cos y_1, 0)\cdot \mathbf{A}^{\varepsilon_j}(T_{\varepsilon_j})\right]\frac{g_{\varepsilon_j}(y_1)^2}{\varepsilon_j^2}\,dy + \\
&\int_{\mathcal{C}} i\left(\overline{U^{\varepsilon_j}}U^{\varepsilon_j}_{y_2} - U^{\varepsilon_j}\overline{U^{\varepsilon_j}}_{y_2}\right)\left[(\cos y_1, \sin y_1, 0)\cdot \mathbf{A}^{\varepsilon_j}(T_{\varepsilon_j})\right]\frac{g_{\varepsilon_j}(y_1)}{\varepsilon_j^2}\,dy + \\
&\int_{\mathcal{C}} i\left(\overline{U^{\varepsilon_j}}U^{\varepsilon_j}_{y_2} - U^{\varepsilon_j}\overline{U^{\varepsilon_j}}_{y_2}\right)\left[(\sin y_1, -\cos y_1, 0)\cdot \mathbf{A}^{\varepsilon_j}(T_{\varepsilon_j})\right]\frac{g_{\varepsilon_j}(y_1)g'_{\varepsilon_j}(y_1)y_2}{\varepsilon_j^2}\,dy + \\
&\int_{\mathcal{C}} i\left(\overline{U^{\varepsilon_j}}U^{\varepsilon_j}_{y_3} - U^{\varepsilon_j}\overline{U^{\varepsilon_j}}_{y_3}\right)\left[(0,0,1)\cdot \mathbf{A}^{\varepsilon_j}(T_{\varepsilon_j})\right]\frac{g_{\varepsilon_j}(y_1)(1+g_{\varepsilon_j}(y_1)y_2)}{\varepsilon_j^2}\,dy + \\
&\int_{\mathcal{C}} i\left(\overline{U^{\varepsilon_j}}U^{\varepsilon_j}_{y_3} - U^{\varepsilon_j}\overline{U^{\varepsilon_j}}_{y_3}\right)\left[(\sin y_1, -\cos y_1, 0)\cdot \mathbf{A}^{\varepsilon_j}(T_{\varepsilon_j})\right]\frac{g_{\varepsilon_j}(y_1)g'_{\varepsilon_j}(y_1)y_3}{\varepsilon_j^2}\,dy = \\
&I + II + III + IV + V.
\end{aligned}
\tag{2.56}
$$

In light of (2.47) and (2.16), we see that

$$
\begin{aligned}
\left|(\cos y_1, \sin y_1, 0)\cdot \mathbf{A}^{\varepsilon_j}(T_{\varepsilon_j}(y))\right| &\le |(\cos y_1, \sin y_1, 0)\cdot \mathbf{A}^e(\cos y_1, \sin y_1, 0)| \\
&+ \left|(\cos y_1, \sin y_1, 0)\cdot (\mathbf{A}^{\varepsilon_j}(T_{\varepsilon_j}(y)) - \mathbf{A}^e(\cos y_1, \sin y_1, 0)\right| \\
&\le \left\|\mathbf{A}^{\varepsilon_j}(T_{\varepsilon_j}(y)) - \mathbf{A}^e(\cos y_1, \sin y_1, 0)\right\|_{L^\infty(\mathcal{C})} \to 0 \text{ as } \varepsilon \to 0
\end{aligned}
$$

and that $|(0,0,1)\cdot \mathbf{A}^{\varepsilon_j}| \to 0$ as well, using (2.7). Hence we can apply Hölder's inequality, (2.12) and (2.37) to see that

$$
|II| + |IV| \le \mu\left(\frac{\|g_{\varepsilon_j}\|_{L^\infty(0,2\pi)}}{\varepsilon_j^2}\right)\cdot o(1)\cdot\left(\int_{\mathcal{C}}\left|U^{\varepsilon_j}_{y_2}\right|^2 + \left|U^{\varepsilon_j}_{y_3}\right|^2\,dy\right)^{1/2} \to 0
\tag{2.57}
$$

as $\varepsilon \to 0$.

Next, observe that since $|(\sin y_1, -\cos y_1, 0)\cdot \mathbf{A}^{\varepsilon_j}|$ is uniformly bounded in $\mathcal{C}$, and since $\left|g'_{\varepsilon_j}\right| = \mathcal{O}(\varepsilon^{1-p})$ (cf. (2.1)), through (2.37) one calculates that

$$
|III + V| \le \mu\left(\frac{\left\|g_{\varepsilon_j}g'_{\varepsilon_j}\right\|_{L^\infty((0,2\pi))}}{\varepsilon_j^2}\right)\left(\int_{\mathcal{C}}\left|U^{\varepsilon_j}_{y_2}\right|^2 + \left|U^{\varepsilon_j}_{y_3}\right|^2\,dy\right)^{1/2} \to 0 \tag{2.58}
$$

as well.

Consequently, we conclude from (2.56), (2.57) and (2.58) that

$$
\lim_{\varepsilon_j\to 0}\int_{\Omega_{\varepsilon_j}} i(\overline{u^{\varepsilon_j}}\nabla u^{\varepsilon_j} - u^{\varepsilon_j}\nabla\overline{u^{\varepsilon_j}})\cdot \mathbf{A}^{\varepsilon_j}\,dx = \lim_{\varepsilon_j\to 0} I. \tag{2.59}
$$

We now split the integral I into two integrals over the regions $\mathcal{C}_\delta$ and $\mathcal{C} \setminus \mathcal{C}_\delta$, which we label as I_1 and I_2 respectively.

We first treat the limit of I_2. Through (2.12) and (2.16), we find that

$$|I_2| \le C \int_{-\delta}^{\delta} \int_{\{y_2^2+y_3^2<1\}} \left|U_{y_1}^{\varepsilon_j}\right| \frac{g_{\varepsilon_j}(y_1)^2}{\varepsilon_j^2}\, dy_2\, dy_3\, dy_1$$

As a consequence of (2.1) and (2.37), we obtain that

$$\int_{-\delta}^{\delta} \int_{\{y_2^2+y_3^2<1\}} \left|U_{y_1}^{\varepsilon_j}\right| \frac{g_{\varepsilon_j}(y_1)^2}{\varepsilon_j^2}\, dy_2\, dy_3\, dy_1 \le$$
$$\left(\int_{-\delta}^{\delta} \int_{\{y_2^2+y_3^2<1\}} \frac{g_{\varepsilon_j}(y_1)^2}{\varepsilon_j^2} \left|U_{y_1}^{\varepsilon_j}\right|^2 dy_2\, dy_3\, dy_1\right)^{1/2} \left(\int_{-\delta}^{\delta} \int_{\{y_2^2+y_3^2<1\}} \frac{g_{\varepsilon_j}(y_1)^2}{\varepsilon_j^2}\, dy_2\, dy_3\, dy_1\right)^{1/2}$$
$$\le C \left(\int_{-\delta}^{\delta} \int_{\{y_2^2+y_3^2<1\}} 1\, dy_2\, dy_3\, dy_1\right)^{1/2} \le C\delta^{1/2}.$$

Hence, we conclude that

$$\lim_{\varepsilon_j \to 0} |I_2| \le C\delta^{1/2}. \tag{2.60}$$

Turning to I_1, through an appeal to (2.2), (2.16), (2.48), (2.51) and (2.52) we may compute

$$\lim_{\varepsilon_j \to 0} I_1 =$$
$$\lim_{\varepsilon_j \to 0} \int_{\{\delta<|y_1|<\pi\}} \int_{\{y_2^2+y_3^2<1\}} i \left(\overline{U^{\varepsilon_j}} U_{y_1}^{\varepsilon_j} - U^{\varepsilon_j} \overline{U^{\varepsilon_j}}_{y_1}\right) \left[(-\sin y_1, \cos y_1, 0) \cdot \mathbf{A}^{\varepsilon_j}\right] \frac{g_{\varepsilon_j}(y_1)^2}{\varepsilon_j^2}\, dy =$$
$$\lim_{\varepsilon_j \to 0} \int_{\{\delta<|y_1|<\pi\}} \int_{\{y_2^2+y_3^2<1\}} i \left(\overline{U^{\varepsilon_j}} U_{y_1}^{\varepsilon_j} - U^{\varepsilon_j} \overline{U^{\varepsilon_j}}_{y_1}\right) A_1^e\, dy =$$
$$\pi \int_{\{\delta<|y_1|<\pi\}} \int_{\{y_2^2+y_3^2<1\}} i \left(\overline{U^0} U_{y_1}^0 - U^0 \overline{U^0}_{y_1}\right) A_1^e\, dy_1.$$

Applying (2.57), (2.58), (2.60) and (2.61) to (2.56), we finally obtain

$$\lim_{\varepsilon_j \to 0} \int_{\Omega_{\varepsilon_j}} i(\overline{u^{\varepsilon_j}} \nabla u^{\varepsilon_j} - u^{\varepsilon_j} \nabla \overline{u^{\varepsilon_j}}) \cdot \mathbf{A}^{\varepsilon_j}\, dx = \pi \int_{\{\delta<|y_1|<\pi\}} i \left(\overline{U^0} U_{y_1}^0 - U^0 \overline{U^0}_{y_1}\right) A_1^e\, dy_1 + \mathcal{O}(\delta^{1/2}). \tag{2.61}$$

Combining (2.50), (2.53), (2.54) and (2.61) and letting $\delta \to 0$, we establish (2.45).

Proof of Claim (2.46)

We may assume $V \in W^{1,2}((-\pi,\pi) \setminus \{0\}; \mathbb{C})$ and $V(-\pi) = V(\pi)$, since otherwise the construction of a sequence satisfying (2.46) is trivial. In order to highlight the fact that, in general, V' will be singular at $y_1 = 0$, we denote by $h \in L^2((-\pi,\pi);\mathbb{C})$ the regular part of the derivative of V, that is, the part which is absolutely continuous with respect to Lebesgue measure. Hence,

$$\int_{-\pi}^{0} h(y_1)\,dy_1 = V^- - V(-\pi) \quad \text{and} \quad \int_{0}^{\pi} h(y_1)\,dy_1 = V(\pi) - V^+, \qquad (2.62)$$

where V^- and V^+ denote the left and right-hand limits of V at $y_1 = 0$ respectively.

We proceed to define a sequence $\{V^\varepsilon\} \subset W^{1,2}((-\pi,\pi);\mathbb{C})$ satisfying $V^\varepsilon(\pi) = V^\varepsilon(-\pi)$ and from this we will define the sequence $\{v^\varepsilon\} \subset W^{1,2}(\Omega_\varepsilon;\mathbb{C})$ verifying (2.46) by viewing the argument of V^ε as the polar angle in a cylindrical coordinate system on $\mathbb{R}^3$. Note that the polar angle is precisely the variable y_1 as defined in (2.4).

To this end, recall the definition of the sequence $\{\lambda_\varepsilon\}$ given in (2.42) and denote by β_ε the quantity

$$\beta_\varepsilon := \int_{-\pi}^{\pi} \lambda_\varepsilon(y_1)\,dy_1.$$

A routine calculation shows that

$$\int_{-\varepsilon^p}^{\varepsilon^p} \lambda_\varepsilon(y_1)\,dy_1 = 1 + \mathcal{O}(\varepsilon^p) \quad \text{while} \quad \int_{\{|y_1|>\varepsilon^p\}} \lambda_\varepsilon(y_1)\,dy_1 = \mathcal{O}(\varepsilon^2), \qquad (2.63)$$

so, in particular, we have $\beta_\varepsilon = 1 + \mathcal{O}(\varepsilon^p)$.

Now we are prepared to define the sequence $\{V^\varepsilon\} \subset W^{1,2}((-\pi,\pi);\mathbb{C})$ via the formula

$$V^\varepsilon(y_1) = \int_{-\pi}^{y_1} \left\{ h(s) + \frac{1}{\beta_\varepsilon}(V^+ - V^-)\lambda_\varepsilon(s) \right\} ds + V(-\pi). \qquad (2.64)$$

This construction follows that found in [4].

With the aid of (2.62) and the periodicity of V, one sees that $V^\varepsilon(-\pi) = V^\varepsilon(\pi)$ and with the aid of (2.63), one readily checks that

$$|V^\varepsilon - V| + |(V^\varepsilon)' - V'| \leq C\varepsilon^2 \quad \text{a.e. on} \quad \{|y_1| > \varepsilon^p\} \qquad (2.65)$$

while

$$|V^\varepsilon - V| \leq \left|V^+ - V^-\right| \quad \text{a.e. on} \quad \{|y_1| < \varepsilon^p\}. \qquad (2.66)$$

We also observe that $V^\varepsilon \to V$ in $L^1((-\pi,\pi);\mathbb{C})$ and that $(V^\varepsilon)' \overset{*}{\rightharpoonup} V'$ as measures on $(-\pi,\pi)$.

In light of the periodicity of V^ε we can now define the sequence $\{v^\varepsilon\} \subset W^{1,2}(\Omega_\varepsilon;\mathbb{C})$ through the relation $v^\varepsilon(x) = V^\varepsilon(\tan^{-1}(x_2/x_1)) = V^\varepsilon(y_1)$. We proceed to verify (2.46) by decomposing the energy $G_\varepsilon(v^\varepsilon, \mathbf{A}^e)$ and studying the limit of each term in the same manner as was done in the proof of Claim (2.45).

We write

$$G_\varepsilon(v^\varepsilon, \mathbf{A}^e) = \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} |\nabla v^\varepsilon|^2 \, dx + \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} i(\overline{v^\varepsilon}\nabla v^\varepsilon - v^\varepsilon \nabla \overline{v^\varepsilon}) \cdot \mathbf{A}^e \, dx$$
$$+\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} |v^\varepsilon|^2 \, |\mathbf{A}^e|^2 \, dx + \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} \frac{\nu^2}{2}(|v^\varepsilon|^2 - \mu^2)^2 \, dx \tag{2.67}$$

First note that through (2.24) one has

$$\begin{aligned}
\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} |\nabla v^\varepsilon|^2 \, dx &= \frac{1}{2\varepsilon^2}\int_{\mathcal{C}} a_{ik}\left(V^\varepsilon_{y_i}\overline{V^\varepsilon}_{y_k} + \overline{V^\varepsilon}_{y_i} V^\varepsilon_{y_k}\right) dy \\
&= \pi \int_{-\pi}^{\pi} \frac{g_\varepsilon(y_1)^2}{\varepsilon^2(1 + g_\varepsilon(y_1) y_2)} \left|\frac{d\, V^\varepsilon}{d\, y_1}\right|^2 dy_1 \\
&= \pi \int_{-\pi}^{\pi} a_\varepsilon(y_1) \left|\frac{d\, V^\varepsilon}{d\, y_1}\right|^2 dy_1 + o(1) \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Thus, from (2.42) and (2.64) we see that

$$\begin{aligned}
&\frac{1}{\pi}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} |\nabla v^\varepsilon|^2 \, dx = \\
&\lim_{\varepsilon\to 0}\int_{-\pi}^{\pi} a_\varepsilon(y_1)\left|h(y_1) + \frac{1}{\beta_\varepsilon}(V^+ - V^-)\lambda_\varepsilon(y_1)\right|^2 dy_1 = \\
&\lim_{\varepsilon\to 0}\int_{-\pi}^{\pi} \left|a_\varepsilon(y_1)h(y_1) + \frac{1}{\beta_\varepsilon}(V^+ - V^-)a_\varepsilon(y_1)\lambda_\varepsilon(y_1)\right|^2 \frac{1}{a_\varepsilon(y_1)}\, dy_1 = \\
&\lim_{\varepsilon\to 0}\int_{-\pi}^{\pi} \left|a_\varepsilon(y_1)h(y_1) + \frac{1}{\beta_\varepsilon}(V^+ - V^-)(1 - a_\varepsilon(y_1))\right|^2 (1 + \lambda_\varepsilon(y_1))\, dy_1 = \\
&\lim_{\varepsilon\to 0}\left\{\int_{-\pi}^{\pi} a_\varepsilon(y_1)^2 \, |h(y_1)|^2 \, dy_1 + \frac{1}{\beta_\varepsilon^2}\left|V^+ - V^-\right|^2 + \int_{-\pi}^{\pi}(1 - a_\varepsilon(y_1)) f_\varepsilon(y_1)\, dy_1\right\}
\end{aligned}$$

where in the last integral we have introduced the real-valued function f_ε so as to include all the remaining terms coming from expanding the square in the previous line. One readily checks that $\|f_\varepsilon\|_{L^\infty((-\pi,\pi))} \le C$ for some C independent of ε. Then, since $a_\varepsilon \to 1$ in $L^1((-\pi,\pi))$ and $\beta_\varepsilon \to 1$, we conclude that

$$\frac{1}{\pi}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} |\nabla v^\varepsilon|^2 \, dx = \int_{-\pi}^{\pi} |V'|^2 \, dy_1 + \left|V^+ - V^-\right|^2. \tag{2.68}$$

Turning to the next integral on the right-hand side of (2.67), we see from

(2.56) with u^ε replaced by v^ε and $\mathbf{A}^\varepsilon$ replaced by $\mathbf{A}^e$ that

$$\int_{\Omega_\varepsilon} i(\overline{v^\varepsilon}\nabla v^\varepsilon - v^\varepsilon\nabla\overline{v^\varepsilon})\cdot\mathbf{A}^e\,dx =$$
$$\int_{\mathcal{C}} i\left(\overline{V^\varepsilon}V^\varepsilon_{y_1} - V^\varepsilon\overline{V^\varepsilon}_{y_1}\right)\mathbf{A}^e(T_\varepsilon)\cdot(-\sin y_1,\cos y_1,0)\frac{g_\varepsilon(y_1)^2}{\varepsilon^2}\,dy =$$
$$\pi\int_{-\pi}^{\pi} i\left(\overline{V^\varepsilon}V^\varepsilon_{y_1} - V^\varepsilon\overline{V^\varepsilon}_{y_1}\right)A^e_1\frac{g_\varepsilon(y_1)^2}{\varepsilon^2}\,dy_1 + \mathcal{O}(\varepsilon)$$

Here we have used the fact that $|\mathbf{A}^e(T_\varepsilon)\cdot(-\sin y_1,\cos y_1,0)| = A^e_1 + \mathcal{O}(\varepsilon)$.

Hence, from (2.65) we have

$$\frac{1}{\pi}\int_{\Omega_\varepsilon} i(\overline{v^\varepsilon}\nabla v^\varepsilon - v^\varepsilon\nabla\overline{v^\varepsilon})\cdot\mathbf{A}^e\,dx =$$
$$\int_{\{|y_1|>\varepsilon^p\}} i\left(\overline{V^\varepsilon}V^\varepsilon_{y_1} - V^\varepsilon\overline{V^\varepsilon}_{y_1}\right)A^e_1\frac{g_\varepsilon(y_1)^2}{\varepsilon^2}\,dy_1 +$$
$$\int_{\{|y_1|<\varepsilon^p\}} \bullet \quad dy_1 \quad + \quad \mathcal{O}(\varepsilon) \quad =$$
$$\int_{\{|y_1|>\varepsilon^p\}} i\left(\overline{V}V_{y_1} - V\overline{V}_{y_1}\right)A^e_1\,dy_1 +$$
$$\int_{\{|y_1|<\varepsilon^p\}} i\left(\overline{V^\varepsilon}V^\varepsilon_{y_1} - V^\varepsilon\overline{V^\varepsilon}_{y_1}\right)A^e_1\frac{g_\varepsilon(y_1)^2}{\varepsilon^2}\,dy_1 + \mathcal{O}(\varepsilon). \qquad (2.69)$$

Now as a consequence of (2.35), (2.42) and (2.64) we can estimate this last term as follows:

$$\left|\int_{\{|y_1|<\varepsilon^p\}} i\left(\overline{V^\varepsilon}V^\varepsilon_{y_1} - V^\varepsilon\overline{V^\varepsilon}_{y_1}\right)A^e_1\frac{g_\varepsilon(y_1)^2}{\varepsilon^2}\,dy_1\right| \le$$
$$C\int_{\{|y_1|<\varepsilon^p\}} a_\varepsilon(y_1)\left|V^\varepsilon_{y_1}\right|\,dy_1 \le$$
$$C\int_{\{|y_1|<\varepsilon^p\}} a_\varepsilon(y_1)\left|h(y_1)\right|\,dy_1 + C\left|V^+ - V^-\right|\int_{\{|y_1|<\varepsilon^p\}} a_\varepsilon(y_1)\frac{1}{\beta_\varepsilon}\left|\lambda_\varepsilon(y_1)\right|\,dy_1 =$$
$$C\int_{\{|y_1|<\varepsilon^p\}} a_\varepsilon(y_1)\left|h(y_1)\right|\,dy_1 + C\left|V^+ - V^-\right|\int_{\{|y_1|<\varepsilon^p\}} \frac{1}{\beta_\varepsilon}\left|1 - a_\varepsilon(y_1)\right|\,dy_1 = \mathcal{O}(\varepsilon^p). \qquad (2.70)$$

Combining (2.69) and (2.70) and passing to the limit as $\varepsilon\to 0$ we conclude that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} i(\overline{v^\varepsilon}\nabla v^\varepsilon - v^\varepsilon\nabla\overline{v^\varepsilon})\cdot\mathbf{A}^e\,dx = \pi\int_{-\pi}^{\pi} i\left(\overline{V}V_{y_1} - V\overline{V}_{y_1}\right)A^e_1\,dy_1. \qquad (2.71)$$

Finally, one finds that the limits of the last two integrals in (2.67) are given by

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon}\left\{|v^\varepsilon|^2|\mathbf{A}^e|^2+\frac{\nu^2}{2}(|v^\varepsilon|^2-\mu^2)^2\right\}dx =$$
$$\pi\int_{-\pi}^{\pi}\left\{|V|^2|A_1^e|^2+\frac{\nu^2}{2}(|V|^2-\mu^2)^2\right\}dy_1 \qquad (2.72)$$

as an easy consequence of (2.65) and (2.66) using a calculation similar to that carried out in (2.53) and (2.54).

Claim (2.46) follows by combining (2.68), (2.71) and (2.72). □

3 Discussion

In this section we discuss some physical implications of the one-dimensional model (2.44). We start by observing that adding a suitable parameter to the geometric characterization of the constriction enables us to control the relative magnitude of the different terms in the limit functional G_0 (2.44). For example, if we replace (2.1)-(2.2) by

$$g_\varepsilon(y_1) = (\varepsilon^{1-p}-2\sqrt{b}\varepsilon)|y_1|+2\sqrt{b}\varepsilon^{1+p} \quad \text{for} \quad 0\le|y_1|\le\varepsilon^p, \qquad (3.1)$$
$$g_\varepsilon(y_1) = \varepsilon \quad \text{for} \quad \varepsilon^p\le|y_1|\le\pi, \qquad (3.2)$$

where b is a fixed positive parameter, then we obtain the modified limit functional

$$G_0(U) = \begin{cases}\int_{(-\pi,\pi)\setminus\{0\}}\left(\left|(i\frac{d}{dy_1}+A_1^e)U\right|^2+\frac{\nu^2}{2}(|U|^2-\mu^2)^2\right)dy_1+b\,|U^+-U^-|^2 \\ \qquad \text{if} \quad U\in W^{1,2}((-\pi,\pi)\setminus\{0\};\mathbb{C}), \quad U(-\pi)=U(\pi), \\ \qquad +\infty \quad \text{otherwise}\end{cases} \qquad (3.3)$$

To simplify the presentation in this section, we replace A_1^e by A and use θ to denote the variable y_1. Equating the first variation of G_0 to zero we obtain the following jump condition at the weak point ($\theta=0$):

$$\left(\frac{d}{d\theta}-iA\right)U^+=\left(\frac{d}{d\theta}-iA\right)U^-=b\left(U^+-U^-\right). \qquad (3.4)$$

Multiplying both sides of (3.4) by $\overline{U^+}$ and taking the imaginary part of the obtained complex-valued expression we find

$$J^+ := Im\left((\frac{dU^+}{d\theta}-iAU^+)\overline{U^+}\right)=-b\,Im\left(U^-\overline{U^+}\right). \qquad (3.5)$$

The object J^+ is the supercurrent immediately after ($\theta = 0^+$) the weak link. Similarly we multiply both sides of (3.4) by $\overline{U^-}$ and obtain

$$J^- := Im\left((\frac{dU^-}{d\theta} - iAU^-)\overline{U^-}\right) = b\, Im\left(U^+\overline{U^-}\right). \tag{3.6}$$

We therefore deduce that $J^+ = J^-$, i.e. the current is conserved across the junction.

In contrast to the continuity of the supercurrent, the order parameter U and its derivative are *not* continuous across the junction. To write the jump in the amplitude's derivative we express U in a polar form $U = \rho e^{i\phi}$. Taking now the real part of (3.4) multiplied by U^+, we obtain after a quick calculation

$$\frac{d\rho^+}{d\theta} = b\left(\rho^+ - \rho^- \cos(\phi^+ - \phi^-)\right), \tag{3.7}$$

where as before, the superscripts $\cdot^+$ and $\cdot^-$ denote evaluation on either side of $\theta = 0$. Similarly we find

$$\frac{d\rho^-}{d\theta} = b\left(\rho^+ \cos(\phi^+ - \phi^-) - \rho^-\right). \tag{3.8}$$

Applying the polar form of U to the current formula (3.5) we get

$$J^+ = J^- = b\rho^+\rho^- \sin(\phi^+ - \phi^-). \tag{3.9}$$

This is the celebrated Josephson formula (cf. (1.1)). Moreover we derived an explicit expression for J_M. It is important to observe that in contrast to most models of Josephson junctions (see e.g. [2] or [19]), in the model being presented here, the order parameter is *not* continuous at the junction. We point out that our equations form a special case of an ad hoc model due to deGennes (cf. equation (7.66) of [7]). In this model, deGennes postulates that the pair $\left(U^+, (\frac{d}{d\theta} - iA)U^+\right)$ is linearly related to the pair $\left(U^-, (\frac{d}{d\theta} - iA)U^-\right)$ through multiplication by a 2×2 matrix he denotes by M. To make the comparison precise, we recast our connection formula across the junction in the form

$$U^+ = U^- + \frac{1}{b}\left(\frac{d}{d\theta} - iA\right)U^-, \tag{3.10}$$

$$\left(\frac{d}{d\theta} - iA\right)U^+ = \left(\frac{d}{d\theta} - iA\right)U^-. \tag{3.11}$$

Using the notation of [7], we can then identify $M_{11} = M_{22} = 1$, $M_{12} = \frac{1}{b}$, $M_{21} = 0$.

The collapse of the three-dimensional domain Ω_ε onto a one-dimensional wire is analogous to similar limits computed in e.g. [5] and [16]. The new feature here is the strong effect of inhomogeneities in the wire. To demonstrate the effect of the term proportional to b in (3.3), we consider the particular problem of phase transition in thin wires with constrictions. It is known that the critical temperature in one-dimensional rings depends upon the magnetic flux threading the hole bounded by the ring. This was discovered experimentally by Little and Parks about 40 years ago, and has been well established theoretically since then, [15]. We proceed to compute this dependency using the model given by G_0. The critical temperature is associated with μ as we explained below (2.5), while μ, in turn, is determined by the eigenvalue problem that is obtained through linearizing the Euler- Lagrange equation associated with G_0 about the normal state $U \equiv 0$. We refer to [11] for a justification of this statement. The linearized equation is

$$\left(\frac{d}{d\theta} - iA\right)^2 U + (\nu\mu)^2 U = 0, \tag{3.12}$$

together with the jump conditions (3.10)-(3.11) enforced at $\theta = 0$ and periodic boundary conditions at the endpoints $\theta = \pm\pi$.

Proceeding as in [11], we make a gauge transformation $U(\theta) \to U(\theta)e^{i\int_0^\theta A}$. This simplifies the eigenvalue problem (3.12) into

$$\frac{d^2}{d\theta^2}U + (\nu\mu)^2 U = 0 \quad \text{for } -\pi < \theta < 0 \text{ and } 0 < \theta < \pi, \tag{3.13}$$

coupled with the boundary conditions

$$\begin{aligned}
&U(-\pi) = U(\pi)e^{i\Phi}, \quad U'(-\pi) = U'(\pi)e^{i\Phi},\\
&U^+ = U^- + \frac{1}{b}(U')^- \quad \text{at } \theta = 0,\\
&(U')^+ = (U')^- \quad \text{at } \theta = 0.
\end{aligned}$$

where $\Phi := \int_{-\pi}^{\pi} A\, d\theta$ is the magnetic flux through the hole bounded by the ring. After writing U in terms of sines and cosines, a simple calculation yields the following transcendental equation for μ:

$$\cos(2\pi\nu\mu) - \frac{\nu\mu}{2b}\sin(2\pi\nu\mu) = \cos\Phi. \tag{3.14}$$

Notice that in the limit $b \to \infty$, the order parameter is continuous, and equation (3.14) reduces to the clean ring limit ([11]).

We wish to compare the equation (3.14) to the analogous one obtained in [11]. In [11], a junction is modeled not through a constriction but rather through a modification of the $1-$d Ginzburg-Landau energy in a thin region corresponding to angular values $0 \leq \theta \leq d$ where $d << 1$. Specifically, one replaces the potential term $\frac{\nu^2}{2}\left(|u|^2-\mu^2\right)^2$ in (2.5) for these θ-values with the term $\frac{\alpha}{d}|u|^2$ where $\alpha > 0$ is a parameter related to the strength of the junction. Then carrying out an asymptotic analysis of the normal/superconducting phase transition in the small d limit, one finds that the corresponding eigenvalue μ solves the transcendental equation

$$\cos 2\pi\mu + \frac{\alpha}{2\mu}\sin 2\pi\mu = \cos\Phi. \tag{3.15}$$

A comparison of the two phase transition curves corresponding to (3.14) and (3.15) with $\nu = b = \alpha = 1$ can be found in Figure 2. We note that qualitatively the two transition curves coming from the two different models are quite similar though the transition temperature for the model from [11] is lower (higher μ). Another distinction is that in the model from [11], the curve does not pass through the origin. That is, even for zero magnetic flux through the ring, the transition temperature in that model is lower than the critical temperature associated with the normal/superconducting phase transition in the absence of any applied magnetic field for a ring without normal inclusions. This is not the case in the constricted model.

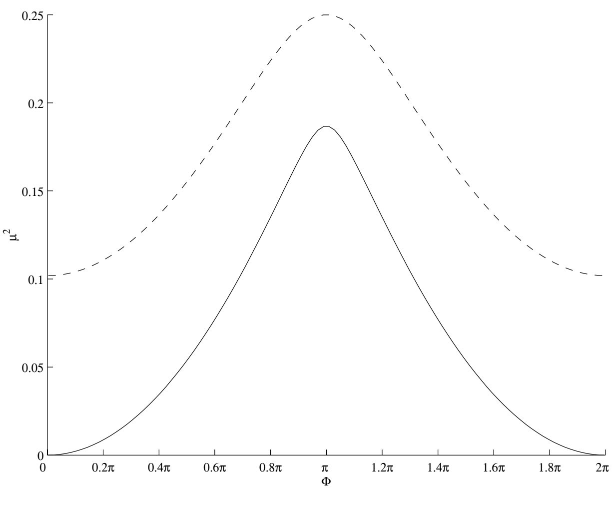

Figure 2: Comparision of phase transition curves μ^2 versus Φ for the constricted model (solid) and the modified GL model (dashed).

Returning to the general question of Josephson junctions, we comment that deGennes did not distinguish between different kinds of junctions. Together with earlier investigations ([11] and [18]) we are able now to identify two distinguished kinds of junctions. The first kind is a classical SNS junction, where a thin normal layer separates two bulk superconducting samples. Under appropriate scaling it can be shown then that the current is proportional to the sin of the magnetic flux threading through the hole bounded by the wire, but the amplitude of the order parameter is continuous, [11]. The second class consists of a geometric weak link (our constriction). Here the current is a periodic function of the phase jump, but the topological constraint implied by the ring is not as strong as in the first class in that the phase is no longer required to jump by a multiple of 2π along the ring.

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