KOLMOGOROV THEORY VIA FINITE-TIME AVERAGES

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Oscar Manley passed away in the Fall of 2001. He was a remarkable scientist, with great physical intuition and mathematical acumen. His influence on his coauthors and other collaborators will be permanent. This article which is our last joint article with him is the direct result of his repeated request to work with the more physical concept of finite-time averages.

Abstract. Several relations from the Kolmogorov theory of fully-developed three-dimensional turbulence are rigorously established for finite-time averages over Leray-Hopf weak solutions of the Navier-Stokes equations. The Navier-Stokes equations are considered with periodic boundary conditions and an external forcing term. The main parameter is the Grashof number associated with the forcing term. The relations rigorously proved in this article include estimates for the energy dissipation rate, the Kolmogorov wavenumber, the Taylor wavenumber, the Reynolds number, and the energy cascade process. For some estimates the averaging time depends on the macroscale wavenumber and the kinematic viscosity alone, while for others such as the Kolmogorov energy dissipation law and the energy cascade, the estimates depend also on the Grashof number. As compared with earlier works by some of the authors the more physical concept of finite-time average is replacing the concept of infinite-time average used before.

1. Introduction

The conventional statistical theory of turbulence is largely concerned with relations between mean quantities, the only ones that can be measured in a turbulent flow. The question of the definition of the mean values, however, is a delicate one as remarked by Monin and Yaglom [30]. In practice time and space averages are the most generally used while in theory the use of averages with respect to a large ensemble of flows avoid some analytical difficulties. The relations between the different forms of averages are either based on an ergodic hypothesis (see e.g. [24]) or through a mathematically esoteric notion of generalized limit (see e.g. [2, 17]). In any

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case, it is important to have relations between the various mean physical quantities pertaining to a turbulent flow. Rigorous estimates have been obtained in terms of ensemble averages in [17, 18, 19]. In this article we derive similar estimates directly for finite-time averages without resorting to any esoteric relation with ensemble averages. Finite-time averages have been considered in the two-dimensional case in [14] with the corresponding two-dimensional estimates for ensemble averages given in [15]. We are here only concerned with the three-dimensional case. For simplicity, we restrict the exposition to the periodic case with zero space average and with the same period in all directions, but the results apply with some minor adjustments to the case of different periods in each direction and to the no-slip case on a bounded, smooth domain.

Our aim in this article is to obtain in a mathematically rigorous way starting from the (Leray-Hopf) weak solutions of the Navier-Stokes equations some estimates which are usually obtained in the conventional theory of turbulence in an heuristic manner and without any explicit reference to the Navier-Stokes (NS) equations. In the conventional statistical theory of turbulence important quantities are the mean rate of energy dissipation per unit mass and unit time \( \epsilon \), the mean kinetic energy per unit mass \( e \), the Kolmogorov wavenumber \( \kappa_\epsilon = (\epsilon/\nu^3)^{1/4} \), where \( \nu \) is the kinetic viscosity, the Taylor wavenumber \( \kappa_\tau = (\epsilon/2\nu e)^{1/2} \), the root-mean-square velocity \( U = \sqrt{2e} \), the Reynolds number \( Re = U/\nu \kappa_0 \), based on the root-mean-square velocity and some large-scale wavenumber \( \kappa_0 \), and the Taylor microscale Reynolds number \( Re_\tau = U/\nu \kappa_\tau \) (see e.g. [1, 12, 21, 23, 26, 27, 33]).

We consider the corresponding quantities based on a finite-time average of a given weak solution of the Navier-Stokes equations. For a given function \( \Phi = \Phi(u(t)) \) of the velocity field \( u = u(t) \) of a given weak solution \( u \) defined for all \( t \geq 0 \), we consider the time average

\[
\langle \Phi(u) \rangle = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \Phi(u(t)) \, dt,
\]

over a given interval \([t_1, t_2] \subset [0, \infty)\). The initial time \( t_1 \) is always taken to be larger than the time at which the solution enters into and remain in a certain absorbing set in the space of finite kinetic energy velocity fields. For the concept of absorbing set and the concepts related to dynamical system theory the reader is refereed to e.g. [17, 37]. The final time \( t_2 \) is taken sufficiently large depending on some physical quantities which are made explicit in the derivations. The function \( \Phi \) varies with the physical quantity we wish to observe. The corresponding finite-time averaged versions of \( \epsilon, e, \kappa_\epsilon, \kappa_\tau, U, Re, \) and \( Re_\tau \) are denoted, respectively, by \( \tilde{\epsilon}, \tilde{e}, \tilde{\kappa}_\epsilon, \tilde{\kappa}_\tau, \tilde{U}, \tilde{Re}, \) and \( \tilde{Re}_\tau \).

In the conventional theory of turbulence it is argued on heuristic grounds that for high-Reynolds-number flows \( \epsilon \sim \kappa_0 U^3 \), which is known as the Kolmogorov energy dissipation law. In this relation, \( \kappa_0 \) is a large-scale wavenumber, which we take to be the square-root of the first eigenvalue of the Stokes operator (of the order of the
inverse of the period of the flow, in the periodic case with the same period in all directions). An important remark is that with this relation the energy dissipation rate becomes independent of the viscosity for high Reynolds numbers. The heuristic argument for this law is simple: $U$ being a characteristic macroscale velocity of the flow, the associated kinetic energy per unit mass is $e = U^2/2$. A characteristic time for the macroscales is $\tau_0 = (\kappa_0 U)^{-1}$, which is called the macroscale circulation time. In statistical equilibrium, the bulk of the energy is in the large scales so that the mean kinetic energy is dissipated over the macroscale circulation time, i.e. $e = \epsilon \tau_0$. Hence, $\epsilon$, the energy dissipation rate per unit mass and unit time should be $e/\tau_0$, from where we infer that $\epsilon \sim \kappa_0 U^3$. In Section 5 of this article, we prove rigorously that $\epsilon \leq C_1 \kappa_0 U^3$, for a sufficiently long averaging interval, and for a suitable parameter $C_1$ which remains bounded when $\text{Re}$ increases. Analogous results have been obtained previously for shear flows and infinite-time averages by Constantin and Doering [4, 9].

Subsequent analogous results in various contexts and still for infinite-time averages have been obtained in [3, 10, 11, 13, 19, 22, 28, 29, 31, 38, 39]. See also [5, 14, 20, 34, 35] for related results.

The Kolmogorov energy dissipation law leads to the relations $\kappa_\epsilon \sim \kappa_0 \text{Re}^{3/4}$ and $\kappa_\tau \sim \kappa_0 \text{Re}^{1/2}$, from which we deduce also $\kappa_\tau \sim \kappa_0^{1/3} \kappa_\epsilon^{2/3}$, $\kappa_\tau \sim \text{Re}^{-1/4} \kappa_\epsilon$, and $\text{Re} \sim \text{Re}_t^2$. In Section 6 of this article, we prove the rigorous estimates $\tilde{\kappa}_\epsilon \leq C_1^{1/4} \kappa_0 \text{Re}^{3/4}$, $\tilde{\kappa}_\tau \leq C_1^{1/2} \kappa_0 \text{Re}^{1/2}$, $\tilde{\kappa}_\tau \leq C_1^{1/3} \kappa_0^{1/3} \kappa_\epsilon^{2/3}$, $\tilde{\kappa}_\tau \leq C_1^{1/4} \text{Re}^{-1/4} \kappa_\epsilon$, and $\text{Re} \leq C_1 \text{Re}_t^2$, with the same $C_1$ as before.

A number of other estimates involve the so-called Grashof number $G^*$, which is a parameter associated with the strength of the forcing term (see (3.5)). It is well-known that the flow is simple when $G^*$ is small (a unique stable stationary solution to which all trajectories converge, see e.g. [36, Chapter 10]); we will assume in general that $G^*$ is sufficiently large (see Remark 4.3). Our current results include the following ones: we give lower and upper bounds for the energy dissipation rate in terms of the Grashof number, namely, $c_6^{-1} \nu^3 \kappa_0^4 (G^* - 1) \leq \tilde{\epsilon}$ and $\epsilon \leq (4\pi)^{-1} \nu^3 \kappa_0^4 G^2$, where $c_6$ is a nondimensional constant. In these estimates, the averaging time is of the order of $\nu^{-1} \kappa_0^{-2}$. This is in contrast with the averaging time sufficient for the previous estimates, which are longer and depend also on $G^*$. We also give lower and upper bounds for the mean energy $\tilde{\epsilon}$ in terms of $G^*$.

Furthermore, in Section 7 we obtain a more involved estimate for $\tilde{\kappa}_\tau$ which depends on $\tilde{\kappa}_\tau$. This estimate may be interpreted together with the heuristic relation $\kappa_\tau \sim \kappa_0^{1/3} \kappa_\epsilon^{2/3}$. If the corresponding relation holds for the time averages, i.e. if $\tilde{\kappa}_\tau \sim \kappa_0^{1/3} \kappa_\epsilon^{2/3}$, then the number of degrees of freedom of the flow can be estimated by (see Remark 7.1)

$$\left( \frac{\tilde{\kappa}_\tau}{\kappa_0} \right)^3 \lesssim G^{*9/8},$$
which improves the known estimate

$$\left( \frac{\tilde{\kappa}_\epsilon}{\kappa_0} \right)^3 \lesssim G^{3/2},$$

valid in the general case in which \( \tilde{\kappa}_\tau \sim \kappa_0^{1/3} \kappa_\epsilon^{2/3} \) is not assumed (see [17, Chapter III, Sec. 3]).

Finally, our last estimate concerns the energy cascade process. This process is one of the most remarkable universal features of turbulence. Through it, within a large range of scales (called the inertial range) going much lower than the energy injection scales, the energy is transferred to smaller scales at a nearly constant rate close to the energy dissipation rate. The injected energy is transported in that way to the very small scales in which it is dissipated by the viscous effects. By working in spectral space, with the help of the Fourier decomposition of vector fields, we associate length scales with wavenumbers. In this association the energy cascade mechanism is interpreted as transferring kinetic energy to higher wavenumbers, with viscosity acting at very high wavenumbers. In this context we show that if the Taylor wavenumber \( \tilde{\kappa}_\tau \) is large enough compared with the wavenumbers in which the forcing term acts then the energy cascade holds, with energy being transferred to higher wavenumbers at a nearly constant rate \( \tilde{\epsilon} \). This is proved for what we call the restricted mean energy flux per unit mass \( \kappa_L^3 \langle \epsilon^*_\kappa(u) \rangle_e \) through wavenumber \( \kappa \), where \( \kappa_L \) is the inverse of the period of the flow. This quantity is the net flux of energy per unit mass transferred per unit time from all the scales to the scales larger than or equal to \( \kappa \) (see Remark 8.1). This restricted energy flux takes into account a possible loss of energy due to the lack of regularity of weak solutions. This lack of regularity implies an energy inequality, which is not sufficient for the estimates. Working with the restricted energy flux we recover an energy equation and are able to proceed with the proper estimates, proving that \( \kappa_L^3 \langle \epsilon^*_\kappa(u) \rangle_e \approx \tilde{\epsilon} \) within the range \( \tilde{\kappa}^2 \leq \kappa^2 \ll \tilde{\kappa}_\tau^2 \), for sufficiently large averaging times, and where \( \tilde{\kappa} \) is an upper bound for the active wavenumber modes in the forcing term.

The results in this article can be seen as an improved version of the results announced in [18, 19], in two different ways: On the one hand, we consider here finite time averages, and by letting the interval of time go to infinity, the present results yield those of [18, 19] using infinite time averages; on the other hand a number of results in this article were not included in [18, 19], even for the infinite time interval.

2. Mathematical setting

We consider the three-dimensional incompressible Navier-Stokes equations with periodic boundary conditions. We assume the flow is periodic with the same period \( L \) in each spatial direction \( x_i, i = 1, 2, 3 \), and we set \( \Omega = (0, L)^3 \). We also assume
that the averages of the flow and of the forcing term over the period $\Omega$ vanish, i.e.

$$
\int_{\Omega} u(x) \, dx = 0, \quad \int_{\Omega} f(x) \, dx = 0,
$$

where $u = (u_1, u_2, u_3)$ denotes the velocity vector, and $x = (x_1, x_2, x_3)$, the space variable.

For the mathematical formulation of the incompressible Navier-Stokes equations and the results mentioned henceforth we refer the reader to \[6, 17, 25, 35, 37\].

The incompressible Navier-Stokes equations in $\Omega$ read as

$$
\frac{\partial u}{\partial t} - \nu \nabla u + (u \cdot \nabla)u + \nabla p = f, \quad \nabla \cdot u = 0,
$$

with the appropriate boundary and initial value conditions. The equation is in the classical form upon division by the mass density $\rho_0$, so that the term $f$ represents the mass density of volume forces applied to the fluid; the parameter $\nu > 0$ is the kinematic viscosity; and $p$ is the kinematic pressure.

We obtain a functional equation formulation for the time-dependent velocity field $u = u(t)$ of the form:

$$
\frac{du}{dt} + \nu A u + B(u, u) = f. \quad (2.1)
$$

This equation holds in an appropriate function space which depends on the boundary conditions. In the periodic case we consider the space of test functions

$$
\mathcal{V} = \left\{ u = w|_{\Omega}; \quad w \in \mathcal{C}^\infty(\mathbb{R}^3), \quad \nabla \cdot w = 0, \quad \int_{\Omega} w(x) \, dx = 0, \quad w(x) \text{ is periodic with period } L \text{ in each direction } x_i \right\}.
$$

The space $H$ is defined as the completion of $\mathcal{V}$ under the $L^2(\Omega)^3$ norm. The space $V$ is the completion of $\mathcal{V}$ under the $H^1(\Omega)^3$ norm. We identify $H$ with its dual and consider the dual space $V'$ of $V$, so that $V \subset H \subset V'$, the injections being continuous, each space dense in the following one. We denote by $H_w$ the space $H$ endowed with its weak topology.

We denote the inner products in $H$ and $V$ respectively by

$$
(u, v) = \int_{\Omega} u(x) \cdot v(x) \, dx, \quad (u, v) = \int_{\Omega} \sum_{i=1,3} \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} \, dx,
$$

and the associated norms by $|u|_0 = (u, u)^{1/2}$, $\|u\| = (\langle u, u \rangle)^{1/2}$.

We denote by $P_{\text{LH}}$ the (Leray-Helmhotz) orthogonal projector in $L^2(\Omega)^3$ onto the subspace $H$. The operator $A$ in (2.1) is the Stokes operator given by $Au = -P_{\text{LH}} \Delta u$. The term $B(u, v) = P_{\text{LH}}((u \cdot \nabla)v)$ is a bilinear operator associated with the inertial term. The Stokes operator is a positive self-adjoint operator on $H$, and we denote by $\lambda_1 > 0$ its first eigenvalue.
Taking the inner product in $H$ of the bilinear term with a third variable yields a trilinear term
\[ b(u, v, w) = (B(u, v), w), \]
which is defined for $u, v, w$ in $V$. An important relation for the trilinear term is the orthogonality property
\[ b(u, v, v) = 0, \]
valid for $u, v \in V$. From this relation follows the anti-symmetric property
\[ b(u, v, w) = -b(u, w, v), \]
for $u, v, w \in V$.

Instead of assuming as usual that $f \in H$ we assume that $f \in V$. This further regularity assumption will be needed for estimates such as the energy dissipation law.

We define a Leray-Hopf weak solution on a time interval $I \subset \mathbb{R}$ to be a function $u = u(t)$ on $I$ with values in $H$ and satisfying the following properties:

(i) $u \in L^\infty_{\text{loc}}(I; H) \cap L^2_{\text{loc}}(I; V)$;
(ii) $\partial u/\partial t \in L^{4/3}_{\text{loc}}(I; V')$;
(iii) $u \in C(I; H_w)$ (i.e. $u(\cdot)$ is weakly continuous from $I$ into $H$);
(iv) $u$ satisfies the functional equation $\text{(2.1)}$ almost everywhere on $I$;
(v) $u$ satisfies the following energy inequality in the distribution sense on $I$:
\[
\frac{1}{2} \frac{d}{dt} |u(t)|_0^2 + \nu \|u(t)\|^2 \leq (f, u(t)).
\]
\[
\text{(2.2)}
\]
(vi) If $I$ is closed at the left end point and its left end point, denoted $t_0$, is finite, then the solution is continuous in $H$ at $t_0$ from the right, i.e. $u(t) \to u(t_0)$ in $H$, as $t \to t_0^+$.

From now on, for notational simplicity, a weak solution will always mean a Leray-Hopf weak solution.

Given a weak solution on an arbitrary interval $I$, it follows that
\[
|u(t)|_0^2 \leq |u(t')|_0^2 e^{-\nu \lambda_1(t-t')} + \frac{1}{\nu^2 \lambda_1^2} |f|^2_0 \left(1 - e^{-\nu \lambda_1(t-t')}\right),
\]
\[
\text{(2.3)}
\]
for all $t$ in $I$ and almost all $t'$ in $I$ with $t' < t$. In the absence of regularity the estimate (2.3) may not be valid for all $t'$ in $I$. The allowed times $t'$ are the Lebesgue points of the function $t \mapsto |u(t)|_0^2$. In the case of a weak solution on an interval of the form $[t_0, t_1]$, the point $t_0$ is a point of continuity of $t \mapsto |u(t)|_0^2$, hence a Lebesgue point, so that the estimate above is also valid for the initial time $t' = t_0$.

Another classical estimate obtained from the energy inequality is
\[
|u(t)|_0^2 + \nu \int_{t'}^t \|u(s)\|^2 \, ds \leq |u(t')|_0^2 + \frac{1}{\nu \lambda_1} |f|^2_0 (t - t'),
\]
\[
\text{(2.4)}
\]
for all $t$ in $I$ and almost all $t'$ in $I$ with $t' < t$, with the set of allowed times $t'$ consisting again of the Lebesgue points of the function $t \mapsto |u(t)|_0^2$. 
A *strong solution* on an arbitrary interval \( I \) is defined as a weak solution on \( I \) satisfying

(vii) \( u \in C(I; V) \).

Any strong solution satisfies the energy equation

\[
\frac{1}{2} \frac{d}{dt} |u(t)|_0^2 + v \| u(t) \|^2 = (f, u(t))
\]  

(2.5)

in the distribution sense on its interval of definition.

It is well established that given any initial time \( t_0 \) and any initial condition \( u_0 \) in \( H \), there exists at least one global weak solution on \([t_0, \infty)\) satisfying \( u(t_0) = u_0 \). It is also known that if \( u_0 \) belongs to \( V \), then there exists a local strong solution, defined on some interval \([t_0, t_1)\), with \( u(t_0) = u_0 \).

The interval of definition \( I \subset \mathbb{R} \) of a weak solution \( u \) can be classified according to the following:

\[
\mathcal{R} = \{ t \in I; u(t) \in V \},
\]

\[
\mathcal{R}^c = \{ t \in I; u(t) \notin V \},
\]

\[
\mathcal{O} = \{ t \in I; \exists \varepsilon > 0, (t - \varepsilon, t + \varepsilon) \subset I, u \in C((t - \varepsilon, t + \varepsilon), V) \}.
\]

The set \( \mathcal{O} \) of “interior regularity” points is necessarily open, dense in \( I \), and has full measure in \( I \), i.e. \( I \setminus \mathcal{O} \) is a Lebesgue set of null measure. Being open, \( \mathcal{O} \) can be written as a countable union of disjoint open intervals, say \( \mathcal{O} = \bigcup_k (\alpha_k, \beta_k) \). By the local existence of regular solutions it follows also that \( \mathcal{R} \subset \bigcup_k [\alpha_k, \beta_k) \).

Other spaces that we consider are the classical Lebesgue spaces \( L^p(\Omega) \), \( 1 \leq p \leq \infty \), and the Sobolev spaces \( H^m(\Omega) \), \( m \in \mathbb{N} \). We denote by \( \| \cdot \|_{L^p} \) the norms in \( L^p(\Omega) \). We consider the seminorms in \( H^m(\Omega) \) given by

\[
|\varphi|_{H^m} = \sum_{|\alpha|=m} \| D^\alpha \varphi \|_{L^2},
\]

where \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \), each \( \alpha_i \) is a nonnegative integer, \( |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \), and \( D^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \). The norms in \( H^m \) are then written as

\[
\| \varphi \|_{H^m} = \sum_{k=0}^{m} |\varphi|_{H^m} = \sum_{0 \leq |\alpha| \leq m} \| D^\alpha \varphi \|_{L^2}.
\]

We also consider the vector-valued spaces \((L^p(\Omega))^d\) and \((H^m(\Omega))^d\), \( d \in \mathbb{N} \), with their appropriate norms denoted in the same way as their scalar counterparts.

Two inequalities that will be useful below are the Agmon inequality

\[
\| \mathbf{v} \|_{L^\infty} \leq c_1 |\mathbf{v}|_0^{1/4} |\mathbf{v}|_{H^2}^{3/4}, \quad \text{for } \mathbf{v} \in (H^2(\Omega))^3,
\]

(2.6)

and the Ladyzhenskaya inequality

\[
\| \mathbf{v} \|_{L^4} \leq c_2 |\mathbf{v}|_0^{1/4} |\mathbf{v}|_{L^4}^{3/4}, \quad \text{for } \mathbf{v} \in V.
\]

(2.7)

The constants \( c_1 \) and \( c_2 \) are universal nondimensional constants.
Since the Stokes operator is a second-order elliptic operator we have the following inequalities which will also be useful:

$$|\nabla \otimes A^{-1/2}v|_0 \leq c_3|v|_0, \quad \text{for } v \in H,$$

(2.8)

and

$$|\nabla \otimes A^{-3/2}v|_{H^2} \leq c_4|v|_0, \quad \text{for } v \in H,$$

(2.9)

where for a given vector valued function \( w = (w_1, w_2, w_3) \) we denote \( \nabla \otimes w = (\partial_{x_i} w_j)_{i,j=1}^3 \), and \( c_3 \) and \( c_4 \) are universal nondimensional constants.

Henceforth, we denote by \( u = u(t) \) (where \( t \geq 0 \)) an arbitrary (Leray-Hopf) weak solution with initial condition \( u(0) = u_0 \in H \).

**Remark 2.1.** The results proved in this article can also be extended to the periodic case with different periods in each direction and to the no-slip case on a bounded domain with smooth boundary (at least of class \( C^3 \)). The constants \( c_i \) would be different but still nondimensional (see Remark 3.1 for more details).

3. **Finite-time averages and some characteristic mean quantities**

We consider a Leray-Hopf weak solution \( u = u(t) \) defined for \( t \geq 0 \), with initial condition \( u(0) = u_0 \), and we take a time interval with endpoints \( 0 \leq t_1 < t_2 < \infty \). We have the classical energy estimate

$$|u(t)|_0^2 \leq e^{-\nu \lambda_1 t}|u_0|_0^2 + \frac{1}{\nu^2 \lambda_1}|A^{-1/2}f|_0^2(1 - e^{-\nu \lambda_1 t}),$$

where \( \lambda_1 \) is the first eigenvalue of the Stokes operator. Introducing \( \kappa_0 = \lambda_1^{1/2} \) as a macroscale wavenumber we write

$$|u(t)|_0^2 \leq e^{-\nu \kappa_0^2 t}|u_0|_0^2 + \frac{1}{\nu^2 \kappa_0^2}|A^{-1/2}f|_0^2(1 - e^{-\nu \kappa_0^2 t}).$$

(3.1)

For \( f \neq 0 \), we assume that \( t_1 \) is sufficiently large so that

$$e^{-\nu \kappa_0^2 t_1}|u_0|_0^2 \leq \frac{1}{\nu^2 \kappa_0^2}|A^{-1/2}f|_0^2,$$

which means

$$t_1 \geq \frac{1}{\nu \kappa_0^2} \ln \frac{\nu^2 \kappa_0^2 |u_0|_0^2}{|A^{-1/2}f|_0^2}.$$  

(3.2)

Thus, (3.1) yields

$$|u(t)|_0^2 \leq \frac{2}{\nu^2 \kappa_0^2} |A^{-1/2}f|_0^2, \quad \text{for } t \geq t_1.$$  

(3.3)

Now from the energy inequality, starting from a Lebesgue point \( t'_1 \) of \( |u(\cdot)|_0^2 \), with \( t_1 \leq t'_1 < t_2 \), we have

$$\nu \int_{t'_1}^{t_2} ||u(t)||^2 dt \leq |u(t'_1)|_0^2 + \frac{t_2 - t'_1}{\nu} |A^{-1/2}f|_0^2 \leq \frac{2}{\nu^2 \kappa_0^2} |A^{-1/2}f|_0^2 + \frac{t_2 - t'_1}{\nu} |A^{-1/2}f|_0^2.$$
Let $t_1'$ go to $t_1$ to find
\[
\nu \int_{t_1'}^{t_2} \|u(t)\|^2 \, dt \leq \frac{2}{\nu^2 \kappa_0^2} |A^{-1/2} f_0|_0^2 + \frac{t_2 - t_1}{\nu} |A^{-1/2} f_0|_0^2.
\] (3.4)

Introducing the Grashof number
\[
G^* = \frac{|A^{-1/2} f_0|_0}{\nu^2 \kappa_0^{1/2}},
\] (3.5)

estimate (3.3) can be written for $G^* > 0$ as
\[
\|u(t)\|_0^2 \leq \frac{2\nu^2}{\kappa_0^2} G^*^2,
\] for $t \geq t_1$. (3.6)

Similarly, (3.4) can be written as
\[
\nu \int_{t_1'}^{t_2} \|u(t)\|^2 \, dt \leq \left( \frac{2\nu^2}{\kappa_0} + (t_2 - t_1) \nu^3 \kappa_0 \right) G^*^2.
\] (3.7)

A useful characteristic wavenumber associated with the forcing term is
\[
\kappa_f = \left( \frac{|A^{1/2} f_0|_0}{|A^{-1/2} f_0|_0} \right)^{1/2}.
\] (3.8)

By interpolation,
\[
|f|_0 \leq |A^{-1/2} f_0|_0^{1/2} |A^{1/2} f_0|_0^{1/2} \leq \kappa_f |A^{-1/2} f_0|_0.
\] (3.9)

For a given real-valued function $\Phi$ defined on the orbit $\{u(t)\}_{t \geq 0}$ we consider its finite-time average
\[
\langle \Phi(u) \rangle = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \Phi(u(t)) \, dt.
\]

In particular we define the finite-time averaged energy dissipation rate per unit mass and unit time as
\[
\bar{\epsilon} = \nu \kappa_L^3 \langle \|u\|^2 \rangle = \frac{\nu \kappa_0^3}{t_2 - t_1} \int_{t_1}^{t_2} \|u(t)\|^2 \, dt,
\]
where
\[
\kappa_L = \frac{1}{L}.
\] (3.10)

Note that the volume of the domain $\Omega$ is $L^3$, so that by multiplying a physical quantity integrated over the spatial domain by $\kappa_3 = 1/L^3$ we are considering the corresponding quantity per unit mass. Note also that the smallest possible wavenumber in the Fourier decomposition of the flow has been denoted by $\kappa_0$, and is given by
\[
\kappa_0 = \lambda_1^{1/2} = \frac{2\pi}{L} = 2\pi \kappa_L.
\] (3.11)
We also define the finite-time averaged kinetic energy per unit mass $\tilde{e}$, root-mean-square velocity $\tilde{U}$, Taylor wavenumber $\tilde{\kappa}_\tau$, and Kolmogorov wavenumber $\tilde{\kappa}_\epsilon$ by

$$\tilde{e} = \frac{1}{2} \kappa_0^3 \langle |u|^2 \rangle_0, \quad \tilde{U} = \sqrt{2 \tilde{e}}, \quad \tilde{\kappa}_\tau = \left( \frac{\tilde{e}}{2 \nu \tilde{e}} \right)^{1/2} = \left( \frac{\langle \|u\|^2 \rangle}{\langle |u|^2 \rangle_0} \right)^{1/2}, \quad \tilde{\kappa}_\epsilon = \left( \frac{\tilde{e}}{\nu^3} \right)^{1/4}. \quad (3.12)$$

With the mean velocity $\tilde{U}$ and the macroscale wavenumber $\kappa_0$ we also define the Reynolds number

$$\text{Re} = \frac{\tilde{U}}{\nu \kappa_0}, \quad (3.13)$$

Another commonly used Reynolds-type number is the Taylor microscale Reynolds number

$$\text{Re}_\tau = \frac{\tilde{U}}{\nu \kappa_\tau}, \quad (3.14)$$

based on the Taylor wavenumber.

Different length scales of the flow are characterized in terms of the eigenvalues of the Stokes operator. The Stokes operator possesses an orthonormal basis of eigenvectors $\{w_j\}_{j \in \mathbb{N}}$, $A w_j = \lambda_j w_j$, with the eigenvalues satisfying $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty$, as $j \to \infty$. In this basis, a vector field $v$ in $H$ can be written as

$$v = \sum_{j \in \mathbb{N}} \hat{v}_j w_j, \quad v_j = (v, w_j).$$

To each eigenvalue $\lambda_j$ we associate a wavenumber $\lambda_j^{1/2}$. The wavenumbers form a discrete set and the smallest wavenumber has been identified as $\kappa_0 = \lambda_1^{1/2}$. Then, we define the component $v_\kappa$ of a vector field $v$, for a single wavenumber $\kappa \geq \kappa_0$, by

$$v_\kappa = \sum_{\lambda_j = \kappa^2} \hat{v}_j w_j.$$

The component $v_{\kappa', \kappa''}$ with a range of wavenumbers $[\kappa', \kappa'']$, $\kappa_0 \leq \kappa' < \kappa'' \leq \infty$, is defined as

$$v_{\kappa', \kappa''} = \sum_{\kappa' \leq \kappa < \kappa''} v_\kappa.$$

With this notation at hand we can write the Navier-Stokes equations projected on those components in the form

$$\frac{d u_{\kappa', \kappa''}}{dt} + \nu A u_{\kappa', \kappa''} + B(u, u)_{\kappa', \kappa''} = f_{\kappa', \kappa''}.$$

The energy-budget equation on those scales can be written, in the case $\kappa'' < \infty$, as

$$\frac{1}{2} \frac{d}{dt} \langle |u_{\kappa', \kappa''}|^2 \rangle_0 + \nu \langle \|u_{\kappa', \kappa''}\|^2 \rangle + b(u, u, u_{\kappa', \kappa''}) = (f_{\kappa', \kappa'', u_{\kappa', \kappa''}}, u_{\kappa', \kappa''}),$$

in the sense of distributions. The trilinear term above can be written as

$$-b(u, u, u_{\kappa', \kappa''}) = -b(u, u, u_{\kappa', \kappa''}) + b(u, u, u_{\kappa'', \infty}) = \epsilon_{\kappa', \kappa''}(u) - \epsilon_{\kappa'', \infty}(u),$$
where
\[ \kappa \mathcal{L} e_{\kappa}(u) = -b(u, u, u_{\kappa, \infty}). \]

The term \( \kappa^3 \mathcal{L} e_{\kappa}(u) \) is interpreted as the net flux of kinetic energy per unit mass transferred per unit time into the modes higher than or equal to \( \kappa \). This is a key quantity related to the redistribution of energy among different scales caused by the advection term. The time averaged version \( \kappa^3 \mathcal{L} \langle e_{\kappa}(u) \rangle \) represents the mean energy flux. If we take \( \kappa'' = \infty \), we obtain in general an energy inequality:
\[
\frac{1}{2} \frac{d}{dt} \| u_{\kappa', \infty'} \|^2 + \nu \| u_{\kappa', \infty'} \|^2 \leq (f_{\kappa', \infty}, u_{\kappa', \infty}) + e_{\kappa}(u),
\]
in the sense of distributions, with equality holding for regular solutions, but possibly not for general weak solutions. For this reason the interpretation of \( \kappa^3 \mathcal{L} e_{\kappa}(u) \) as the net energy flux per unit time into the modes higher than or equal to \( \kappa \) may be correct only for regular solutions. For general weak solutions we need to account for a possible loss of energy due to the lack of regularity. This is accomplished with the help of a technical result (see Lemma 8.1 below) which asserts that the limit \( \lim_{\kappa \to \infty} \langle e_{\kappa}(u) \rangle \) exists, and we denote this limit by \( \langle e(u) \rangle_{\infty} = \lim_{\kappa \to \infty} \langle e_{\kappa}(u) \rangle \).

This quantity which vanishes if \( u \) is smooth is otherwise associated with the loss of energy caused by the possible development of singularities, which prevent the energy equation to hold. Based on this term the restricted energy flux
\[ e_{\kappa}^*(u) = e_{\kappa}(u) - \langle e(u) \rangle_{\infty} \] (3.15)
is defined. The term \( \kappa^3 \mathcal{L} \langle e_{\kappa}^*(u) \rangle \) is interpreted as the the mean net flux of kinetic energy per unit mass transferred per unit time into the modes higher than or equal to \( \kappa \), and this interpretation is valid even for general weak solutions. The energy cascade is obtained for the restricted mean energy flux per unit mass \( \kappa^3 \mathcal{L} \langle e_{\kappa}^*(u) \rangle \) (see Theorem 8.1).

**Remark 3.1.** In the periodic case with different periods in each direction or in the no-slip case the quantities per unit mass should be obtained by dividing the original quantity by the whole volume of the domain. This amounts to taking \( \kappa L = 1/|\Omega|^{1/3} \).

Then the ratio \( \kappa L/\kappa_0 \) is no longer constant and depends on the shape of \( \Omega \) but it is still a nondimensional parameter. All the results obtained in this article can be adapted to those cases by modifying the constants \( c_i \) below to contain explicitly the ratio \( \kappa L/\kappa_0 \).

**4. Lower bounds for the mean kinetic energy and the mean energy dissipation rate**

In this section we derive lower bound estimates for the mean kinetic energy \( \bar{\epsilon} \) and the mean rate of energy dissipation \( \bar{\mathcal{E}} \) in terms of the Grashof number \( G^* \).

We first show that
Lemma 4.1. For any \( t_2 \geq t_1 \geq 0 \),
\[
(t_2 - t_1)|A^{-1/2}f_0^2|_0^2 = (u(t_2) - u(t_1), A^{-1}f) + \nu \int_{t_1}^{t_2} (u(t), f) \, dt + \int_{t_1}^{t_2} b(u(t), u(t), A^{-1}f) \, dt.
\] (4.1)

Proof. As a weak solution between \( t_1 \) and \( t_2 \), the interior regularity points of \( u = u(t) \) form an open dense set \( O \) of full measure in \((t_1, t_2)\), with \( O \) the union of a countable set of nonoverlapping intervals
\[
O = \bigcup_{j \in \mathbb{N}} (\alpha_j, \beta_j).
\]
In particular on each interval \((\alpha_j, \beta_j)\) the solution \( u \) is continuous in the topology of \( V \), and
\[
\frac{du(t)}{dt} + \nu Au(t) + B(u(t), u(t)) = f,
\]
holds continuously in \( V' \) for \( t \in (\alpha_j, \beta_j) \). Taking the inner product in \( H \) with \( A^{-1}f \) yields
\[
|A^{-1/2}f_0^2|_0^2 = \frac{d}{dt}(u(t), A^{-1}f) + \nu (u(t), f) + b(u(t), u(t), A^{-1}f).
\]
Integrating between \( \alpha \) and \( \beta \), with \( \alpha_j < \alpha < \beta < \beta_j \), we obtain
\[
(\beta - \alpha)|A^{-1/2}f_0^2|_0^2 = (u(\beta) - u(\alpha), A^{-1}f) + \nu \int_{\alpha}^{\beta} (u(t), f) \, dt + \int_{\alpha}^{\beta} b(u(t), u(t), A^{-1}f) \, dt.
\]
Using only the weak continuity of the weak solutions we let \( \alpha \) and \( \beta \) go to the endpoints of the interval of regularity to find
\[
(\beta_j - \alpha_j)|A^{-1/2}f_0^2|_0^2 = (u(\beta_j) - u(\alpha_j), A^{-1}f) + \nu \int_{\alpha_j}^{\beta_j} (u(t), f) \, dt + \int_{\alpha_j}^{\beta_j} b(u(t), u(t), A^{-1}f) \, dt.
\]
Adding up the intervals within \( t_1 \) and \( t_2 \) we arrive at
\[
(t_2 - t_1)|A^{-1/2}f_0^2|_0^2 = \sum_{j \in \mathbb{N}} (u(\beta_j) - u(\alpha_j), A^{-1}f) + \nu \int_{t_1}^{t_2} (u(t), f) \, dt + \int_{t_1}^{t_2} b(u(t), u(t), A^{-1}f) \, dt.
\]
In order to complete the proof let us now argue that
\[
\sum_{j \in \mathbb{N}} (u(\beta_j) - u(\alpha_j), A^{-1}f) = (u(t_2) - u(t_1), A^{-1}f).
\] (4.2)
This follows from the facts that \( t \mapsto u(t) \) is absolutely weakly continuous and that the set of singular points has measure zero. Indeed, given \( N \in \mathbb{N} \), consider the intervals associated with \( j = 1, \ldots, N \). Let the intervals be reordered increasingly, so that

\[
\alpha_{1}^{N} < \beta_{1}^{N} < \alpha_{2}^{N} < \beta_{2}^{N} < \cdots < \alpha_{N}^{N} < \beta_{N}^{N} \leq t_{2}.
\]

Then, \( \Sigma_{N} = [t_{1}, \alpha_{1}^{N}] \cup [\beta_{1}^{N}, t_{2}] \cup \left( \bigcup_{k=1}^{N-1} [\beta_{k}^{N}, \alpha_{k+1}^{N}] \right) \) covers the set of singular points, which has measure zero. Thus,

\[
\sum_{k=1}^{N-1} (\alpha_{k+1}^{N} - \beta_{k}^{N}) \to 0, \quad \text{as } N \to \infty.
\]

Since \( t \mapsto u(t) \) is absolutely weakly continuous, the scalar function \( t \mapsto (u(t), A^{-1}f) \) is absolutely continuous, and by (4.3),

\[
\sum_{k=1}^{N-1} (u(\alpha_{k+1}^{N}) - u(\beta_{k}^{N}), A^{-1}f) \to 0, \quad \text{as } N \to \infty.
\]

Again since the set of singular points is of measure zero, we must have \( \alpha_{1}^{N} \to t_{1} \) and \( \beta_{N}^{N} \to t_{2} \), as \( N \to \infty \).

Then, by the weak continuity of the weak solutions we find

\[
(u(\alpha_{1}^{N}), A^{-1}f) \to (u(t_{1}), A^{-1}f) \text{ and } (u(\beta_{N}^{N}), A^{-1}f) \to (u(t_{2}), A^{-1}f), \quad \text{as } N \to \infty.
\]

Finally, since

\[
\sum_{j=1}^{N} (u(\beta_{j}) - u(\alpha_{j}), A^{-1}f)
\]

\[
= (u(\beta_{N}^{N}), A^{-1}f) - (u(\alpha_{1}^{N}), A^{-1}f) + \sum_{k=1}^{N-1} (u(\alpha_{k+1}^{N}) - u(\beta_{k}^{N}), A^{-1}f)
\]

we obtain at the limit as \( N \to \infty \) that

\[
\sum_{j \in \mathbb{N}} (u(\beta_{j}) - u(\alpha_{j}), A^{-1}f) = (u(t_{2}) - u(t_{1}), A^{-1}f).
\]

This completes the proof of (4.2) and, hence, of the lemma. \( \square \)

**Remark 4.1.** An alternative way to prove Lemma 4.1 is to use the weak formulation of the Navier-Stokes equations with the test function \( A^{-1}f \), which belongs to \( V \). Note also that Lemma 4.1 holds even for \( f \) in \( V' \).

We are now in a position to prove our first lower bound estimate, which is for the mean kinetic energy.
Proposition 4.1. For $t_1$ satisfying (3.2) and for

$$ t_2 - t_1 \geq \frac{4\sqrt{2}}{\nu \kappa_0}, $$

we have

$$ |A^{-1/2}f|_0 \leq \nu^2 \kappa_f^{1/2} + (2c_1 c_3^{1/4} c_4^{3/4} + 1) \kappa_f^{3/2} (|u|_0^{2})^2, $$

and, hence,

$$ \tilde{e} \geq \nu \frac{c_5}{c_6} \left( \frac{\kappa_0}{\kappa_f} \right)^{3/2} \left( \kappa_f^{-1} \left( \frac{\kappa_f}{\kappa_0} \right)^{1/2} \right), $$

with $c_5 = 16\pi^3 (2c_1 c_3^{1/4} c_4^{3/4} + 1)$.

Proof. We estimate the terms in the right hand side of (4.1). First, from (3.3),

$$ (u(t_2) - u(t_1), A^{-1}f) \leq \frac{2\sqrt{2}}{\nu \kappa_0} |A^{-1/2}f|_0 |A^{-1}f|_0 \leq \frac{2\sqrt{2}}{\nu \kappa_0} |A^{-1/2}f|_0^2. $$

Secondly, using (3.9) and the Cauchy-Schwarz inequality,

$$ \nu \int_{t_1}^{t_2} (u(t), f) \, dt \leq \nu \int_{t_1}^{t_2} |u(t)|_0 |f|_0 \, dt \leq \nu \kappa_f |A^{-1/2}f|_0 \int_{t_1}^{t_2} |u(t)|_0 \, dt $$

$$ \leq \nu \kappa_f |A^{-1/2}f|_0 (t_2 - t_1)^{1/2} \left( \int_{t_1}^{t_2} |u(t)|_0^2 \, dt \right)^{1/2}. $$

Using now the orthogonality property of the nonlinear term and the Hölder inequality we find

$$ \int_{t_1}^{t_2} b(u(t), u(t), A^{-1}f) \, dt = - \int_{t_1}^{t_2} b(u(t), A^{-1}f, u(t)) \, dt $$

$$ \leq \int_{t_1}^{t_2} |u(t)|_0^2 \| \nabla \otimes A^{-1}f \|_{L^\infty} \, dt. $$

Applying the Agmon inequality (2.6) to the term involving $f$ and then using (2.8) and (2.9) we have

$$ \| \nabla \otimes A^{-1}f \|_{L^\infty} \leq c_1 |\nabla \otimes A^{-1}f|_0^{1/4} |\nabla \otimes A^{-1}f|_{H^2}^{3/4} $$

$$ \leq c_1 |\nabla \otimes A^{-1/2}A^{-1/2}f|_0^{1/4} |\nabla \otimes A^{-3/2}A^{1/2}f|_{H^2}^{3/4} $$

$$ \leq c_1 c_3^{1/4} c_4^{3/4} |A^{-1/2}f|_0^{1/4} |A^{1/2}f|_0^{3/4}. $$

Inserting this estimate into (4.7) and using the definition of $\kappa_f$ in (3.8) we obtain

$$ \int_{t_1}^{t_2} b(u(t), u(t), A^{-1}f) \, dt \leq c_1 c_3^{1/4} c_4^{3/4} \kappa_f^{3/2} |A^{-1/2}f|_0 \int_{t_1}^{t_2} |u(t)|_0^2 \, dt. $$
Dividing (4.1) by $t_2 - t_1$ and inserting the above estimates, we obtain

$$|A^{-1/2}f|_0^2 \leq \frac{1}{t_2 - t_1 \nu \kappa_0^2} |A^{-1/2}f|_0^2 + \nu \kappa_f |A^{-1/2}f|_0 \left( \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |u(t)|_0^2 \, dt \right)^{1/2}$$

$$+ c_1 c_3^{1/4} c_4^{3/4} \kappa_f^{3/2} |A^{-1/2}f|_0 \left( \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |u(t)|_0^2 \, dt \right).$$

Thus,

$$|A^{-1/2}f|_0 \leq \frac{1}{t_2 - t_1 \nu \kappa_0^2} |A^{-1/2}f|_0 + \nu \kappa_f \left( \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |u(t)|_0^2 \, dt \right)^{1/2}$$

$$+ c_1 c_3^{1/4} c_4^{3/4} \kappa_f^{3/2} \left( \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |u(t)|_0^2 \, dt \right).$$

From (4.4), we have

$$\frac{1}{t_2 - t_1 \nu \kappa_0^2} \leq \frac{1}{2},$$

so that

$$|A^{-1/2}f|_0 \leq 2 \nu \kappa_f \left( \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |u(t)|_0^2 \, dt \right)^{1/2}$$

$$+ 2 c_1 c_3^{1/4} c_4^{3/4} \kappa_f^{3/2} \left( \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |u(t)|_0^2 \, dt \right).$$

Thus,

$$|A^{-1/2}f|_0 \leq 2 \nu \kappa_f \left( \langle |u|_0^2 \rangle \right)^{1/2} + 2 c_1 c_3^{1/4} c_4^{3/4} \kappa_f^{3/2} \langle |u|_0^2 \rangle$$

$$\leq \nu^2 \kappa_f^{1/2} + (2 c_1 c_3^{1/4} c_4^{3/4} + 1) \kappa_f^{3/2} \langle |u|_0^2 \rangle,$$

which proves the first estimate. Divide this estimate by $\nu^2 \kappa_0^{1/2}$ and use (3.11) to find

$$G^* \leq \left( \frac{\kappa_f}{\kappa_0} \right)^{1/2} + \frac{c_5}{\nu^2 \kappa_0^2} \left( \frac{\kappa_f}{\kappa_0} \right)^{3/2} \tilde{e},$$

which yields the second estimate. \hfill \Box

**Remark 4.2.** Since $\tilde{\text{Re}} = 2\frac{\tilde{\epsilon}}{(\nu^2 \kappa_0^2)}$, the lower bound (4.5) for the mean kinetic energy in Proposition 4.1 yields the following lower bound for the Reynolds number

$$\tilde{\text{Re}} \geq C_0 \left( G^* - \left( \frac{\kappa_f}{\kappa_0} \right)^{1/2} \right)^{1/2},$$

(4.8)
where

\[ C_0 = C_0 \left( \frac{\kappa_0}{\kappa_f} \right) = \left( \frac{2}{c_5} \right)^{1/2} \left( \frac{\kappa_0}{\kappa_f} \right)^{1/4}. \]  

(4.9)

We now prove a lower bound estimate for the mean rate of energy dissipation \( \tilde{\epsilon} \) in terms of the Grashof number.

**Proposition 4.2.** For \( t_1 \) satisfying (3.2) and for \( t_2 - t_1 \) satisfying (4.4) we have

\[ |A^{-1/2}f_0|_0 \leq \nu^2 \kappa_0^{1/2} + \frac{2c_2^2 + 1}{\kappa_0^{1/2}} (\|u\|^2), \]  

(4.10)

and, hence,

\[ \tilde{\epsilon} \geq \frac{\nu^3 \kappa_0}{c_6} (G^* - 1), \]  

(4.11)

with \( c_6 = 8\pi^3(2c_2^2 + 1) \).

**Proof.** We estimate in a different way the second and third terms in the right hand side of (4.1), namely

\[ \nu \int_{t_1}^{t_2} (u(t), f) \ dt \leq \nu \int_{t_1}^{t_2} \|u(t)\| \|A^{-1/2}f_0\|_0 \ dt \leq \nu \|A^{-1/2}f_0\|_0 \int_{t_1}^{t_2} \|u(t)\| \ dt \]

\[ \leq \nu \|A^{-1/2}f_0\|_0 (t_2 - t_1)^{1/2} \left( \int_{t_1}^{t_2} \|u(t)\|^2 \ dt \right)^{1/2}, \]

and

\[ \int_{t_1}^{t_2} b(u(t), u(t), A^{-1}f) \ dt = - \int_{t_1}^{t_2} b(u(t), A^{-1}f, u(t)) \ dt \]

\[ \leq \int_{t_1}^{t_2} \|u(t)\|_0^2 |A^{-1/2}f_0|_0 \ dt \]

\[ \leq c_2^2 \int_{t_1}^{t_2} |u(t)|_0^2 \|u(t)\|^{3/2} |A^{-1/2}f_0|_0 \ dt \]

\[ \leq \frac{c_2^2}{\kappa_0^{1/2}} |A^{-1/2}f_0|_0 \int_{t_1}^{t_2} \|u(t)\|^2 \ dt. \]

Dividing each side of (4.1) by \( t_2 - t_1 \) and inserting the above estimates along with (4.6), we find

\[ |A^{-1/2}f_0|^2 \leq \frac{1}{t_2 - t_1} \frac{2\sqrt{2}}{\nu \kappa_0^2} |A^{-1/2}f_0|^2 + \nu |A^{-1/2}f_0|_0 \left( \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \|u(t)\|^2 \ dt \right)^{1/2} \]

\[ + \frac{c_2^2}{\kappa_0^{1/2}} |A^{-1/2}f_0|_0 \left( \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \|u(t)\|^2 \ dt \right). \]
Thus,

\[
|A^{-1/2}f|_0 \leq \frac{1}{t_2 - t_1} \frac{2\sqrt{2}}{\nu \kappa_0} |A^{-1/2}f|_0 + \nu \left( \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \|u(t)\|^2 \, dt \right)^{1/2} + \frac{c_2^2}{\kappa_0^{1/2}} \left( \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \|u(t)\|^2 \, dt \right).
\]

Using (4.4), we conclude that

\[
|A^{-1/2}f|_0 \leq 2\nu \langle \|u\|^2 \rangle^{1/2} + \frac{2c_2^2}{\kappa_0^{1/2}} \langle \|u\|^2 \rangle \leq \nu^2 \kappa_0^{1/2} + \frac{2c_2^2 + 1}{\kappa_0^{1/2}} \langle \|u\|^2 \rangle,
\]

which proves (4.10). Divide this estimate by \(\nu \kappa_0^{1/2}\) and use (3.11) to find

\[
G^* \leq 1 + \frac{c_6}{\nu^{3/4}} \tilde{\epsilon},
\]

which yields (4.11).

\[\square\]

**Remark 4.3.** Of course (4.5) and (4.11) give useful information provided \(G^*\) is sufficiently large:

\[
G^* > \max \left( 1, \left( \frac{\kappa_f}{\kappa_0} \right)^{1/2} \right) \quad (4.12)
\]

We will often assume henceforth that condition (4.12) holds.

\[\square\]

**Remark 4.4.** Observe that the estimates in Proposition 4.2 hold also if we assume \(f\) in \(H\).

5. Upper bounds for the mean kinetic energy and the mean energy dissipation rate

In this section we derive upper bound estimates for the mean kinetic energy \(\tilde{e}\) and the mean rate of energy dissipation \(\tilde{\epsilon}\). The first estimates are in terms of the Grashof number. Then we estimate the mean energy dissipation rate in terms of the root-mean-square velocity \(\tilde{U}\). The latter provides one half of the relation \(\tilde{\epsilon} \sim \kappa_0 \tilde{U}^3\) known as the Kolmogorov energy dissipation law, and obtained heuristically in the conventional theory of turbulence. An analogous upper bound has been rigorously proved in the case of shear flows and infinite-time averages by Constantin and Doering [4, 9]. Subsequent analogous results in various contexts and also for infinite-time averages have been obtained in [3, 10, 11, 13, 19, 22, 28, 29, 31, 38, 39].

First, we present the following result, which bounds the mean energy dissipation rate in terms of the Grashof number.
Proposition 5.1. For $t_1$ satisfying (3.2) and for
\[ t_2 - t_1 \geq \frac{2}{\nu \kappa_0^2}, \tag{5.1} \]
we have
\[ \bar{\epsilon} \leq \frac{\nu^3 \kappa_0^4}{4\pi^3} G^*^2. \tag{5.2} \]

Proof. Multiplying (3.7) by $\kappa_L^3$ and dividing by $t_2 - t_1$ yields
\[ \bar{\epsilon} \leq \kappa_L^3 \left( \frac{2\nu^2}{\kappa_0(t_2 - t_1)} + \nu^3 \kappa_0 \right) G^*^2. \]
From the assumption (5.1) on the interval of integration we find
\[ \frac{2\nu^2}{\kappa_0(t_2 - t_1)} \leq \nu^3 \kappa_0, \]
so that
\[ \bar{\epsilon} \leq 2\nu^3 \kappa_0 \kappa_L^3 G^*^2. \]
Using (3.11), we find
\[ \bar{\epsilon} \leq \frac{\nu^3 \kappa_0^4}{4\pi^3} G^*^2, \]
which concludes the proof. \qed

A simple corollary of this result is the following.

Corollary 5.1. For $t_1$ satisfying (3.2) and for $t_2 - t_1$ satisfying (5.1) we have
\[ \bar{\epsilon} \leq \nu^2 \kappa_0^3 G^*^2. \tag{5.3} \]

Proof. This follows from the inequality $\kappa_0 |u|_0 \leq \|u\|$, which leads to $\nu \kappa_0^2 \bar{\epsilon} \leq \bar{\epsilon}$ and the result then follows from (5.2). \qed

Remark 5.1. Observe that the estimates in Proposition 5.1 and Corollary 5.1 hold also if we assume $f$ in $H$.

We now obtain an estimate of the mean energy dissipation rate in terms of the root-mean-square velocity. As mentioned in the Introduction, in the statistical theory of turbulence this relation is known as the Kolmogorov energy dissipation law.

Theorem 5.1. Assuming (4.12), we have for $t_1$ satisfying (3.2) and for
\[ t_2 - t_1 \geq \max \left\{ \frac{4\sqrt{2}}{\nu \kappa_0^2}, \frac{G^*}{\nu \kappa_0^2} \right\}, \tag{5.4} \]
that
\[ \bar{\epsilon} \leq C_1 \kappa_0 \bar{U}^3, \tag{5.5} \]
where
\[ C_1 = C_1 \left( \frac{1}{G^*}, \frac{\kappa_f}{\kappa_0} \right) \]
\[ = \frac{1}{C_0^3 G^{3/2}} \left( \frac{1}{2\pi^3} \left( \frac{\kappa_f}{\kappa_0} \right)^{1/2} + \frac{c_5^3}{2^{6/5}3^3\pi^9} \left( \frac{\kappa_0}{\kappa_f} \right)^{1/2} + \frac{1}{2^{1/2}3^{1/4}\pi^{9/4}} \frac{\kappa_f}{\kappa_0} \right) \]
\[ \left( 1 - \left( \frac{\kappa_f}{\kappa_0} \right)^{1/2} \frac{1}{G^*} \right)^{-3/2} + c_7 \left( \frac{\kappa_f}{\kappa_0} \right)^{5/2}, \quad (5.6) \]

with \( c_7 = 2^{-3/2} + 2^{-5/2} \pi^{-3/2} c_5 \) and \( C_0 = C_0(\kappa_0/\kappa_f) \) as defined in (4.8).

**Proof.** Integration of the energy inequality yields
\[ \frac{1}{2} \|u(t_2)\|^2 + \nu \int_{t_1}^{t_2} \|u(t)\|^2 \, dt \leq \frac{1}{2} \|u(t_1')\|^2 + \int_{t_1'}^{t_2} \langle f, u(t) \rangle \, dt, \]
for \( t_1 \leq t_1' < t_2 \), \( t_1' \) being a Lebesgue point of \( t \mapsto \|u(t)\|_0^2 \). We have
\[ \int_{t_1'}^{t_2} \langle f, u(t) \rangle \, dt \leq \int_{t_1'}^{t_2} |f|_0 |u(t)|_0 \, dt \leq (t_2 - t_1')^{1/2} |f|_0 \left( \int_{t_1'}^{t_2} |u(t)|_0^2 \, dt \right)^{1/2}, \]
so that
\[ \nu \int_{t_1'}^{t_2} |u(t)|_0^2 \, dt \leq \frac{1}{\nu^{2/3} \kappa_0^2} |A^{-1/2} f\|_0^2 + (t_2 - t_1')^{1/2} |f|_0 \left( \int_{t_1'}^{t_2} |u(t)|_0^2 \, dt \right)^{1/2}. \]
Use (3.3) to write
\[ \nu \int_{t_1'}^{t_2} |u(t)|_0^2 \, dt \leq \frac{1}{\nu^{2/3} \kappa_0^2} |A^{-1/2} f\|_0^2 + (t_2 - t_1')^{1/2} |f|_0 \left( \int_{t_1'}^{t_2} |u(t)|_0^2 \, dt \right)^{1/2}. \]
Let \( t_1' \) go to \( t_1 \) and divide by \( t_2 - t_1 \) to find
\[ \nu \langle |u|^2 \rangle \leq \frac{1}{(t_2 - t_1)\nu^{2/3} \kappa_0^2} |A^{-1/2} f\|_0^2 + |f|_0 (\langle |u|^2 \rangle_0^2)^{1/2}. \quad (5.7) \]
Multiply by \( \kappa_L^3 \), introduce \( G^* \), and use (3.9) to find
\[ \dot{\epsilon} \leq \frac{\nu^2 \kappa_L^3}{\kappa_0(t_2 - t_1)} G^{*2} + 2^{1/2} \nu^2 \kappa_0^{1/2} \kappa_L^{3/2} \kappa_f \dot{\epsilon}^{1/2} G^*. \]
From assumption (5.4) we have
\[ \frac{\nu^2 G^*}{\kappa_0(t_2 - t_1)} \leq \nu^{3} \kappa_0, \]
so that
\[ \dot{\epsilon} \leq G^* \left( \nu^3 \kappa_0 \kappa_L^3 + 2^{1/2} \nu^2 \kappa_0^{1/2} \kappa_L^{3/2} \kappa_f \dot{\epsilon}^{1/2} \right). \]
Using (3.11) we find
\[ \tilde{\epsilon} \leq G^* \left( \frac{\nu^3 \kappa_0^4}{8\pi^3} + \frac{\nu^2 \kappa_0^2 \kappa_f \tilde{\epsilon}^{1/2}}{2\pi^{3/2}} \right). \]

Relation (4.5) can be rewritten as
\[ G^* \leq \left( \frac{\kappa_f}{\kappa_0} \right)^{1/2} + \frac{c_5}{\nu^2 \kappa_0^2} \left( \frac{\kappa_f}{\kappa_0} \right)^{3/2} \tilde{\epsilon}. \] (5.8)

Using (5.8) and the Young inequality, we find
\[ \tilde{\epsilon} \leq \left( \left( \frac{\kappa_f}{\kappa_0} \right)^{1/2} + \frac{c_5}{\nu^2 \kappa_0^2} \left( \frac{\kappa_f}{\kappa_0} \right)^{3/2} \tilde{\epsilon} \right) \left( \frac{\nu^3 \kappa_0^4}{8\pi^3} + \frac{\nu^2 \kappa_0^2 \kappa_f \tilde{\epsilon}^{1/2}}{2\pi^{3/2}} \right) \]
\[ \leq \nu^3 \kappa_0^4 \left( \frac{1}{8\pi^3} \right)^{1/2} + \frac{c_5}{\nu^2 \kappa_0^2} \left( \frac{\kappa_f}{\kappa_0} \right)^{3/2} \tilde{\epsilon} \left( \frac{1}{8\pi^3} + \frac{1}{2\pi^{3/2}} \right) \left( \frac{\kappa_f}{\kappa_0} \tilde{\epsilon}^{1/2} \right) \]
\[ \leq \nu^3 \kappa_0^4 \left( \frac{1}{8\pi^3} \right)^{1/2} + \frac{c_5}{\nu^2 \kappa_0^2} \left( \frac{\kappa_f}{\kappa_0} \right)^{1/2} + \frac{1}{2\pi^{3/2}} \frac{\kappa_f}{\kappa_0} \left( \frac{\kappa_f}{\kappa_0} \right)^{1/2} \kappa_0 \tilde{\epsilon}^{3/2}. \]

Since \( \tilde{U} = \sqrt{2} \tilde{\epsilon} \) we obtain
\[ \tilde{\epsilon} \leq \nu^3 \kappa_0^4 \left( \frac{1}{8\pi^3} \right)^{1/2} + \frac{c_5}{2^{93/4} \pi^2} \left( \frac{\kappa_0}{\kappa_f} \right)^{1/2} + \frac{1}{2^{1/2} \pi^{3/2} \kappa_0} \frac{\kappa_f}{\kappa_0} \tilde{\epsilon}^{3/2}. \]

where \( c_7 = 2^{-3/2} + 2^{-5/2} \pi^{-3/2} c_5 \). Using (4.8) it is straightforward to see that this implies inequality (5.5).

\[ \square \]

6. Estimates for other characteristic quantities

The aim in this section is to present a number of other rigorous estimates for time averages which are related to the statistical theory of turbulence. More precisely, in the conventional theory of turbulence it is argued heuristically that for homogeneous turbulent flows the relations \( \kappa_\epsilon \sim \kappa_0 \text{Re}^{3/4}, \kappa_\tau \sim \kappa_0 \text{Re}^{1/2}, \kappa_\tau \sim \kappa_0^{1/3} \kappa_\epsilon^{2/3}, \kappa_\tau \sim \text{Re}^{-1/4} \kappa_\epsilon, \text{and Re} \sim \text{Re}^2 \) hold in the mean. We prove rigorous upper bound estimates for the corresponding relations for the mean quantities with respect to finite-time averages, as long as the averaging time is sufficiently long. Furthermore we obtain several estimates for \( \text{Re} \) and \( \tilde{\kappa}_\epsilon \) in terms of the Grashof number, assuming that \( G^* \) is sufficiently large (see Remark 4.3).
From the estimate (5.5) for the energy dissipation rate we can readily estimate the Kolmogorov wavenumber according to

$$\tilde{\kappa}_\epsilon = \left( \frac{\tilde{\epsilon}}{\nu^3} \right)^{1/4} \leq C_1^{1/4} \left( \frac{k_0 \tilde{U}^3}{\nu^3} \right)^{1/4} = C_1^{1/4} k_0 \tilde{\text{Re}}^{3/4}. $$

We can also rewrite the estimate (5.5) as

$$\tilde{\epsilon} \leq C_1 k_0 (2\tilde{\epsilon})^{3/2}, \quad (6.1)$$

so that

$$\frac{\tilde{\epsilon}}{\epsilon} \leq C_1 k_0 2(2\tilde{\epsilon})^{1/2} = C_1 k_0 \tilde{U}. $$

Thus, the Taylor wavenumber can be estimated as

$$\tilde{\kappa}_\tau = \left( \frac{\tilde{\epsilon}}{2\nu^3} \right)^{1/2} \leq C_1^{1/2} k_0^{1/2} \tilde{U}^{1/2} = C_1^{1/2} k_0 \tilde{\text{Re}}^{1/2}. \quad (6.2)$$

Similarly,

$$\frac{\tilde{\epsilon}}{\epsilon^3/2} \leq C_1 k_0 2\sqrt{2}, $$

so that

$$\tilde{\kappa}_\tau = \left( \frac{\tilde{\epsilon}}{2\nu^3} \right)^{1/2} \leq \frac{1}{\sqrt{2}} \left( \frac{\tilde{\epsilon}}{\nu^3} \right)^{1/6} \left( \frac{\tilde{\epsilon}}{\epsilon^{3/2}} \right)^{1/3} \leq C_1^{1/3} k_0^{1/3} \tilde{\kappa}_\epsilon^{2/3}. $$

Moreover, by (6.1),

$$\tilde{\kappa}_\tau = \left( \frac{\tilde{\epsilon}}{2\nu^3} \right)^{1/2} \left( \frac{\nu^3}{\tilde{\epsilon}} \right)^{1/4} = C_1^{1/4} k_0^{1/4} (2\tilde{\epsilon})^{3/8} \nu^{1/4} \leq C_1^{1/4} k_0^{1/4} \frac{2^{1/2} \tilde{\epsilon}^{1/2}}{2^{1/2} \tilde{\epsilon}^{1/2}} = C_1^{1/4} k_0^{1/4} \nu^{1/4} \frac{\tilde{U}^{1/4}}{\tilde{\text{Re}}^{1/4}} = C_1^{1/4} \frac{\tilde{U}^{1/4}}{\tilde{\text{Re}}^{1/4}}. $$

Finally, using (6.2), we find

$$\tilde{\text{Re}} \leq C_1 \tilde{\text{Re}}_\tau. $$

Hence, we have proved the following result.

**Theorem 6.1.** Under the assumptions of Theorem 5.1, we have

$$\tilde{\kappa}_\epsilon \leq C_1^{1/4} k_0 \tilde{\text{Re}}^{3/4}, \quad (6.3)$$

$$\tilde{\kappa}_\tau \leq C_1^{1/2} k_0 \tilde{\text{Re}}^{1/2}, \quad (6.4)$$

$$\tilde{\kappa}_\tau \leq C_1^{1/3} k_0^{1/3} \tilde{\kappa}_\epsilon^{2/3}, \quad (6.5)$$

$$\tilde{\kappa}_\tau \leq C_1^{1/4} \tilde{\text{Re}}^{-1/4} \tilde{\kappa}_\epsilon, \quad (6.6)$$

$$\tilde{\text{Re}} \leq C_1 \tilde{\text{Re}}_\tau. \quad (6.7)$$
Remark 6.1. In particular, it follows from (6.5) that
\[ \frac{\tilde{\kappa}_0}{\kappa_0} \leq C_1^{1/3} \left( \frac{\tilde{\kappa}_e}{\kappa_0} \right)^{2/3}. \]

We now derive further estimates involving the Grashof number. First, we recall that Remark 4.2 yields a lower bound for \( \tilde{\text{Re}} \) in terms of the Grashof number, a result which we restate below in the form of a proposition. As for an upper bound, we have, using (3.6),
\[ \tilde{\text{Re}}^2 = \frac{\tilde{U}^2}{\nu^2 \kappa_0^2} = \frac{2\tilde{\epsilon}}{\nu^2 \kappa_0^2} = \frac{\kappa_0^3}{8\pi^3 \nu^2} \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |u(t)|^2 \, dt \leq \frac{1}{4\pi^3} G^*^2. \]
Thus,
\[ \tilde{\text{Re}} \leq \frac{1}{2\pi^{3/2}} G^*. \]

Hence, we have the following results.

**Proposition 6.1.** For \( t_1 \) satisfying (3.2) and \( t_2 > t_1 \) we have
\[ \tilde{\text{Re}} \leq \frac{1}{2\pi^{3/2}} G^*. \tag{6.8} \]

If, moreover, \( t_2 - t_1 \) satisfies (4.4), and (4.12) holds, then
\[ \tilde{\text{Re}} \geq C_0 \left( G^* - \left( \frac{\kappa_x}{\kappa_0} \right)^{1/2} \right)^{1/2}, \tag{6.9} \]
where \( C_0 = C_0(\kappa_0/\kappa_f) \) is defined by (4.8).

Now, from Proposition 4.2, we have
\[ G^* \leq 1 + \frac{c_6}{\nu^3 \kappa_0^2} \tilde{\epsilon} = 1 + c_6 \left( \frac{\tilde{\kappa}_e}{\kappa_0} \right)^4. \]

Moreover, using (5.2) we have
\[ \tilde{\kappa}_e = \left( \frac{\tilde{\epsilon}}{\nu^3} \right)^{1/4} \leq \left( \frac{\nu^3 \kappa_0^4 G^*^2}{4\pi^3 \nu^3} \right)^{1/4} = \frac{\kappa_0}{2^{1/2} \pi^{3/4} G^*^{1/2}}. \]

Hence, we have the following proposition.
Proposition 6.2. For $t_1$ satisfying (3.2) and $t_2 - t_1$ satisfying (5.1) we have
\[ \tilde{\kappa}_\epsilon \leq \frac{\kappa_0}{2^{1/2} \pi^{3/4}} G^{*1/2}, \]  
(6.10)
Moreover, if $t_1 - t_2$ satisfies (4.4), and (4.12) holds, then
\[ G^* \leq 1 + c_6 \left( \frac{\tilde{\kappa}_\epsilon}{\kappa_0} \right)^4, \]  
(6.11)
which is equivalent to
\[ \tilde{\kappa}_\epsilon \geq c_6^{-1/4} (G^* - 1)^{1/4} \kappa_0. \]  
(6.12)

\[ \square \]

Remark 6.2. Observe that the estimate (6.8) and all of those in Proposition 6.2 hold also if we assume $f$ in $H$.

7. Improved estimate for the number of degrees of freedom

In the Kolmogorov theory of turbulence, the wavenumber $\kappa_\epsilon$ is associated with the smallest scales relevant for the flow, which leads to the heuristic inference that the ratio $(\kappa_\epsilon/\kappa_0)^3$ is an appropriate estimate for the number of degrees of freedom of the flow (see [7, 8, 16, 17, 26, 37]). From (6.10) one deduces that this number of degrees of freedom defined in terms of finite-time averages satisfies
\[ \left( \frac{\tilde{\kappa}_\epsilon}{\kappa_0} \right)^3 \leq \frac{1}{2^{3/2} \pi^{9/4}} G^{*3/2}. \]  
(7.1)

We now deduce a sharper estimate for $\tilde{\epsilon}$ which allows us to improve the above estimate (see Remark 7.1).

Proposition 7.1. Let $t_1$ satisfy (3.2) and $t_2 > t_1$. Then
\[ \tilde{\epsilon} = \nu \kappa_\epsilon^3 \langle \|u\|_0^2 \rangle \leq \frac{1}{8 \pi^3} \left( \frac{2 \nu^2 \kappa_0^2}{t_2 - t_1} + \frac{\nu \kappa_0^4 \kappa_\epsilon^2 f}{\kappa_\epsilon^2} \right) G^* \]  
(7.2)
and
\[ \tilde{\kappa}_\epsilon = \left( \frac{\tilde{\epsilon}}{\nu^3} \right)^{1/4} \leq \frac{1}{2^{3/4} \pi^{3/4}} \left( \frac{2 \kappa_0^2}{\nu (t_2 - t_1)} + \frac{\kappa_0^4 \kappa_\epsilon^2 f}{\kappa_\epsilon^2} \right)^{1/4} G^{*1/2}. \]  
(7.3)

Proof. We saw in (5.7) that
\[ \nu \langle \|u\|_0^2 \rangle \leq \frac{1}{(t_2 - t_1) \nu^2 \kappa_0^2} |A^{-1/2} f|_0^2 + |f|_0 (\langle |u|_0^2 \rangle)^{1/2}. \]  
(7.4)
From the definition in (3.12) of $\tilde{\kappa}_\tau$ we have
\[ \langle |u|_0^2 \rangle = \frac{\langle \|u\|_0^2 \rangle}{\tilde{\kappa}_\epsilon^2}. \]
Inserting this relation into (7.4) yields
\[ \nu \langle \| u \|_2 \rangle^2 \leq \frac{1}{(t_2 - t_1)\nu^2k_0^2} \left| A^{-1/2}f_0^2 \right| + \nu \langle \| u \|_2 \rangle^2 \leq \frac{1}{(t_2 - t_1)\nu^2k_0^2} \left| A^{-1/2}f_0^2 \right| + \frac{1}{2\nu^2k_0^2} \left| f_0^2 \right| + \frac{\nu \langle \| u \|_2 \rangle^2}{2}. \]

Thus, using (3.9),
\[ \nu \langle \| u \|_2 \rangle^2 \leq \frac{2}{(t_2 - t_1)\nu^2k_0^2} \left| A^{-1/2}f_0^2 \right| + \frac{1}{\nu k_0^2} \langle \| u \|_2 \rangle^2 \]
\[ \leq \frac{2}{(t_2 - t_1)\nu^2k_0^2} \left| A^{-1/2}f_0^2 \right| \]
\[ \leq \frac{2}{(t_2 - t_1)\nu^2k_0^2} + \frac{\nu^2k_0^2}{\nu k_0^2} \nu^4 \kappa_0 G^{*2}. \]

Finally,
\[ \tilde{\epsilon} = \nu \kappa_0^3 \langle \| u \|_2 \rangle^2 \leq \left( \frac{2\nu\kappa_0^3}{\kappa_0(t_2 - t_1)} + \frac{\nu^3 \kappa_0^3 \kappa_f^2}{\nu k_0^2} \right) G^{*2}. \]

Using (3.11) we prove (7.2). Applying this estimate to the definition of the Kolmogorov wavenumber as \( \tilde{\kappa}_\epsilon = (\tilde{\epsilon}/\nu^3)^{1/4} \) completes the proof. \( \square \)

**Remark 7.1.** The estimate of Proposition 7.1 implies that by averaging on a sufficiently long time interval, namely with
\[ t_2 - t_1 \geq \frac{\tilde{\kappa}_\epsilon^2}{\nu k_0^2 \kappa_f^2}, \]
the Kolmogorov wavenumber is bounded as
\[ \tilde{\kappa}_\epsilon \lesssim \left( \frac{\kappa_0^4 \kappa_f^2 \tilde{\kappa}_\epsilon^2}{\nu k_0^2 \kappa_f^2} \right)^{1/4} G^{*1/2}. \]

This means that the estimate for the number of degrees of freedom of a turbulent flow is of order (see [7, 8, 17, 26, 37])
\[ \left( \frac{\tilde{\kappa}_\epsilon}{k_0} \right)^3 \lesssim \left( \frac{\kappa_f}{\tilde{\kappa}_\epsilon} \right)^{3/2} G^{*3/2}. \]

Then, if we consider the Kolmogorov theory and assume that \( \kappa_f \sim \kappa_0 \) and that \( \tilde{\kappa}_\epsilon \sim \kappa_0^{1/3} \tilde{\kappa}_\epsilon^{2/3} \) holds for the flow then
\[ \tilde{\kappa}_\epsilon \lesssim \left( \frac{\kappa_0}{\tilde{\kappa}_\epsilon} \right)^{1/3} G^{*1/2}. \]
so that

\[
\left( \frac{\tilde{\kappa}}{\kappa_0} \right)^{4/3} \lesssim G^{1/2}.
\]

In this case, the number of degrees of freedom can be better estimated as

\[
\left( \frac{\tilde{\kappa}}{\kappa_0} \right)^3 \lesssim G^{9/8}. \tag{7.5}
\]

In terms of the Reynolds number, however, there is no improvement. In fact, on the one hand we have using \( G^* \lesssim \tilde{\text{Re}}^2 \) (from (6.9)) and the previous estimate (7.5) that \( (\tilde{\kappa}/\kappa_0)^3 \lesssim (\text{Re})^{9/4}\), matching the heuristic estimate for the number of degrees of freedom of a turbulent flow. On the other hand, we already had from (6.3), without the assumptions \( \kappa_f \sim \kappa_0 \) and \( \tilde{\kappa}_\tau \sim \kappa_0^{1/3} \kappa_\tau^{2/3} \), that \( (\tilde{\kappa}/\kappa_0)^3 \lesssim (\text{Re})^{9/4}\).

Remark 7.2. In [19] an estimate similar to (7.5) was given without proof for the case of ensemble averages associated with infinite-time averages. The main difference was the power 9/7 instead of 9/8. Upon reworking the estimate in the present context we ended up improving the power.

Remark 7.3. Observe that the estimates in Proposition 7.1 and in Remark 7.1 hold also if we assume \( f \) in \( H \) and replace the characteristic wavenumber \( \kappa_f \) by \( |f|_0/A^{-1/2}f|_0 \).

8. The energy cascade

As mentioned in the Introduction, one of the most remarkable universal features of turbulence is the energy cascade process. The existence of an inertial range characterizes fully developed turbulent flows. The aim in this section is to give a general existence result for that range in spectral space. More precisely, we show that if the Taylor wavenumber \( \tilde{\kappa}_\tau \) is sufficiently large compared with the scale \( \kappa \) up to which the forcing term acts then the energy cascade holds, with energy being transferred to higher wavenumbers at a nearly constant rate \( \tilde{\epsilon} \), for wavenumbers \( \kappa \) within the range \( \kappa_0^2 \leq \kappa^2 \ll \tilde{\kappa}_\tau^2 \) (see Remark 8.1). The assumption that \( \tilde{\kappa}_\tau \) must be relatively large is consistent with the Kolmogorov theory of turbulence.

First, we need the following technical result.

Lemma 8.1. For \( \kappa_0 \leq \kappa' < \kappa'' < \infty \), we have the equation

\[
\frac{|u_{\kappa',\kappa''}(t_2)|_0^2 - |u_{\kappa',\kappa''}(t_1)|_0^2}{2(t_2 - t_1)} + \nu \langle \|u_{\kappa',\kappa''}\|^2 \rangle = \langle (f, u_{\kappa',\kappa''}) \rangle + \langle \epsilon_{\kappa'}(u) \rangle - \langle \epsilon_{\kappa''}(u) \rangle, \tag{8.1}\]
from which we infer that the following limit exists:

$$\langle \mathbf{e}(\mathbf{u}) \rangle_\infty \overset{\text{def}}{=} \lim_{\kappa \to \infty} \langle \mathbf{e}_\kappa(\mathbf{u}) \rangle_\infty,$$

and the following relations hold:

$$\langle \mathbf{e}(\mathbf{u}) \rangle_\infty \geq 0,$$

and

$$\frac{|\mathbf{u}_{\kappa',\kappa''}(t_2)|_0^2 - |\mathbf{u}_{\kappa',\kappa''}(t_1)|_0^2}{2(t_2 - t_1)} + \nu \langle \|\mathbf{u}_{\kappa',\kappa''}(t)\|^2 \rangle = \langle (\mathbf{f}, \mathbf{u}_{\kappa',\kappa''}) \rangle + \langle \mathbf{e}_\kappa(\mathbf{u}) \rangle - \langle \mathbf{e}(\mathbf{u}) \rangle_\infty. \quad (8.4)$$

Proof. Take the inner product in $H$ of the Navier-Stokes equations with $\mathbf{u}_{\kappa',\kappa''}$, which is the portion of $\mathbf{u}$ with wavenumber in the interval $[\kappa', \kappa'']$. We find that

$$\frac{1}{2} \frac{d}{dt}|\mathbf{u}_{\kappa',\kappa''}(t)|_0^2 + \nu \langle \|\mathbf{u}_{\kappa',\kappa''}(t)\|^2 \rangle = \langle (\mathbf{f}, \mathbf{u}_{\kappa',\kappa''}) \rangle + \langle \mathbf{e}_\kappa(\mathbf{u}) \rangle - \langle \mathbf{e}(\mathbf{u}) \rangle_\infty. \quad (8.5)$$

in the distribution sense. Here, we have an equality since by projecting the solution onto the space of finite wavenumbers in the interval $[\kappa', \kappa'']$, the resulted projection $\mathbf{u}_{\kappa',\kappa''}$ is smooth in the sense that its time derivative is square-integrable in time with values in $H$. In this case we have (see, e.g. [35, Chapter 3, Lemma 1.2]),

$$\frac{1}{2} \frac{d}{dt}|\mathbf{u}_{\kappa',\kappa''}(t)|_0^2 = \langle \frac{d}{dt}\mathbf{u}_{\kappa',\kappa''}(t), \mathbf{u}_{\kappa',\kappa''}(t) \rangle$$

in the distribution sense, and using the Navier-Stokes equations for $d\mathbf{u}_{\kappa',\kappa''}/dt$ we obtain (8.5).

Integrating (8.5) from $t_1$ to $t_2$ yields

$$\frac{1}{2} |\mathbf{u}_{\kappa',\kappa''}(t_2)|_0^2 - \frac{1}{2} |\mathbf{u}_{\kappa',\kappa''}(t_1)|_0^2 + \nu \int_{t_1}^{t_2} \|\mathbf{u}_{\kappa',\kappa''}(t)\|^2 \, dt$$

$$= \int_{t_1}^{t_2} \langle (\mathbf{f}, \mathbf{u}_{\kappa',\kappa''}(t)) \rangle \, dt - \int_{t_1}^{t_2} \langle b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{u}_{\kappa',\kappa''}(t)) \rangle \, dt.$$

Since $\mathbf{u}_{\kappa',\kappa''} = \mathbf{u}_{\kappa',\infty} - \mathbf{u}_{\kappa'',\infty}$, we can write

$$-b(\mathbf{u}, \mathbf{u}, \mathbf{u}_{\kappa',\kappa''}) = -b(\mathbf{u}, \mathbf{u}, \mathbf{u}_{\kappa',\infty}) + b(\mathbf{u}, \mathbf{u}, \mathbf{u}_{\kappa'',\infty}) = \mathbf{e}_{\kappa'}(\mathbf{u}) - \mathbf{e}_{\kappa''}(\mathbf{u}).$$

Thus,

$$\frac{1}{2} |\mathbf{u}_{\kappa',\kappa''}(t_2)|_0^2 - \frac{1}{2} |\mathbf{u}_{\kappa',\kappa''}(t_1)|_0^2 + \nu \int_{t_1}^{t_2} \|\mathbf{u}_{\kappa',\kappa''}(t)\|^2 \, dt$$

$$= \int_{t_1}^{t_2} \langle (\mathbf{f}, \mathbf{u}_{\kappa',\kappa''}(t)) \rangle \, dt + \int_{t_1}^{t_2} \langle \mathbf{e}_{\kappa'}(\mathbf{u}(t)) \rangle \, dt - \int_{t_1}^{t_2} \langle \mathbf{e}_{\kappa''}(\mathbf{u}(t)) \rangle \, dt.$$

Divide by $t_2 - t_1$ to find

$$\frac{|\mathbf{u}_{\kappa',\kappa''}(t_2)|_0^2 - |\mathbf{u}_{\kappa',\kappa''}(t_1)|_0^2}{2(t_2 - t_1)} + \nu \langle \|\mathbf{u}_{\kappa',\kappa''}\|^2 \rangle = \langle (\mathbf{f}, \mathbf{u}_{\kappa',\kappa''}) \rangle + \langle \mathbf{e}_{\kappa'}(\mathbf{u}) \rangle - \langle \mathbf{e}_{\kappa''}(\mathbf{u}) \rangle.$$
which proves (8.1)

We now solve this equation for $\langle \epsilon_{\kappa'}(u) \rangle_{\kappa'}$, and we see that all the other terms in the equation converge as $\kappa''$ goes to infinity. So this term must have a limit as $\kappa''$ goes to infinity, and we thus define

$$
\langle \epsilon(u) \rangle_{\kappa'} = \lim_{\kappa'' \to \infty} \langle \epsilon_{\kappa'}(u) \rangle_{\kappa''} = \lim_{\kappa'' \to \infty} \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \epsilon_{\kappa'}(u(t)) \, dt,
$$

which proves (8.2). Taking the limit as $\kappa'' \to \infty$ in the equation (8.1) yields

$$
\frac{|u_{\kappa',\infty}(t_2)|^2 - |u_{\kappa',\infty}(t_1)|^2}{2(t_2 - t_1)} + \nu \langle \|u_{\kappa',\infty}\|^2 \rangle = \langle (f, u_{\kappa',\infty}) \rangle + \langle \epsilon_{\kappa'}(u) \rangle - \langle \epsilon(u) \rangle_{\kappa'}.
$$

(8.6)

This proves (8.4)

Now, taking the inner product in $H$ of the Navier-Stokes equations with $u_{\kappa',\infty}$ yields

$$
\frac{1}{2} \frac{d}{dt} |u_{\kappa',\infty}|_0^2 + \nu \|u_{\kappa',\infty}\|^2 + b(u, u, u_{\kappa',\infty}) \leq (f, u_{\kappa',\infty}),
$$
in the distribution sense. Integrating from $t'_1$ to $t_2$, where $t'_1$ is a Lebesgue point of $t \mapsto |u(t)|_0^2$, with $t_1 \leq t'_1 < t_2$, yields

$$
\frac{1}{2} |u_{\kappa',\infty}(t_2)|_0^2 - \frac{1}{2} |u_{\kappa',\infty}(t'_1)|_0^2 + \nu \int_{t'_1}^{t_2} \|u_{\kappa',\infty}(t)\|^2 \, dt
$$

$$
= \int_{t'_1}^{t_2} (f, u_{\kappa',\infty}(t)) \, dt + \int_{t'_1}^{t_2} \epsilon_{\kappa'}(u(t)) \, dt.
$$

Now let $t'_1$ converge to $t_1$ and divide the result by $t_2 - t_1$ to find

$$
\left( \frac{u_{\kappa',\infty}(t_2)}{u_{\kappa',\infty}(t_1)} \right)_{t'=t_1}^t + \nu \langle \|u_{\kappa',\infty}\|^2 \rangle \leq \langle (f, u_{\kappa',\infty}) \rangle_1 + \langle \epsilon_{\kappa'}(u) \rangle_1.
$$

(8.7)

By comparing (8.4) with (8.7) we see that

$$
\langle \epsilon(u) \rangle_{\kappa'} \geq 0.
$$

(8.8)

which completes the proof.

\[ \square \]

**Theorem 8.1.** Suppose that

$$
f = f_{\kappa, \pi},
$$

(8.9)

for some $\kappa_0 \leq \pi < \infty$, and that (4.12) holds. Assume that $t_1$ satisfies (3.2) and that for some $\delta > 0$,

$$
t_2 - t_1 \geq \max \left\{ \frac{4\sqrt{2}}{\nu \kappa_0^2}, \frac{c_{\text{G}} G^2 s}{8\pi^3 \nu \kappa_0^2 (G^* - 1) \delta} \right\}.
$$

(8.10)

Then, for $\kappa \geq \pi$,

$$
1 - \left( \frac{\kappa}{\kappa_{\tau}} \right)^2 - \delta \leq \frac{k^2}{\tilde{c}} \langle \epsilon_{\kappa}(u) \rangle_{\kappa'} \leq 1 + \delta.
$$

(8.11)
where \( e^\kappa_*(u) = e_\kappa(u) - \langle e(u) \rangle_\infty \).

**Proof.** Using (8.9) in equation (8.4), with \( \kappa' = \kappa \), and using (3.6) and (3.11), we write

\[
\kappa^3_L \langle e^\kappa_*(u) \rangle = \frac{\kappa^3_L |u_{\kappa,\infty}(t_2)|^2_0}{2(t_2 - t_1)} - \frac{\kappa^3_L |u_{\kappa,\infty}(t_1)|^2_0}{2(t_2 - t_1)} + \nu \kappa^3_L \langle \|u_{\kappa,\infty}\|^2 \rangle^{-}
\]

\[
\geq - \frac{\nu^2 \kappa^2_0 G^{*2}}{8 \pi^3 (t_2 - t_1)} + \nu \kappa^3_L \langle \|u\|^2 \rangle^{-} - \nu \kappa^3_L \langle \|u_{\kappa,\infty}\|^2 \rangle^{-}
\]

\[
\geq - \frac{\nu^2 \kappa^2_0 G^{*2}}{8 \pi^3 (t_2 - t_1)} + \bar{\epsilon} - \nu \kappa^3_L \langle \|u_{\kappa,\infty}\|^2 \rangle^{-}
\]

\[
\geq - \frac{\nu^2 \kappa^2_0 G^{*2}}{8 \pi^3 (t_2 - t_1)} + \bar{\epsilon} - \nu \kappa^3_L \frac{\kappa^2_0}{\kappa_\tau^2} \langle \|u\|^2 \rangle^{-}.
\] (8.12)

Thus,

\[
\kappa^3_L \langle e^\kappa_*(u) \rangle^{-} \geq \left(1 - \left(\frac{\kappa}{\kappa_\tau}\right)^2\right) \bar{\epsilon} - \frac{\nu^2 \kappa^2_0 G^{*2}}{8 \pi^3 (t_2 - t_1)}. \tag{8.13}
\]

From the estimate (4.11) in Proposition 4.2 we have

\[
G^* - 1 \leq \frac{c_6 \bar{\epsilon}}{\nu^3 \kappa^3_0},
\]

so that using assumption (4.12) we find

\[
1 \leq \frac{c_6 \bar{\epsilon}}{\nu^3 \kappa^3_0 (G^* - 1)}.
\]

Then, under assumption (8.10) we obtain

\[
\frac{\nu^2 \kappa^2_0 G^{*2}}{8 \pi^3 (t_2 - t_1)} \leq \frac{c_6 \bar{\epsilon} G^{*2}}{8 \pi^3 \nu \kappa^2_0 (G^* - 1)(t_2 - t_1)} \leq \delta \bar{\epsilon}.
\]
On the other hand,
\[
\kappa_3^2 \langle L_\epsilon^* (\epsilon_\kappa^* (u)) \rangle = \frac{\kappa_3^2 |u_\kappa,\infty (t_2)|^2}{2(t_2 - t_1)} - \frac{\kappa_3^2 |u_\kappa,\infty (t_1)|^2}{2(t_2 - t_1)} + \nu \kappa_3^3 \langle \|u_\kappa,\infty\|_2^2 \rangle
\]
\[
\leq \frac{\nu^2 \kappa_3^3 G^* v^2}{\kappa_0 (t_2 - t_1)} + \nu \kappa_3^3 \langle \|u\|_2^2 \rangle
\]
\[
\leq \frac{\nu^2 \kappa_3^3 \pi^2}{8 \pi^2 (t_2 - t_1)} + \nu \kappa_3^3 \langle \|u\|_2^2 \rangle
\]
\[
\leq \delta \hat{\epsilon} + \hat{\epsilon}.
\]

Putting this estimate together with (8.13) yields
\[
\left(1 - \left(\frac{\kappa}{\kappa_\tau}\right)^2 - \delta\right) \hat{\epsilon} \leq \kappa_3^3 \langle \epsilon_\kappa^* (u) \rangle \leq (\delta + 1) \hat{\epsilon}.
\]

Divide by \(\hat{\epsilon}\) to find (8.11) and conclude the proof. \(\square\)

**Remark 8.1.** The previous theorem can be interpreted in the following sense: With a forcing term acting up to some wavenumber \(\kappa\) and for a sufficiently long time interval \([t_1, t_2]\), it follows from (8.11) that if \(\kappa_\tau^2 \gg \pi^2\), then \(\kappa_3^3 \langle \epsilon_\kappa^* (u) \rangle \approx \hat{\epsilon}\) within the range \(\kappa^2 \leq \kappa_\tau^2 \ll \kappa^2\). This means that the energy flux to higher modes within the above range occurs at a nearly constant rate which is close to the mean energy dissipation rate \(\hat{\epsilon}\), a phenomenon known as the energy cascade. The assumption that \(\kappa_\tau^2 \gg \pi^2\) is a natural one since \(\kappa_\tau\) is the inverse of the Taylor microscale, which in turbulent flows is expected to be small compared with the large scales driving the flow, which we take here to be bounded below by the inverse of \(\pi\).

**Remark 8.2.** The assumption (4.12) in Theorem 8.1 is not a strong restriction. In fact, for turbulent flows \(G^*\) is expected to be large. One way to see this is through the estimate (6.10) for \(\hat{\kappa}^*\). Since \(\hat{\kappa}^*\) is large for turbulent flows, as expected on heuristic as well as empirical grounds, \(G^*\) must also be large.

**Remark 8.3.** Observe the use in the proof of Theorem 8.1 of the estimate (4.11), which essentially bounds one power of \(G^*\) by \(\hat{\epsilon}\). In the terms involving the time difference \(t_2 - t_1\) in (8.12) and (8.14) there are two powers of \(G^*\) coming from the bound on the kinetic energy at times \(t_1\) and \(t_2\). Since these terms are expected to be small relative to \(\hat{\epsilon}\) the use of (4.11) introduces \(\hat{\epsilon}\) along with a \(G^* - 1\) in the denominator. Hence, for large \(G^*\), it suffices to bound \(t_2 - t_1\) in terms of a single power of \(G^*\).
Remark 8.4. The result of Theorem 8.1 can be extended to include forces which act on higher wavenumbers as long as they are mostly concentrated on low wavenumbers. More precisely, given $\delta > 0$, and choosing $t_1$ and $t_2$ as in Theorem 8.1, we defined $\kappa_c$ as the lowest wavenumber for which
\[
|\kappa^3_L(\langle (f, u_{\kappa, \infty}) \rangle)| \leq \delta \tilde{\epsilon}, \quad \text{for } \kappa \geq \kappa_c. \tag{8.15}
\]
For $\delta$ small, this characterizes $\kappa_c$ as a wavenumber below which most of the energy injection occurs. Similarly, we define $\kappa_c$ as the highest wavenumber for which
\[
\kappa^3_L(\|u_{\kappa_c, \infty}\|^2) \geq (1 - \delta)\tilde{\epsilon}. \tag{8.16}
\]
This characterizes $\kappa_c$ as a wavenumber above which most of the energy dissipation occurs. If $\kappa_c \leq \kappa_c$, then within the range $\kappa_c \leq \kappa \leq \kappa_c$ it follows that $\kappa^3_L(\langle e_\kappa(u) \rangle) \approx \tilde{\epsilon}$. This follows from reworking estimates (8.12) and (8.14) to account for the forcing terms, which do not vanish in this case, and from using (8.16) for the lower bound in (8.12) without resorting to $\kappa_\tau$. The range $[\kappa_c, \kappa_c]$ is the energy cascade range. These definitions are the finite-time average versions of the corresponding definitions for ensemble averages first introduced in [32].

Remark 8.5. Consider now the definition of $\kappa_c$ and $\kappa_c$ given in Remark 8.4. Theorem 8.1 asserts that in the case $f = f_{\kappa_0, \kappa}$, then $\kappa_c \leq \kappa$ and $\kappa_c \geq \delta^{1/2}\kappa_\tau$. Thus, the condition $\tilde{\kappa}_\tau^2 \gg \tilde{\kappa}^2$, which reads rigorously as $\tilde{\kappa}^2/\tilde{\kappa}_\tau^2 \leq \delta$, implies that $\kappa_c \geq \kappa_c$, so that the energy cascade range exists. Now, if $\tilde{\kappa}_\tau^2 \gg \tilde{\kappa}_f^2$, which reads rigorously as $\tilde{\kappa}_f^2/\tilde{\kappa}_\tau^2 \leq \delta$, then $\kappa_c/\kappa_c \leq \delta$, which means that $\kappa_c \gg \kappa_c$, so that a wide energy cascade range exists. In other words, the condition $\tilde{\kappa}_\tau^2 \gg \tilde{\kappa}^2$ implies the existence of the energy cascade range, while the condition $\tilde{\kappa}_\tau^2 \gg \tilde{\kappa}_f^2$ implies the existence of a wide energy cascade range. The order of precision in those relations is determined by $\delta$.

References


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