CONVECTION-DIFFUSION EQUATIONS IN A CIRCLE: THE COMPATIBLE CASE

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Abstract. In this article we aim to study the boundary layer generated by a convection-diffusion equation in a circle. In the model problem that we consider two characteristic points appear. To the best of our knowledge such boundary layer problems have not been studied in a systematic way yet and we indeed know that very complex situations can occur. In the cases that we consider in the present article certain simplifying compatibility conditions are assumed. Other situations will be studied in forthcoming articles which involve noncompatible data, more general domains or higher order operators.

1. Introduction

Our aim in this article is to study singularly perturbed problems of the form

\[
\begin{align*}
L_\varepsilon u^\varepsilon &:= -\varepsilon \Delta u^\varepsilon - u^\varepsilon_y = f(x,y) \text{ in } D, \\
u^\varepsilon &= 0 \text{ on } \partial D,
\end{align*}
\]

where \(0 < \varepsilon \ll 1\), \(D\) is the unit disk with center \((0,0)\), and the function \(f\) is assumed to be as smooth as needed. Here we note that \((-1,0)\) are the characteristic points for the limit operator, that is when \(\varepsilon = 0\).

We denote the upper and lower half parts of the unit circle by \(C_u(x) = \sqrt{1-x^2}\) and \(C_l(x) = -\sqrt{1-x^2}\), respectively. The limit problem (i.e. when \(\varepsilon = 0\)) is then formally defined by

\[
\begin{align*}
-u^0_y &= f(x,y) \text{ in } D, \\
u^0 &= 0 \text{ on } \Gamma_u,
\end{align*}
\]

where \(\Gamma_u = \{(x,y)\mid x^2 + y^2 = 1, y > 0\}\). The choice of the boundary condition (1.2) \((u^0 = 0\) on \(\Gamma_u\)) rather than say, \(u^0 = 0\) on \(\Gamma_l\)) is justified by the convergence theorems below (see e.g. Theorem 4.1). The solution of (1.2) is explicitly found

\[
u^0(x,y) = \int_y^{C_u(x)} f(x,s)ds, \ (x,y) \in D.
\]

The convergence of \(u^\varepsilon\) to \(u^0\) in \(L^2(D)\), as \(\varepsilon \to 0\), has been studied in the context of the linear semigroup theory by C. Bardos [Ba70]. N. Levinson [Le50] and W. Eckhaus and E. M. de Jager [EJ66] study the boundary layers generated by (1.1) when \(\varepsilon\) is small but these results are not valid in the vicinity of the characteristic points. In [EJ66] the authors announced future work explaining the boundary layers in the vicinity of the characteristic points, but we did not find any subsequent article by these authors. More recently, Temme [Te07] studies the same problem as (1.1) in the same domain \(D\); he expresses the solutions using the Bessel functions and approximate the solutions and the boundary layers by approximating the Bessel functions. We are not aware of
any other work related to such problems and hence a large number of questions remain unanswered to the best of our knowledge. In this article, and as explained below, we first want to study the level of regularity of $u^0$ which depends on the existence or not of some compatibility properties of the data ($f$ and $D$). Depending on the compatibility properties of the data, one will want to study the boundary layers generated by this singular perturbation problem, similar to the Prandtl boundary layer of fluid mechanics. There is a vast literature available on singular perturbation problems for partial differential equations, abstract elliptic or parabolic equations, or equations related to fluid mechanics; see e.g. [Ba88], [DR04], [Ec72], [Li73], [Ol66], [Oma08], [Ma05], [SK87], [TW02], and [VL57]. The difficulties generated by the type of singular perturbation problems under consideration in this article are different and have some analogies with problems in optics and acoustics that have been studied with various levels of mathematical rigor; see e.g. [ABM05], [BL10], [BM94], [KR60], [MAB05], and the references therein. Our aim in this article and in forthcoming works is to combine some of the tools developed in these articles with the concept of correctors and matched boundary layers as studied in e.g. [Li73], [JT05], [JT07].

This article is organized as follows. In Section 2 we study the regularity of $u^0$; we introduce certain compatibility conditions satisfied by $f$ and $D$, and their effect on the regularity of $u^0$. In Sections 3 and 4 we consider cases where some compatibility conditions are satisfied. In Section 3 we introduce the corrector for problem (1.1) and study its regularity. In Section 4 we infer some convergence results for the difference between $u^\varepsilon$, $u^0$ and the correctors.

Before we proceed we introduce here the full formal asymptotic expansion of $u^\varepsilon$, $u^\varepsilon \sim \sum_{j=0}^{\infty} \varepsilon^j w^j$, also known as the outer expansion of $u^\varepsilon$ in boundary layer theory. Inserting this expansion in (1.1), we obtain, for $j = 0, 1, \cdots$,

$$
\begin{align*}
-u^j_y & = \Delta u^{j-1} \text{ in } D, \\
   u^j & = 0 \text{ on } \Gamma_u.
\end{align*}
$$

(1.4)

Here, for convenience, we have written $\Delta u^{-1} = f(x, y)$. The solutions $w^j$ of (1.4) vanishing on $C_u$ are easily found

$$
u^j(x, y) = \int_y^{C_u(x)} \Delta u^{j-1}(x, s)ds, \ (x, y) \in D.
$$

(1.5)

Our aim in Section 2 is to study the regularity of the $u^j$. A convenient expression of $w^j$ is given in Lemma 2.2.

We conclude this Introduction with two additional remarks. Firstly we observe that unlike earlier works on the subject and in particular [Le50] we do not use the maximum principle in this article, so that our methods have the potential for extension to systems and higher order equations which do not admit a maximum principle. Secondly there is some remote connection between the problem considered here and the issue of the Hamilton Jacobi equations considered by P. L. Lions and others (see e.g. [CL83], [Li83]). However unlike the case of the Hamilton Jacobi equations the problems studied here are linear and the solution to the inviscid limit equation is unique.
2. Compatibility conditions and the regularity of the $u^j$

2.1. First compatibility conditions. We notice that the tangential component of $\nabla u^0$ on $\Gamma_u$ vanishes, so that, as $(x, y) \to (\pm 1, 0)$, with $(x, y) \in \Gamma_u$, the limit of $u^0_y(x, y)$ will be zero. However, if $f(x, y) \neq 0$ at $(\pm 1, 0)$, this is incompatible with (1.2) and we expect certain singularities near those points. Hence, to avoid such singularities, it is natural to assume, in a first step, that $f$ satisfies compatibility conditions of the following type, for some $m \geq 0$:

$$
(2.1) \quad \frac{\partial^k f}{\partial y^k} = 0 \text{ at } (\pm 1, 0), \ k = 0, 1, \ldots, m;
$$

$m$ will be specified when needed. From (1.3) we note that $\partial_l u^0/\partial y_l(\pm 1, 0) = 0$ for $l = 0, 1, \ldots, m+1$, when (2.1) is satisfied.

As we said the compatibility conditions (2.1) are meant to prevent some singularities for $u^0$. Their role appears when we try to estimate the derivatives of $u^0$. We impose the compatibility condition (2.1) with $m=0$ and using the fact that $C_u'(x) = -xC_u^{-1}(x)$, we find that $u^0_y$ is bounded near $(x, y) = (\pm 1, 0)$ and thus $|u^0_y|_{L^2} \leq \kappa$.

Continuing the estimates we first find that $|u^0_y|_{L^2} \leq |f_y|_{L^2} \leq \kappa$. To estimate $u^0_{xx}$, we differentiate (2.2) in $x$ and we obtain

$$
(2.3) \quad u^0_{xx} = 2f_x(x, C_u(x))C_u'(x) + f_y(x, C_u(x))(C_u'(x))^2
$$

Assuming the compatibility condition (2.1) with $m = 0$ and using the fact that $C_u'(x) = -xC_u^{-1}(x)$, we first obtain that $u^0 \in C(\bar{D})$ and $|u^0_{yy}|_{L^2} = |f|_{L^2} \leq \kappa$. To estimate $u^0_{xy}$, we differentiate (2.2) in $x$ and $y$ and we obtain

$$
(2.4) \quad \left| \int_D \frac{w}{C_u(x)} \, dx \right| \leq \kappa \left( \int_D \frac{w}{1-x^2} \, dx \right)^{\frac{1}{2}}
$$

and using $C_u'(x) = -xC_u^{-1}(x)$ we then find

$$
(2.5) \quad \left| \int_D u^0_{xy} \, w \, dx \right| \leq \kappa |w|_{L^2(D)}
$$
Remark 2.1. Notice from the limit problem (1.4), due to the boundary conditions \( u^j = 0 \) on \( \Gamma_u \), that \( 0 = -u^j_y = \Delta u^{j-1} \) at \((\pm 1, 0)\). Since \( u^{j-1}_{yy} \) is already zero, it is required that \( u^{j-1}_{xx} = 0 \) at \((\pm 1, 0)\). Hence, we need more compatibility properties in the \( x \)-direction to attain smoothness of the high order terms \( u^j, j \geq 1 \) as shown in (2.6) below.

2.2. More compatibility conditions. In order to derive additional regularity properties of the \( u^j \), we will need the following type of compatibility conditions:

\[
\frac{\partial^{p+q} f}{\partial x^p \partial y^q} = 0 \quad \text{at} \quad (\pm 1, 0), \quad 0 \leq 2p + q \leq 1 + 3j, \quad p, q \geq 0.
\]

Note that the conditions (2.6) are similar to the conditions (2.1) when \( p = 0 \) and \( q = k, \cdots, m \); however, the conditions (2.1) with \( k = 0 \) is not included in (2.6) and will usually produce a slightly different analysis.

The regularity properties resulting from such compatibility conditions will be clear after the following two technical lemmas.

Lemma 2.1. We assume that

\[
\frac{\partial^{p+q} g(x, y)}{\partial x^p \partial y^q} = 0 \quad \text{at} \quad (\pm 1, 0), \quad 0 \leq 2\alpha + \beta \leq \gamma - 1, \quad \gamma \geq 1, \quad \alpha, \beta \geq 0,
\]

where \( g(x, y) \) belongs to \( C^\gamma(\bar{D}) \) and \( D \) is as in (1.1), i.e. the unit disk. Then the following function

\[
g(x, C_u(x))
\]

is bounded for all \( x \in (-1, 1) \).

Proof. It suffices to show that \( \frac{g(x, C_u(x))}{C_u'(x)} \) is bounded (has a finite limit) as \( x \to 1^- \). Since the case \( x = 1^- \) is similar, we just consider the limit as \( x \to 1^- \). To prove that \( \lim_{x \to 1^-} \frac{g(x, C_u(x))}{C_u'(x)} \) is bounded, we proceed by induction on \( m \), and use L’Hopital’s rule and the fact that \( C_u'(x) = -xC_u^{-1}(x) \). For \( m = 1 \), we just observe that\(^1\)

\[
\lim_{x \to 1^-} \frac{g(x, C_u(x))}{C_u'(x)} = \lim_{x \to 1^-} \frac{g_x(x, C_u(x)) + g_y(x, C_u(x))C_u'(x)}{C_u'(x)} = \lim_{x \to 1^-} g_y(x, C_u(x)).
\]

Assuming that the result holds for \( \gamma \leq k, k \geq 1 \), we then verify the claim for \( \gamma = k + 1 \) observing that

\[
\lim_{x \to 1^-} \frac{g(x, C_u(x))}{C_u^{k+1}(x)} = -\lim_{x \to 1^-} \frac{g_x(x, C_u(x))}{(k+1)C_u^{k-1}(x)} + \lim_{x \to 1^-} \frac{g_y(x, C_u(x))}{(k+1)C_u^k(x)}.
\]

Since \( \gamma \) is replaced by \( k + 1 \), we are assuming that \( g \in C^{k+1}(\bar{D}) \) and (2.7) holds for \( 0 \leq 2\alpha + \beta \leq k - 1 \) and \( g_x \) belongs to \( C^{k}(\bar{D}) \) and satisfies (2.7) for \( 0 \leq 2\alpha + \beta \leq k - 2 \). It follows from the induction assumption that each of the terms in the right-hand side of (2.10) has a finite limit as \( x \to 1^- \), and thus so does the term in the left-hand side. The lemma is proved. \( \square \)

\(^1\)We use L’Hopital’s rule in the form \( \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f(x)}{g(x)} \), with \( f, g \in C^1 \), \( f(a) = g(a) = 0 \) and \( g'(a) = 0 \).
Our aim in the next lemma is to derive a suitable form of the \( u^j \), their derivatives and their primitives. Then, using some of the compatibility conditions (2.1) and (2.6), we will infer regularity properties of the \( u^j \) such as (2.16) below. We first introduce the notation:

\[
\frac{\partial^{-1}g}{\partial y^{-1}}(x,y) = \int_y^C u(x)g(x,s)ds;
\]

so that for instance

\[
u^0 = \frac{\partial^{-1}f}{\partial y^{-1}}.
\]

However, this notation should be used in a careful way because \( \frac{\partial^{-1}}{\partial y^{-1}} \circ \frac{\partial}{\partial x} \) is not the identity and \( \frac{\partial^{-1}}{\partial y^{-1}} \) does not commute with \( \frac{\partial}{\partial x} \); more precisely,

\[
\frac{\partial^{-1}}{\partial y^{-1}} \left[ \frac{\partial}{\partial x} g \right](x,y) = g(x,Cu(x)) - g(x,y).
\]

With this in mind we have the following technical lemma whose proof is deferred to the Appendix; see Sec. 5.1.

**Lemma 2.2.** For all \( i, j \geq 0 \) and \( m \in \mathbb{Z} \), the \( u^j \) and their derivatives are expressed as follows:

\[
\left\{ \frac{\partial^m}{\partial y^m} \left[ \frac{\partial^i u^j}{\partial x^i} \right] \right\}(x,y) = \sum_{l+s \leq i+2j, l,s \geq 0} g^j_{lims}(x,y) \frac{\partial^{j+s} f}{\partial x^l \partial y^s}(x,Cu(x))
\]

\[
+ \sum_{k=0}^{i} c^j_{kim} \left\{ \frac{\partial^{m-j+2k-1}}{\partial y^{m-j+2k-1}} \left[ \frac{\partial^{j+2j-2k} f}{\partial x^{2j+2k-2k}} \right] \right\}(x,y),
\]

where

\[
\left| \frac{\partial^{m-q}}{\partial x^{l} \partial y^{q}} g^j_{lims}(x,y) \right| \leq \kappa |Cu(x)|^{-(-1+3j+2i-2l-s)}, \ \forall r, q \geq 0.
\]

Furthermore, if \( m \geq 0 \), \( g^j_{lims}(x,y) = g^j_{lims}(x) \), and the coefficients \( c^j_{kim} \), \( \kappa \) are constants, \( \kappa \) depending on \( r, j, l, i, m, s, q \).

Note that \( \frac{\partial^m}{\partial y^m} \left[ \frac{\partial^{i} u^j}{\partial x^{i}} \right] = \frac{\partial^{i} u^j}{\partial x^{i}} \frac{\partial^m}{\partial y^m} \) for \( m \geq 0 \) and the second sum in the right-hand side of (2.13a) is then regular and bounded in this case.

We now show how Lemmas 2.1 and 2.2 can be used to derive regularity properties and a priori estimates when we assume some compatibility conditions. In the a priori estimates in (4.23) below, we will encounter the term \( \int_D \Delta u^j \text{w} dxdy \). Let us estimate it (as we did in (2.3) and see what compatibility conditions are needed.

Setting \( i = 2, m = 0 \) or \( i = 0, m = 2 \) in Lemma 2.2, \( w \in H_0^1(D) \) we find that

\[
\left| \int_D \Delta u^j \text{w} dxdy \right| \leq \kappa |w|_{L^2(D)}
\]

\[
+ \kappa \sum_{l+s \leq 2j+1, l,s \geq 0} \left| \int_D \frac{\partial^{j+s} f}{\partial x^{l} \partial y^{s}}(x,Cu(x))Cu(x)^{-(2+3j-2l-s)} \cdot Cu(x)^{-1} \text{w} dxdy \right|,
\]
Here, we would like to bound the function $\frac{\partial^{p+q} f}{\partial x^p \partial y^q}(x, C_u(x))C_u(x)^{-(2+3j-2l-s)}$. For this, thanks to Lemma 2.1, we just require that $\frac{\partial^{p+q} f}{\partial x^p \partial y^q}(\frac{\partial^{l+s} l}{\partial x^l \partial y^s}) = 0$ at $(\pm 1, 0)$, $0 \leq 2p + q \leq 1 + 3j - 2l - s$, which is guaranteed by the following compatibility conditions

\begin{equation}
\frac{\partial^{p+q} f}{\partial x^p \partial y^q} = 0 \text{ at } (\pm 1, 0), \quad 0 \leq 2p + q \leq 1 + 3j - 2l - s, \quad p, q \geq 0. \tag{2.15}
\end{equation}

Hence, applying the Hardy inequality as in (2.4) we obtain under the assumption (2.15):

\begin{equation}
\left| \int_D \Delta u^j w \, dx \, dy \right| \leq \kappa |w|_{L^2(D)} + \kappa |w_x|_{L^2(D)}^\frac{1}{2} |w|_{L^2(D)}^\frac{3}{2}. \tag{2.16}
\end{equation}

Remark 2.2. Further regularity properties for $u^j$ on the boundary $\partial D$, i.e. for $u^j(\cos \eta, \sin \eta)$, will appear in Section 3 when we introduce the boundary fitted coordinates (see Lemmas 3.1 and 3.2 below).

Remark 2.3. An important remark is in order here: when the compatibility conditions (2.1), (2.6) are not satisfied, we expect $u^0$ and the $u^j$ to display various singularities at the characteristic points $(\pm 1, 0)$, which will make more difficult the study of the boundary layer problem; see the first results in this direction in [JT11].

Now our aim in Section 3 is to introduce the boundary fitted coordinates and study the correctors (boundary layer solutions) for our problem. The resulting convergence results are derived in Section 4.

3. Boundary fitted coordinates and the correctors

3.1. Boundary fitted coordinates. Let $\xi$ be the distance to the boundary $\partial D$ counted positively in the inward normal direction and let $\eta$ be the arc length of $\partial D$ starting at $(x, y) = (1, 0)$.

We will use the boundary-fitted coordinates, $x = (1 - \xi) \cos \eta, \quad y = (1 - \xi) \sin \eta, \quad \xi = 1 - r$ where $r$ is the distance to the center $(0, 0)$ and $\eta$ is the polar angle from $Ox$. Hence $D$ is mapped onto the domain:

\begin{equation}
D^* = \{(\xi, \eta) \in (0, 1) \times (0, 2\pi)\}. \tag{3.1}
\end{equation}

We have

\begin{equation}
\frac{\partial}{\partial x} = -\cos \eta \frac{\partial}{\partial \xi} + \sin \eta \frac{\partial}{1 - \xi \frac{\partial}{\partial \eta}}, \quad \frac{\partial}{\partial y} = -\sin \eta \frac{\partial}{\partial \xi} + \cos \eta \frac{\partial}{1 - \xi \frac{\partial}{\partial \eta}}, \tag{3.2}
\end{equation}

and we can transform the differential operators of (1.1) to:

\begin{equation}
L_\varepsilon u^\varepsilon = -\varepsilon \Delta u^\varepsilon - u^\varepsilon_y \tag{3.3}
\end{equation}

\begin{align*}
= -\varepsilon \frac{\varepsilon^2 u^\varepsilon}{(1 - \xi)^2 \frac{\partial}{\partial \eta}^2} + \varepsilon \frac{\partial u^\varepsilon}{1 - \xi \frac{\partial}{\partial \eta}} - \varepsilon \frac{\partial^2 u^\varepsilon}{\xi^2 \frac{\partial}{\partial \xi}} + \sin \eta \frac{\partial u^\varepsilon}{\partial \xi} - \cos \eta \frac{\partial u^\varepsilon}{1 - \xi \frac{\partial}{\partial \eta}}.
\end{align*}

We construct a corrector $\theta^0$ which corrects the boundary values at $\xi = 0, \pi < \eta < 2\pi$. Considering the stretched variable $\xi = \xi/\varepsilon$ we identify the dominating differential operators and we are
Figure 1. The thickness of the boundary layers $\theta^j$ in the circle domain $D$ which are measured by $\varepsilon /(-\sin \eta)$, $\pi < \eta < 2\pi$ with (a) $\varepsilon = 10^{-1.5}$, and (b) $\varepsilon = 10^{-2}$.

led to the following equation for the first corrector $\theta^0$:

$$
\begin{align*}
-\frac{\partial^2 \theta^0}{\partial \xi^2} + \sin \eta \frac{\partial \theta^0}{\partial \xi} &= 0, \quad \text{for } 0 < \tilde{\xi} < \infty, \quad \pi < \eta < 2\pi, \\
\theta^0 &= -u^0(\cos \eta, \sin \eta) \text{ at } \tilde{\xi} = 0, \\
\theta^0 \to 0 &\text{ as } \tilde{\xi} \to \infty.
\end{align*}
$$

(3.4)

Hence we are able to obtain an explicit solution:

$$
\theta^0 = -u^0(\cos \eta, \sin \eta) \exp \left( \frac{\sin \eta}{\varepsilon} \tilde{\xi} \right) \chi_{[\pi, 2\pi]}(\eta),
$$

(3.5)

where $\chi_A$ is the characteristic function of $A$. Using a cut-off function we write an approximate form of $\theta^0$:

$$
\theta^0 = -u^0(\cos \eta, \sin \eta) \exp \left( \frac{\sin \eta}{\varepsilon} \tilde{\xi} \right) \delta(\xi) \chi_{[\pi, 2\pi]}(\eta),
$$

(3.6)

where $\delta(\xi)$ is a smooth cut-off function such that $\delta(\xi) = 1$ for $\xi \in [0, 1/4]$ and $0$ for $\xi \in [1/2, 1]$.

Since $\theta^0$ vanishes like $u^0$ at $\eta = \pi, 2\pi$, $\theta^0$ is continuous and piecewise smooth on $D$, and thus we conclude that $\theta^0, \bar{\theta}^0 \in H^1(D)$. From (1.3) we note here that

$$
u^0(\cos \eta, \sin \eta) = \int_{-\sin \eta}^{\sin \eta} f(\cos \eta, s) ds, \quad \pi < \eta < 2\pi,
$$

(3.7)

and $u^0 + \bar{\theta}^0 \in H^1_0(D)$.

3.2. Higher order correctors. As we did in (3.4), we now introduce the boundary layer correctors $u^\varepsilon \sim \sum_{j=0}^\infty \varepsilon^j \theta^j$. Using the facts that $(1 - \xi)^{-1} = \sum_{k=0}^\infty \xi^k$, $(1 - \xi)^{-2} = \sum_{k=0}^\infty (k + 1)\xi^k$ with
\(\xi = \xi \epsilon, \bar{\xi} = O(1)\), we rewrite the differential operators (3.3):

\[
- \epsilon \Delta u^r - u_y^r \sim - \epsilon \left( \sum_{l=0}^{\infty} (l+1) \epsilon \xi^l \right) \sum_{k=0}^{\infty} \frac{\epsilon^k \theta^k}{\eta^2} + \epsilon \left( \sum_{l=0}^{\infty} \epsilon^l \xi^l \right) \sum_{k=0}^{\infty} \epsilon^{k-1} \frac{\partial \theta^k}{\partial \xi} - \epsilon \sum_{j=0}^{\infty} \epsilon^{j-2} \frac{\partial^2 \theta^j}{\partial \xi^2} + \sin \eta \sum_{j=0}^{\infty} \epsilon^{j-1} \frac{\partial \theta^j}{\partial \xi} - \cos \eta \left( \sum_{l=0}^{\infty} \epsilon^l \xi^l \right) \sum_{k=0}^{\infty} \epsilon^k \frac{\partial \theta^k}{\partial \eta}
\]

(3.8)

Balancing at each order of \(\epsilon^j\) with \(\bar{\xi} = \xi / \epsilon, \bar{\xi} = O(1)\), we deduce that, for \(0 < \bar{\xi} < \infty, \pi < \eta < 2\pi, j = 0, 1, \cdots\),

\[
\left\{ \begin{array}{l}
- \frac{\partial^2 \theta^j}{\partial \xi^2} + \sin \eta \frac{\partial \theta^j}{\partial \xi} = \sum_{k=0}^{\infty} \left( \sum_{k=0}^{\infty} (j-k-1) \xi^{j-k-2} \frac{\partial^2 \theta^k}{\partial \eta^2} - \sum_{k=0}^{\infty} \xi^{j-k-1} \frac{\partial \theta^k}{\partial \xi} + \cos \eta \sum_{k=0}^{\infty} \xi^{j-k-1} \frac{\partial \theta^k}{\partial \eta} \right),
\theta^j = v^j(\eta) := -u^j(\cos \eta, \sin \eta) \text{ at } \bar{\xi} = 0,
\theta^j \to 0 \text{ as } \bar{\xi} \to \infty.
\end{array} \right.
\]

(3.9)

Knowing \(\theta^0\) as in (3.5) we inductively obtain \(\theta^j\) in the form \(\theta^j = \theta^j_o + \theta^j_p\) where \(\theta^j_o\) is the solution of Eq. (3.9) where the right-hand side of (3.9)_1 is replaced by 0 and \(\theta^j_p\) is the solution of (3.9) with (3.9)_1 replaced by \(v^j = 0\) at \(\bar{\xi} = 0\). The solutions \(\theta^j_o\) for each \(j\) are easily found, as for \(\theta^0\):

(3.10) \(\theta^j_o = v^j(\eta) \exp \left( (\sin \eta) \xi \right) \chi_{[\pi, 2\pi]}(\eta), \text{ for } 0 \leq \bar{\xi} < \infty,\)

where \(v^j(\eta) = -u^j(\cos \eta, \sin \eta)\). Obtaining a particular solution \(\theta^j_p\) of (3.10), we find inductively that the correctors \(\theta^j = \theta^j_o + \theta^j_p\) have the following form:

**Lemma 3.1.** The boundary layer correctors \(\theta^j\) are explicitly given by:

(3.11) \(\theta^j = P^j(\eta, \bar{\xi}) \exp \left( (\sin \eta) \xi \right) \chi_{[\pi, 2\pi]}(\eta), j \geq 0,\)

where

(3.12) \(P^j(\eta, \bar{\xi}) = \sum_{i=0}^{j} \sum_{k=0}^{2j-2i} a_{i,3j-3i-k}(\eta) \bar{\xi}^k,\)

(3.13) \(a_{i,q}(\eta) = \sum_{m+r \leq q, m,r \geq 0} \frac{c_{m,r}}{\sin^m \eta} \frac{d^r v^i(\eta)}{d\eta^r},\)

and \(v^j(\eta) = -u^j(\cos \eta, \sin \eta)\). Here the coefficients \(c_{m,r} = c_{m,r}(\eta) \in C^\infty([0, 2\pi])\) may be different at different occurrences.

Note that \(P^j(\eta, \bar{\xi})\) is a polynomial in \(\bar{\xi}\) of degree \(2j\) with coefficients depending only on \(\eta\).

We defer the proof of this lemma to the Appendix; see Sec. 5.2.
In the sequel we are not interested in the precise form of the $a_{i,q}$ but in their level of singularity which comes from the negative powers of $\sin \eta$ and some compensations coming possibly from the $c_{m,r} = c_{m,r}(\eta)$ and the functions $d^r v^i/d\eta^r(\eta)$. Note that these functions are themselves smooth functions of $\eta$ and that, as indicated above, the functions $c_{m,r}$ may be different at different occurrences.

For example, we have

$$
\frac{d}{d\eta} a_{i,q}(\eta) = \sum_{m+r \leq q, m,r \geq 0} \frac{c_{m,r}(\eta)}{(\sin \eta)^{m+1}} \frac{d^r v^i}{d\eta^r}(\eta) - \sum_{m+1+r \leq q+1, m+1 \geq 0} m c_{m,r} \frac{d^r v^i}{d\eta^r}(\eta)
$$

\begin{equation}
(3.14)
\end{equation}

$$
+ \sum_{m+r+1 \leq q+1, m \geq 0, r+1 \geq 1} \frac{c_{m,r}}{(\sin \eta)^{m+1}} \frac{d^{r+1} v^i}{d\eta^{r+1}}(\eta) = (\text{changing the } c_{m,r})
$$

$$
= a_{i,q}(\eta) + a_{i,q+1}(\eta) = a_{i,q+1}(\eta).
$$

We thus write

\begin{equation}
(3.15)
\end{equation}

$$
\frac{d}{d\eta} a_{i,q}(\eta) = a_{i,q+1}(\eta).
$$

Similarly, for $k \geq 0$,

$$
a_{i,q-k}(\eta) = \sum_{m+r \leq q-k, m,r \geq 0} \frac{c_{m,r}}{(\sin \eta)^{m}} \frac{d^r v^i}{d\eta^r} = \sum_{m+k+r \leq q, m+k \geq 0} \frac{c_{m,r}}{(\sin \eta)^{m+k}} \frac{d^{r+1} v^i}{d\eta^{r+1}}
$$

\begin{equation}
(3.16)
\end{equation}

$$
= (\text{changing the } c_{m,r}) = (\sin \eta)^k a_{i,q}(\eta).
$$

To prove the boundedness of the coefficients $a_{i,q}(\eta)$ as in (3.13), we first need to show that, for $r, i \geq 0$, $\pi \leq \eta \leq 2\pi$,

\begin{equation}
(3.17)
\end{equation}

$$
\frac{d^r v^i(\eta)}{d\eta^r} = \sum_{l+s \leq r, l,s \geq 0} c_{l,s}(\sin \eta)^{2l-r+s} \frac{\partial^{l+s} u^i}{\partial x^l \partial y^s}(\cos \eta, \sin \eta) + \sum_{0 \leq r' \leq r-1} c_{r'}(\sin \eta)^{r'-r} \frac{d^{r'} v^i(\eta)}{d\eta^{r'}}
$$

where the functions $c_{l,s} = c_{l,s}(\eta)$, $c_{r'} = c_{r'}(\eta)$ belong to $C^\infty([0, 2\pi])$ and may be different at different occurrences. To verify this, we proceed by induction on $r$. We calculate the derivatives at order $r = 0, 1, 2$. Since $v^i(\eta) = -u^i(\cos \eta, \sin \eta)$, we find that

\begin{equation}
(3.18)
\end{equation}

$$
\frac{dv^i(\eta)}{d\eta} = \sin \eta \frac{\partial u^i}{\partial x}(\cos \eta, \sin \eta) - \cos \eta \frac{\partial u^i}{\partial y}(\cos \eta, \sin \eta),
$$

and, using (3.18),

\begin{equation}
(3.19)
\end{equation}

$$
\frac{d^2 v^i(\eta)}{d\eta^2} = \sum_{l+s=2, l,s \geq 0} c_{l,s}(\sin \eta)^l \frac{\partial^2 u^i}{\partial x^l \partial y^s}(\cos \eta, \sin \eta) + (\sin \eta)^{-1} \frac{\partial u^i}{\partial y}(\cos \eta, \sin \eta) + \frac{\cos \eta}{\sin \eta} \frac{dv^i(\eta)}{d\eta}.
$$
Hence, (3.17) is obviously true at order \( r = 0, 1, 2 \). Assuming that (3.17) holds at orders 0, 1, \( \cdots \), \( r \), we want to prove (3.17) at order \( r + 1 \). We then write with the induction assumption:

\[
\frac{d^{r+1}v^i(\eta)}{d\eta^{r+1}} = \frac{d}{d\eta} \left( \frac{d^r v^i(\eta)}{d\eta^r} \right)
\]

\[(3.20)\]

\[
\sum_{l+s = r+1, l,s \geq 0} \tilde{c}_{ls}(\sin \eta)^{2l-(r+1)}+1 \frac{\partial^{l+s-1} f}{\partial x^l \partial y^s} (\cos \eta, \sin \eta) + \sum_{0 \leq r' \leq r} \tilde{c}_{r'}(\sin \eta)^{r'-(r+1)} \frac{d^r v^i(\eta)}{d\eta^r},
\]

where \( \tilde{c}_{ls} = \tilde{c}_{ls}(\eta) \), \( \tilde{c}_{r'} = \tilde{c}_{r'}(\eta) \) \( \in C^\infty([0, 2\pi]) \). Hence, all terms in the right-hand side of (3.20) can be written as in the right-hand side of (3.17) with \( r \) replaced by \( (r + 1) \), and thus (3.17) is verified at all orders \( r \).

We then verify the following lemma.

**Lemma 3.2.** We assume that the following conditions hold:

\[(3.21) \quad \frac{\partial^{p_1+p_2} f}{\partial x^{p_1} \partial y^{p_2}} = 0 \text{ at } (\pm 1, 0) \text{ for } 0 \leq 2p_1 + p_2 \leq -2 + 3i + q, p_1, p_2 \geq 0, i, q, 0, \]

where no conditions on \( f \) are needed, if \( 3i + q \leq 1 \). Then the functions

\[(3.22) \quad \frac{1}{\sin \eta} \frac{d^r v^i}{d\eta^r} \text{ are bounded for } \eta \in [\pi, 2\pi], r = 0, \cdots, q, \]

and thus \( a_{i,q}(\eta) \), defined in (3.13), is bounded. Here, \( v^i(\eta) = -u^i(\cos \eta, \sin \eta) \).

**Proof.** We first consider the case where \( 3i + q \geq 2 \). Inserting (2.13a) in (3.17) we can write that

\[(3.23) \quad \frac{1}{\sin \eta} \frac{d^r v^i}{d\eta^r}(\eta) = \sum_{l+s \leq r, l,s \geq 0} \tilde{g}^i_{ls}(\cos \eta, \sin \eta) \frac{\partial^{l+s} f}{\partial x^l \partial y^s}(\cos \eta, \sin \eta) + \sum_{l+s \leq r, l,s \geq 0} \tilde{c}_{ls}(\cos \eta, \sin \eta) \frac{\partial^{l+s} f}{\partial x^l \partial y^s}(\cos \eta, \sin \eta) + \sum_{0 \leq r' \leq r} \tilde{c}_{r'}(\sin \eta)^{r'-(r+1)} \frac{d^r v^i(\eta)}{d\eta^r}, \]

where \( |\tilde{g}^i_{ls}(\cos \eta, \sin \eta)| \leq \kappa(\sin \eta)^{-1} \) and \( |\tilde{c}_{ls}(\cos \eta, \sin \eta)| \leq \kappa(\sin \eta)^{-1} \).

To bound the first sum in (3.23), thanks to Lemma 2.1, we require that \( \frac{\partial^{p'+s'+q'}(\sin \eta)^{(p'+s'+q')}}{\partial x^{p'} \partial y^{s'} \partial \eta^{q'}} = 0 \) at \((\pm 1, 0)\) for \( 0 \leq 2p' + q' \leq -1 + 3i - 2l' - s' + q - s - 1 \). We then note that \( 0 \leq 2(p' + l') + q' + s' \leq -1 + 3i + q - s - 1 \leq -2 + 3i + q \), the condition (3.21) thus guarantees the boundedness of the first sum. To bound the second sum in (3.23), we require that \( \frac{\partial^{p'+s'+q'}(\sin \eta)^{(p'+s'+q')}}{\partial x^{p'} \partial y^{s'} \partial \eta^{q'}} = 0 \) at \((\pm 1, 0)\) for \( 2p' + q' \leq -2l' - s' - 1 \). Since \( 0 \leq 2(p' + l' + 2i - 2k) + q' + s' + i + 2k - 1 \leq -2 + 3i - 2k + q \), the condition (3.21) thus guarantees the boundedness of the second sum.

There remains to show that the third sum in the right-hand side of (3.23) is bounded. This is done by induction on \( r \); the sum disappears and there is nothing to prove for \( r = 0 \). We then assume that it holds at order \( r \geq 0 \) and prove it at order \( r + 1 \). At order \( r + 1 \), we note that the last sum is bounded by our induction assumption at order \( r \) and then the lemma holds at order \( r + 1 \) (and thus for all \( r \geq 0 \)).

For \( 3i + q \leq 1 \), i.e., \( i = 0, q = 0, 1 \), from (1.3) we note that \( v^0(\eta) = \int_{\sin \eta}^{-\sin \eta} f(\cos \eta, s)ds \), \( \eta \in [\pi, 2\pi] \), and we can easily verify (3.22). \( \square \)
3.3. Regularity properties of the $\theta^j, \bar{\theta}^j$. As we did before, using the cut-off function $\delta(\xi)$ we use the approximate forms of the functions $\theta^j$ in (3.11):

$$\bar{\theta}^j = P^j(\eta, \bar{\xi}) \exp((\sin \eta \bar{\xi})\delta(\xi)\chi_{[\pi, 2\pi]}(\eta)).$$

Remark 3.2. The compatibility conditions (2.6) guarantee the assumption of Lemma 3.2 with $q = 3j - 3i + 3$. We then note that the $a_{i,3j-3i+3}(\eta)$, $0 \leq i \leq j + 1$, are bounded and thus $a_{i,3j-3i-k}(\eta) = (\sin \eta)^{k+3} a_{i,3j-3i+3}(\eta) \to 0$, as $\eta \to \pi, 2\pi$ for $k \geq 0$. Hence, $P^j(\eta, \bar{\xi}) \to 0$ as $\eta \to \pi, 2\pi$ (and so does $\bar{\theta}^j$). These results imply that $\bar{\theta}^j \in H^1(D)$ and $\bar{\theta}^j \in H^1_0(D)$.

Let us then estimate $\bar{\theta}^j$ for all $j \geq 0$ which will be used below. From Lemma 3.1, (3.24), (3.15) and (3.16) we may first write that, for $\eta \in (\pi, 2\pi),

$$\bar{\theta}^j = \sum_{i=0}^{j} a_{i,3j-3i}(\eta) \sum_{k=0}^{2j-2i} ((\sin \eta \bar{\xi})^k \exp((\sin \eta \bar{\xi})\delta(\xi)),$$

$$\frac{\partial \bar{\theta}^j}{\partial \xi} = \frac{\epsilon}{\theta^j} \frac{\partial \bar{\theta}^j}{\partial \xi} = \left[ \sum_{i=0}^{j} a_{i,3j-3i-1}(\eta) \sum_{k=0}^{2j-2i} ((\sin \eta \bar{\xi})^k \exp((\sin \eta \bar{\xi})\delta(\xi),$$

$$\frac{\partial \bar{\theta}^j}{\partial \eta} = \sum_{i=0}^{j} a_{i,3j-3i+1}(\eta) \sum_{k=0}^{2j-2i+1} ((\sin \eta \bar{\xi})^k \exp((\sin \eta \bar{\xi})\delta(\xi).$$

We can estimate the corrector $\theta^j$ and the difference with its approximate form $\bar{\theta}^j$ as follows.

Lemma 3.3. There exist a constant $\kappa > 0$ such that, for integers $l, m \geq 0$, $1 \leq p \leq \infty$, setting $\frac{\epsilon}{\theta^j} = 1$,

$$\left| (\sin \eta)^{-l} \left( \frac{\epsilon}{\theta^j} \right)^m \frac{\partial \theta^j}{\partial \xi} \right|_{L^p(D^*)} \leq \kappa \max_{i=0, \ldots, j} \left\{ \sup_{\eta} |a_{i,3j-3i+1+l+m}(\eta)| \right\} \frac{1}{p},$$

$$\left| (\sin \eta)^{-l} \left( \frac{\epsilon}{\theta^j} \right)^m \frac{\partial \bar{\theta}^j}{\partial \xi} \right|_{L^p(D^*)} \leq \kappa \max_{i=0, \ldots, j} \left\{ \sup_{\eta} |a_{i,3j-3i+1+l+m}(\eta)| \right\} \frac{1}{p-1},$$

$$+ \kappa \max_{i=0, \ldots, j} \left\{ \sup_{\eta} |a_{i,3j-3i+1+l+m}(\eta)| \right\} \frac{1}{p},$$

$$\left| (\sin \eta)^{-l} \left( \frac{\epsilon}{\theta^j} \right)^m \frac{\partial \bar{\theta}^j}{\partial \eta} \right|_{L^p(D^*)} \leq \kappa \max_{i=0, \ldots, j} \left\{ \sup_{\eta} |a_{i,3j-3i+2+l+m}(\eta)| \right\} \frac{1}{p},$$

where $a_{i,q}(\eta)$ is defined in (3.13), and $D^*$ is as in (3.1).
Proof. To obtain the first inequality, we multiply (3.25) by \((\sin \eta)^{-l}\left(\frac{\xi}{\varepsilon}\right)^m\), take the \(L^p\)-norm, \(1 \leq p < \infty\), and we find

\[
\left| (\sin \eta)^{-l}\left(\frac{\xi}{\varepsilon}\right)^m \hat{\theta}^j \right|_{L^p(D^*)} = \sum_{i=0}^{j} a_{i,3j-3l}(\eta) (-\sin \eta)^{-1-l-m} \sum_{k=0}^{2j-2i} (-\sin \eta)^{(k+1)(m+1)} \exp((-\sin \eta)\bar{\xi})\delta^j(\xi) \left|_{L^p(D^*)} \right.
\]

\[
\leq \kappa \max_{i=0,\ldots,j} \left\{ \sup_{\eta} |a_{i,3j-3l+1+l+m}(\eta)| \right\} \int_0^{2\pi} \int_0^{2\pi} (-\sin \eta)^p (-\sin \eta)\bar{\xi}^{p(k+m)} \exp(p(\sin \eta)\bar{\xi})d\xi d\eta \right.^{\frac{1}{p}}.
\]

Since \(x^k \exp(-x) \leq \kappa \exp(-\frac{x^2}{2})\) for \(x > 0\), we note that \((-\sin \eta)\bar{\xi}^{p(k+m)} \exp(p(\sin \eta)\bar{\xi}) \leq \kappa \exp(c(\sin \eta)\bar{\xi})\)

for some constants \(\kappa, c > 0, \pi < \eta < 2\pi\), and \(\int_0^{2\pi} \int_0^{2\pi} (-\sin \eta)^p \exp(c(\sin \eta)\bar{\xi})d\xi d\eta \leq \kappa \varepsilon\) and thus we find (3.28).1

For \(p = \infty\), we just let \(p \rightarrow \infty\) in (3.28)1 where \(\kappa\) is independent of \(p\). That is,

\[
(3.30) \left| (\sin \eta)^{-l}\left(\frac{\xi}{\varepsilon}\right)^m \hat{\theta}^j \right|_{L^\infty(D^*)} \leq \kappa \lim_{p \rightarrow \infty} \max_{i=0,\ldots,j} \left\{ \sup_{\eta} |a_{i,3j-3l+1+l+m}(\eta)| \right\} \varepsilon^{\frac{1}{p}}.
\]

The other two inequalities are proven in a similar way. \(\Box\)

To estimate the \(\theta^j\) and their derivatives, setting \(\delta(\xi) = 1\) in (3.25)-(3.27) (note then that \(\delta'(\xi) = 0\) in (3.26)) and following the same reasoning we immediately conclude the lemma below.

**Lemma 3.4.** There exist a constant \(\kappa > 0\) such that, for integers \(l, m \geq 0\), and for \(1 \leq p \leq \infty\), and setting \(\varepsilon^{\frac{1}{p}} = 1\),

\[
(3.31) \left| (\sin \eta)^{-l}\left(\frac{\xi}{\varepsilon}\right)^m \frac{\partial \theta^j}{\partial \xi} \right|_{L^p(D^*)} \leq \kappa \max_{i=0,\ldots,j} \left\{ \sup_{\eta} |a_{i,3j-3l+1+l+m}(\eta)| \right\} \varepsilon^{\frac{1}{p}-1},
\]

where \(a_{i,j}(\eta)\) are coefficients of the form (3.13).

We will later need the following lemma to estimate the difference \(\theta^j - \bar{\theta}^j\).
Lemma 3.5. Let $A = A^\sigma = (0, 1) \times (\pi + \sigma, 2\pi - \sigma)$, $0 < \sigma < \frac{\pi}{2}$. For $i, l, m \geq 0$, there exist constants $\kappa, c > 0$ such that

$$
\left| (\theta^i - \tilde{\theta}^i) \right|_{L^2(A)} \leq \kappa \max_{i = 0, \ldots, j} \left\{ \sup_{\eta} |a_{i,3j-3i+1}(\eta)| \right\} \varepsilon^\frac{j}{2} \exp(-c \sin \sigma \frac{1}{\varepsilon}),
$$

$$
\left| \frac{\partial}{\partial \xi} (\theta^i - \tilde{\theta}^i) \right|_{L^2(A)} \leq \kappa \max_{i = 0, \ldots, j} \left\{ \sup_{\eta} |a_{i,3j-3i+1}(\eta)| \right\} \varepsilon^{-\frac{j}{2}} \exp(-c \sin \sigma \frac{1}{\varepsilon}),
$$

$$
+ \kappa \max_{i = 0, \ldots, j} \left\{ \sup_{\eta} |a_{i,3j-3i+1}(\eta)| \right\} \varepsilon^\frac{j}{2} \exp(-c \sin \sigma \frac{1}{\varepsilon}),
$$

and

$$
\left| (\theta^i - \tilde{\theta}^i) \right|_{L^2(A)} \leq \kappa \max_{i = 0, \ldots, j} \left\{ \sup_{\eta} |a_{i,3j-3i+2+l}(\eta)| \right\} \sigma^{\frac{2l+3}{2}},
$$

$$
\left| \frac{\partial}{\partial \eta} (\theta^i - \tilde{\theta}^i) \right|_{L^2(A)} \leq \kappa \max_{i = 0, \ldots, j} \left\{ \sup_{\eta} |a_{i,3j-3i+2+l}(\eta)| \right\} (\varepsilon^{-1} \sigma^{\frac{2l+5}{2}} + \sigma^{\frac{2l+3}{2}}),
$$

$$
\left| \frac{\partial}{\partial \eta} (\theta^i - \tilde{\theta}^i) \right|_{L^2(A)} \leq \kappa \max_{i = 0, \ldots, j} \left\{ \sup_{\eta} |a_{i,3j-3i+2+l}(\eta)| \right\} \sigma^{\frac{2l+1}{2}}.
$$

Proof. The difference $\theta^i - \tilde{\theta}^i$ and its derivatives are exactly as in (3.25)-(3.27) with $\tilde{\theta}^i$ replaced by $\theta^i - \tilde{\theta}^i$ and $\delta(\xi)$ by $1 - \delta(\xi)$, which is zero for $0 < \xi < 1/4$. We similarly follow (3.29) with $l = m = 0$. On the subdomain $A$, we just change the interval of integration, i.e. we replace $\int_0^{\frac{\pi}{2}}$ by $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}}$ and $\int_0^{\pi}$ by $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}}$. We thus find that

$$
\left| \theta^i - \tilde{\theta}^i \right|_{L^2(A)} \leq \kappa \max_{i = 0, \ldots, j} \left\{ \sup_{\eta} |a_{i,3j-3i+1}(\eta)| \right\} \varepsilon^{\frac{j}{2}} \exp(-c \sin \sigma \frac{1}{\varepsilon}), c > 0.
$$

The other two inequalities in (3.32) similarly follow. On the subdomain $D \setminus A$, thanks to Lemmas 3.3, 3.4, we note that

$$
\left| \theta^i - \tilde{\theta}^i \right|_{L^2(A)} \leq \left| (\sin \eta)^{l-1}(\theta^i - \tilde{\theta}^i) \right|_{L^2(D \setminus A)} \left| (\sin \eta)^{l+1} \right|_{L^2(D \setminus A)}
$$

$$
\leq \kappa \max_{i = 0, \ldots, j} \left\{ \sup_{\eta} |a_{i,3j-3i+2+l}(\eta)| \right\} \left[ \int_{\pi}^{\pi+\sigma} (\eta - \pi)^{2l+2} d\eta + \int_{2\pi-\eta}^{2\pi} (2\pi - \eta)^{2l+2} d\eta \right]^{\frac{1}{2}}
$$

$$
\leq \kappa \max_{i = 0, \ldots, j} \left\{ \sup_{\eta} |a_{i,3j-3i+2+l}(\eta)| \right\} \sigma^{\frac{2l+3}{2}}.
$$

The other two inequalities in (3.33) follow similarly. $\square$

4. Convergence results

4.1. First orders. Writing $w = u^\varepsilon - u^0 - \tilde{\theta}^0$ from (1.1), (1.2), and (3.4) we deduce that

$$
\left\{ \begin{array}{l}
-\varepsilon \Delta w - w_y = R.H.S, \\
w = 0 \text{ on } \partial D.
\end{array} \right.
$$

Using the approximate form $\tilde{\theta}^0$ for $\theta^0$ we can write $L_\varepsilon \tilde{\theta}^0 = L_\varepsilon \theta^0 + L_\varepsilon (\tilde{\theta}^0 - \theta^0)$, and then $< L_\varepsilon \tilde{\theta}^0, \varphi > = < L_\varepsilon \theta^0, \varphi \delta(\xi) >$ for all $\varphi \in H^1_0(D)$ where $<, >$ is the duality product in the space
\(H^1_0(D)\) and \(\tilde{\delta}(\xi)\) is a smooth function such that \(\tilde{\delta}(\xi) = 1\) if \(\xi \leq 1/2\) and \(\tilde{\delta}(\xi) = 0\) if \(\xi \geq 3/4\). Note that \(\tilde{\theta}^0 = 0\) for \(\xi \geq 1/2\).

Thanks to (3.3), we first observe that

\[(4.2) \quad R.H.S. = \varepsilon \Delta u^0 - L_\varepsilon(\theta^0) + L_\varepsilon(\theta^0 - \tilde{\theta}^0).\]

Taking the duality product \(\langle , \rangle\) of (4.1) with \(e^y w\) we find with \(D^*\) as in (3.1):

\[
\varepsilon |w|_{H^1(D)}^2 + |w|_{L^2(D)}^2 \\
\leq | < u^0, e^y w \tilde{\delta}(\xi) > | + | < \varepsilon L_\varepsilon(\theta^0), \varepsilon e^y w \tilde{\delta}(\xi) > | + | < L_\varepsilon(\theta^0 - \tilde{\theta}^0), \varepsilon e^y w \tilde{\delta}(\xi) > | \\
\leq \kappa \varepsilon |u^0|_{H^1(D)} |w|_{H^1(D)} + \left| \frac{\varepsilon}{(1 - \xi)^2} \frac{\partial^2 \theta^0}{\partial \eta^2} - \frac{\varepsilon}{1 - \xi} \frac{\partial \theta^0}{\partial \xi} + \cos \eta \frac{\partial \theta^0}{\partial \eta} \right| \\
+ | < L_\varepsilon(\theta^0 - \tilde{\theta}^0), \varepsilon e^y w \tilde{\delta}(\xi) > | \\
\leq \kappa \varepsilon |u^0|_{H^1(D)} |w|_{H^1(D)} + \kappa \varepsilon \left| \frac{\partial \theta^0}{\partial \eta} \right|_{L^2(D^*)} \left| \frac{\partial w}{\partial \eta} \right|_{L^2(D^*)} \left| \tilde{\delta}(\xi) \right|_{L^2(D^*)} \\
+ \kappa \varepsilon |\nabla_{\xi, \eta}(\theta^0 - \tilde{\theta}^0)|_{L^2(D^*)} + |\theta^0 - \tilde{\theta}^0|_{L^2(D^*)} \left| \nabla_{\xi, \eta}(w \tilde{\delta}(\xi)) \right|_{L^2(D^*)}.
\]

The following estimates for the derivatives of \(\theta^0\) are particular cases of results stated and proved in Lemmas 3.2 and 3.4\(^2\):

\[(4.4) \quad \left| \frac{\partial \theta^0}{\partial \xi} \right|_{L^2(D^*)} \leq \kappa \varepsilon^{-\frac{1}{2}},\]

and, assuming (2.1) with \(m = 0\),

\[(4.5) \quad \left| \frac{\partial \theta^0}{\partial \eta} \right|_{L^2(D^*)} \leq \kappa \varepsilon^\frac{1}{2}.
\]

Assuming (2.1) with \(m = 0\), from Lemmas 3.2 and 3.5 below with \(j = 0, \sigma = \varepsilon^{\frac{3}{2}}\), we find that

\[(4.6) \quad |\theta^0 - \tilde{\theta}^0|_{L^2(D^*)} \leq \kappa \varepsilon, \quad \left| \frac{\partial}{\partial \xi}(\theta^0 - \tilde{\theta}^0) \right|_{L^2(D^*)} \leq \kappa \varepsilon^\frac{1}{2}, \quad \left| \frac{\partial}{\partial \eta}(\theta^0 - \tilde{\theta}^0) \right|_{L^2(D^*)} \leq \kappa \varepsilon^\frac{1}{2}.
\]

Hence, since \(\left| \frac{\partial w}{\partial \eta} \tilde{\delta}(\xi) \right|_{L^2(D^*)}, \left| \nabla_{\xi, \eta}(w \tilde{\delta}(\xi)) \right|_{L^2(D^*)} \leq \kappa |w|_{H^1(D)} \) and \(\left| \tilde{\delta}(\xi) \right|_{L^2(D^*)} \leq \kappa |w|_{L^2(D)},\)

we conclude from (4.3) that

\[(4.7) \quad \sqrt{\varepsilon} |w|_{H^1(D)} + |w|_{L^2(D)} \leq \kappa \sqrt{\varepsilon}.
\]

The following theorem then holds:

**Theorem 4.1.** Let \(u^0\) and \(u^0\) be the solutions of Eq. (1.1) and (1.2), respectively. Assuming the \(m\) compatibility conditions (2.1) with \(m = 0\), that is,

\[(4.8) \quad f = 0 \text{ at } (\pm 1, 0),\]

\(^2\)From Lemma 3.2, even if no compatibilities on \(f\) are assumed, \(a_{0,1}(\eta)\), defined in (3.13), is bounded; see Remark 3.1. If (2.1) holds with \(m = 0, 1, a_{0,2+k}(\eta)\) is bounded, \(k = 0, 1\). Hence, from Lemmas 3.3 and 3.5 we can obtain the estimates as in (4.4)-(4.6) and (4.12)-(4.14).
the following estimates hold:

\[(4.9) \quad |u^\varepsilon - u^0 - \bar{\theta}^{0}|_{L^2(D)} + \sqrt{\varepsilon}|u^\varepsilon - u^0 - \bar{\theta}^{0}|_{H^1(D)} \leq \kappa \varepsilon^{\frac{3}{2}},\]

and thus

\[(4.10) \quad |u^\varepsilon - u^0|_{L^2(D)} \leq \kappa \varepsilon^{\frac{1}{2}},\]

where \(\bar{\theta}^{0}\) is the corrector given in (3.6).

Requiring now the compatibility conditions (2.1) with \(m = 1\) and applying the Hardy inequality we can improve the estimates (4.9). As in (4.3) we can obtain that

\[(4.12) \quad \left| \frac{\partial \theta^0}{\partial \xi} \right|_{L^2(D^*)} \leq \kappa \varepsilon^{\frac{3}{2}}.\]

and

\[(4.13) \quad \left| \frac{\partial \theta^0}{\partial \eta} \right|_{L^2(D^*)} \leq \kappa \varepsilon^{\frac{3}{2}}.\]

We continue to assume (2.1) with \(m = 1\), and we infer from Lemmas 3.2 and 3.5 with \(j = 0\), \(\sigma = \varepsilon^{\frac{3}{2}}\) that

\[(4.14) \quad |\bar{\theta}^{0} - \theta^{0}|_{L^2(D^*)} \leq \kappa \varepsilon^{\frac{3}{2}}, \quad \left| \frac{\partial (\bar{\theta}^{0} - \theta^{0})}{\partial \xi} \right|_{L^2(D^*)} \leq \kappa \varepsilon^{\frac{3}{2}}, \quad \left| \frac{\partial}{\partial \eta} (\bar{\theta}^{0} - \theta^{0}) \right|_{L^2(D^*)} \leq \kappa \varepsilon.

We can then rewrite (4.11) in the form:

\[(4.15) \quad \varepsilon |w|^2_{H^1(D)} + |w|^2_{L^2(D)} \leq \kappa \varepsilon |w|_{L^2(D)} + \kappa \varepsilon |w|^\frac{1}{2}_{L^2(D)} |w|^\frac{1}{2}_{L^2(D)} + \kappa \varepsilon^2 |w|_{H^1(D)}

\[
\leq \frac{\varepsilon}{4} |w|^2_{H^1(D)} + \frac{1}{4} |w|^2_{L^2(D)} + \frac{\varepsilon^\frac{3}{2}}{2} |w|_{L^2(D)} |w|_{L^2(D)} + \kappa \varepsilon^{\frac{3}{2}}.
\]

Hence, the following theorem gives a result of \(H^1\)-convergence:

**Theorem 4.2.** Let \(u^\varepsilon\) and \(u^0\) be the solutions of Eq. (1.1) and (1.2), respectively. We assume the compatibility conditions (2.1) with \(m = 1\), that is,

\[(4.16) \quad f = \frac{\partial f}{\partial y} = 0 \text{ at } (\pm 1, 0).
\]

Then the following estimates hold:

\[(4.17) \quad |u^\varepsilon - u^0 - \bar{\theta}^{0}|_{L^2(D)} + \sqrt{\varepsilon}|u^\varepsilon - u^0 - \bar{\theta}^{0}|_{H^1(D)} \leq \kappa \varepsilon^{\frac{3}{2}},\]

where \(\bar{\theta}^{0}\) is the corrector given in (3.6).

\(^{3}m = 0\) suffices here.
Remark 4.1. If we impose more compatibility conditions, that is, if \( f = f_x = f_y = f_{yy} = 0 \) at \((\pm 1, 0)\), thanks to Lemma 2.1, from (2.3) we find \(|u_0^0|_{L^2(D)} \leq \kappa\) and thus \(|\Delta u^0|_{L^2(D)} \leq \kappa\). Hence, from (4.11) we can improve the asymptotic error (4.17):

\[
|u^\varepsilon - u^0 - \bar{\theta}^\varepsilon|_{L^2(D)} + \sqrt{\varepsilon}|u^\varepsilon - u^0 - \bar{\theta}^\varepsilon|_{H^1(D)} \leq \kappa \varepsilon.
\]

4.2. Higher orders. We now look for improved approximations of \( u^\varepsilon \) using the higher order functions and correctors \( w^j, \theta^j, \bar{\theta}^j, j \geq 1 \).

Setting \( u_{en} = \sum_{j=0}^n \varepsilon^j w^j \) and \( \theta_{en} = \sum_{j=0}^n \varepsilon^j \theta^j \) and writing \( w_n = u^\varepsilon - u_{en} - \theta_{en} \) we deduce that

\[
\begin{align*}
-\varepsilon \Delta w_n - w_{ny} &= R.H.S, \\
w_n &= 0 \text{ on } \partial D.
\end{align*}
\]

We now derive the explicit expression of the R.H.S. Multiplying (1.4)_1 by \( \varepsilon^j \) and summing from \( j = 0 \) to \( n \) we find that \( L_c u_{en} = -\varepsilon^{n+1} \Delta u^n \). We now multiply (3.9)_1 by \( \varepsilon^{j+1} \) and sum from \( j = 0 \) to \( n \) and we thus find that

\[
A := -\varepsilon \frac{\partial^2 \theta_{en}}{\partial \xi^2} + \sin \eta \frac{\partial \theta_{en}}{\partial \xi} = \varepsilon \sum_{j=0}^n \sum_{k=0}^j (j-k)(-1)^{j-k} \varepsilon^{j-k} \frac{\partial^2 \theta^k}{\partial \eta^2}
\]

\[
(4.20)
\]

\[
(4.21)
\]

where \( \theta_{en} = \sum_{j=0}^n \varepsilon^j \theta^j \).

Using (3.3) and (4.20) we can then write

\[
\begin{align*}
L_c \theta_{en} &= -\frac{\varepsilon}{(1-\xi)^2} \frac{\partial^2 \theta_{en}}{\partial \eta^2} + \frac{\varepsilon}{1-\xi} \frac{\partial \theta_{en}}{\partial \xi} + A - \cos \eta \frac{\partial \theta_{en}}{\partial \eta} \\
&= -\frac{\varepsilon}{(1-\xi)^2} \left( \frac{\partial^2 \theta_{en}}{\partial \eta^2} + \varepsilon^n \frac{\partial^2 \theta_{en}}{\partial \eta^2} \right) \\
&\quad + \frac{\varepsilon}{1-\xi} \frac{\partial \theta_{en}}{\partial \xi} - \cos \eta \frac{\partial \theta_{en}}{\partial \eta} - \varepsilon \sum_{j=0}^n \varepsilon^j \frac{\partial^2 \theta^j}{\partial \eta^2} \sum_{k=n-j}^\infty (k+1) \xi^k \\
&\quad + \varepsilon \sum_{j=0}^n \varepsilon^j \frac{\partial \theta^j}{\partial \xi} \sum_{k=n-j}^\infty \xi^k - \cos \eta \sum_{j=0}^n \varepsilon^j \frac{\partial \theta^j}{\partial \eta} \sum_{k=n-j}^\infty \xi^k =: \text{Rem}
\end{align*}
\]

Using the approximate forms \( \bar{\theta}^j \) for \( \theta^j \) we can write \( L_c \bar{\theta}_{en} = L_c \theta_{en} + L_c (\bar{\theta}_{en} - \theta_{en}) := \text{Rem} + L_c (\bar{\theta}_{en} - \theta_{en}) \). Hence, subtracting from (1.1) the R.H.S. in (4.19) can be written as

\[
R.H.S. = \varepsilon^{n+1} \Delta u^n - \text{Rem} + L_c (\theta_{en} - \bar{\theta}_{en}).
\]
Taking the duality product $<,>$ of (4.19) with $e^\theta w_n$, we infer from (2.16) that

$$
\varepsilon |w_n|_{H^1(D)}^2 + |w_n|_{L^2(D)}^2 \leq \kappa \varepsilon^{n+1} |w_n|_{L^2(D)} + \kappa \varepsilon^{n+1} |w_n|_{H^1(D)} |w_n|_{L^2(D)}^2 + \kappa \varepsilon^{n+1} \sum_{j=0}^{n} \left| \left( \frac{\varepsilon}{2\eta} \right)^{n-j} \frac{\partial \beta_j}{\partial \eta} \right|_{L^2(D^*)} \frac{\partial w_n}{\partial \eta} \delta(\varepsilon) |L^2(D^*)|
$$

(4.23)

$$
+ \kappa \varepsilon^{n+2} \sum_{j=0}^{n} \left| \left( \frac{\varepsilon}{2\eta} \right)^{n+1-j} \frac{\partial \beta_j}{\partial \xi} \right|_{L^2(D^*)} \xi^{-1} w_n \delta(\varepsilon) |L^2(D^*)|
$$

We first estimate the terms, $| \left( \frac{\varepsilon}{2\eta} \right)^{n-j} \frac{\partial \beta_j}{\partial \eta} |_{L^2(D^*)}$, $| \left( \frac{\varepsilon}{2\eta} \right)^{n+1-j} \frac{\partial \beta_j}{\partial \xi} |_{L^2(D^*)}$, and $| \left( \frac{\varepsilon}{2\eta} \right)^{n+1-j} \frac{\partial \beta_j}{\partial \eta} |_{L^2(D^*)}$.

As indicated in Lemma 3.4 with $p = 2, l = 0$, we need to bound $a_k(q_k(\eta))$ for $k = 0, \ldots, j$, $0 \leq q_k \leq 2j - 3k + n + 3$, $j = 0, \ldots, n$, which only requires the condition (3.21) with $i = k, q = q_k$ and thus $0 \leq 2p^i + q^i \leq -2 + 3k + q_k \leq 1 + 2j + n \leq 1 + 3n$, or (2.15) with $j = n$. Hence, the three terms are respectively bounded by $\kappa \varepsilon^{\frac{3}{2}}$, $\kappa \varepsilon^{-\frac{3}{2}}$ and $\kappa \varepsilon^{\frac{3}{2}}$.

Using (3.2) and Lemma 3.5 we now estimate the following term:

$$
\left| < L_\varepsilon(\theta_{en} - \bar{\theta}_{en}), e^\theta w_n \delta(\varepsilon) > \right|
$$

(4.24)

$$
\leq \kappa (\varepsilon |\nabla_{\varepsilon, \eta}(\theta_{en} - \bar{\theta}_{en})|_{L^2(D^*)} + |\theta_{en} - \bar{\theta}_{en}|_{L^2(D^*)}|\nabla_{\varepsilon, \eta}(w_n \delta(\varepsilon))|_{L^2(D^*)})
$$

$$
\leq \kappa \sum_{j=0}^{n} \varepsilon^j \max_{i=0, \ldots, j} \{ \sup_{\eta} |a_{i,3j-3i+2+l}(\eta)| \} \left( \frac{\sigma^{2i+3}}{\varepsilon^{\frac{3}{2}}} + \varepsilon^\sigma^{2i+4} + \exp(-\varepsilon^\sigma \frac{1}{\varepsilon}) \right) |w_n|_{H^1(D)}
$$

We assume the compatibility condition (2.15) with $j = n$. In Lemma 3.2, the conditions (3.21) then hold for $q = 3 + 3(n - i)$. Setting $l = 1 + 3n - 3j$ we find that $a_{i,3j-3i+2+l}(\eta) = a_{i,3+3(n-i)}(\eta)$, $i = 0, \ldots, j$, in (4.24) are all bounded. Setting $\sigma = \varepsilon^\frac{3}{2}$ we thus find that

$$
\left| < L_\varepsilon(\theta_{en} - \bar{\theta}_{en}), e^\theta w_n \delta(\varepsilon) > \right| \leq \kappa \sum_{j=0}^{n} \varepsilon^j (\frac{\sigma^{2i+3}}{\varepsilon^{\frac{3}{2}}} + \varepsilon^\sigma^{2i+4} + \exp(-\varepsilon^\sigma \frac{1}{\varepsilon})) |w_n|_{H^1(D)}
$$

(4.25)

$$
\leq \kappa \sum_{j=0}^{n} \varepsilon^{\frac{3}{2} + 2n - j} |w_n|_{H^1(D)} \leq \kappa \varepsilon^{\frac{3}{2} + n} |w_n|_{H^1(D)}
$$

Hence, thanks to the Hardy inequality, the estimate (4.23) is simplified to:

$$
\varepsilon |w_n|_{H^1(D)}^2 + |w_n|_{L^2(D)}^2 \leq \kappa \varepsilon^{n+1} |w_n|_{L^2(D)} + \kappa \varepsilon^{n+1} |w_n|_{H^1(D)} |w_n|_{L^2(D)}^2 + \kappa \varepsilon^{n+2} \sum_{j=0}^{n} \left| \left( \frac{\varepsilon}{2\eta} \right)^{n-j} \frac{\partial \beta_j}{\partial \eta} \right|_{L^2(D^*)} \frac{\partial w_n}{\partial \eta} \delta(\varepsilon) |L^2(D^*)| + \kappa \varepsilon^{n+1} \sum_{j=0}^{n} \left| \left( \frac{\varepsilon}{2\eta} \right)^{n+1-j} \frac{\partial \beta_j}{\partial \xi} \right|_{L^2(D^*)} \xi^{-1} w_n \delta(\varepsilon) |L^2(D^*)| + \kappa \varepsilon^{n+1} \sum_{j=0}^{n} \left| \left( \frac{\varepsilon}{2\eta} \right)^{n+1-j} \frac{\partial \beta_j}{\partial \eta} \right|_{L^2(D^*)}
$$

and we conclude with the following theorem generalizing Theorems 4.1 and 4.2:

**Theorem 4.3.** Let $u^\varepsilon$ and $u_{en} = \sum_{j=0}^{n} \varepsilon^j u^j t$ be the solutions of Eq. (1.1) and (1.4), respectively, and let $\bar{\theta}^i$ be the corrector described in (3.24). We assume that the compatibility conditions (2.15) hold with $j = n$, that is,

$$
\frac{\partial u^p + q}{\partial x^p \partial y^q} = 0 \quad \text{at} \quad (\pm 1, 0), \quad 0 \leq 2p + q \leq 1 + 3n, \quad p, q \geq 0.
$$

(4.27)
The following estimate then is verified:

\begin{equation}
|u^\varepsilon - u_{en} - \bar{\theta}_{en}|_{L^2(D)} + \sqrt{\varepsilon}|u^\varepsilon - u_{en} - \bar{\theta}_{en}|_{H^1(D)} \leq \kappa \varepsilon^{n+\frac{3}{4}}.
\end{equation}

Here we note that Theorem 4.3 does not improve the convergence of $u^\varepsilon$ to $u^0$ in established in (4.10).

Remark 4.2. If we require

\begin{equation}
\frac{\partial^{p+q}f}{\partial x^p \partial y^q} = 0 \quad \text{at } (\pm 1, 0), \quad 0 \leq 2p + q \leq 2 + 3n, \quad p, q \geq 0,
\end{equation}

in the estimate (2.16) we find that \( |\int_D \Delta u^I w\,dx\,dy| \leq \kappa |w|_{L^2(D)} \) which improves the estimate in (4.26), i.e., the term \( \kappa \varepsilon^{n+1} |w_{n,1}|_{H^1(D)} |w_{n,1}|_{L^2(D)} \) is removed. We thus obtain the following estimate better than (4.28):

\begin{equation}
|u^\varepsilon - u_{en} - \bar{\theta}_{en}|_{L^2(D)} + \sqrt{\varepsilon}|u^\varepsilon - u_{en} - \bar{\theta}_{en}|_{H^1(D)} \leq \kappa \varepsilon^{n+1}.
\end{equation}

5. Appendix

5.1. Proof of Lemma 2.2. We prove (2.13a)-(2.13b) by induction on $j$, starting from (1.3) and
the induction formula (1.5). Additional induction on $i$ or $m$ may be needed at each state.

1) We first consider the case where $j = 0$. In this case we prove (2.13a)-(2.13b) for all $m \in \mathbb{Z}$, $i, r, q \geq 0$. We distinguish three cases: $m = 0$, $m \geq 1$ and $m \leq -1$.

1.a) For $m = 0$, starting from (1.3), (2.2), (2.3) we see that (2.13a)-(2.13b) with $i = 0, 1, 2$ hold for all $r \geq 0$. Hence, we have, for $i = 0, 1, 2$,

\begin{equation}
\frac{\partial^i u^0}{\partial x^i} = \sum_{l+s \leq 1, l, s \geq 0}^0 g_{l|s}^0(x) \frac{\partial^{l+s} f}{\partial x^l \partial y^s}(x, C_u(x)) + c_i^0 \frac{\partial^{-1}}{\partial y^{-1}} \left( \frac{\partial^j f}{\partial x^j} \right)(x, y),
\end{equation}

where $g_{l|s}^0 = g_{l|0s}^0$, $c_1^0 = c_{00}^0$ and

\begin{equation}
\left| \frac{d^r}{dx^r} g_{l|s}^0(x) \right| \leq \kappa C_u(x)^{-(-1+2r+2i-2l-s)}.
\end{equation}

We conclude this case by induction on $i$. Indeed by differentiating (5.1) in $x$ we also verify (2.13a)-(2.13b) at order $i+1$ (thus at all orders $i \geq 0$) for $m = 0$, $r \geq 0$ and $j = 0$.

1.b) The case $m \leq -1$ is immediate by applying the operator $\partial^{-1}/\partial y^{-1}$ repeatedly to (5.1) and observing that this operator is smooth, it does not induce any additional singularity. Note also that a polynomial dependence on $y$ of the coefficients $g_{l|s}^{0}$, $c_i^0$ appears at this stage.

1.c) Finally, for $m \geq 1$, since $-u_y^0 = f$, we see that $\frac{\partial m}{\partial y} \left[ \frac{\partial^m f}{\partial x^m} \right] = \frac{\partial^{m+1} f}{\partial x^{m+1}} + \frac{\partial^{m-1}}{\partial y^{-1}} \left[ \frac{\partial^m f}{\partial x^m} \right]$, and thus (2.13a)-(2.13b) with $j = 0$ and $m \geq 1$ hold with $g_{l|s}^{0}(x, y) = 0$ for all $i, r \geq 0$. Hence, for all $m \in \mathbb{Z}$, (2.13a)-(2.13b) with $j = 0$ hold for all $i, r, q \geq 0$.

2) We now verify the lemma for $j \geq 1$. Let us assume that (2.13a)-(2.13b) hold at order $j$ with $i, r \geq 0$, $m \in \mathbb{Z}$. We again distinguish the cases $m = 0$, $m \leq -1$, $m \geq 1$. 

2.a) For \( m = 0 \), thanks to the expression (1.5) and the notation (2.11), we obtain an expression similar to (5.1). That is, replacing \( f \), \( u^0 \) by \( \Delta u^j \), \( u^{j+1} \), respectively, we find that

\[
\frac{\partial^ju^{l+1}}{\partial x^l} = \sum_{l+s \leq i-1, l,s \geq 0} g^{l,s}_{l,s}(x) \left\{ \frac{\partial^{l+s+2}u^j}{\partial x^{l+2}\partial y^s}(x,u(x)) + \frac{\partial^{l+s+2}u^j}{\partial x^{l}\partial y^{s+2}}(x,u(x)) \right\}
\]

\[
+ c_i \left[ \frac{\partial^{l+1}u^j}{\partial x^l\partial y}(x,y) - c_i \frac{\partial^{l+1}u^j}{\partial x^{l+2}}(x,y) + c_i \frac{\partial^{l+1}u^j}{\partial x^l\partial y}(x,u(x)) \right].
\]

(5.3)

Here we used the fact that, for \( g \) similar to (5.1). That is, replacing \( \partial_x \) by \( \partial_x \), \( \partial_y \) by \( \partial_y \), where

\[
\frac{\partial}{\partial y^{-1}} \left[ \frac{\partial^2 g}{\partial y^2} \right](x,y) = - \frac{\partial^{l+1}u^j}{\partial y\partial x^l}(x,y) + \frac{\partial^{l+1}u^j}{\partial y\partial x^l}(x,u(x)).
\]

(5.4)

Let us proceed with the first term in the right-hand side of (5.3). Thanks to the assumption at order \( j \), for \( l+s \leq i-1, l,s \geq 0 \), i.e. evaluating the expression (2.13a) at \( y = C_u(x) \) for \( i = l+2, m = s \), we observe that

\[
\frac{\partial^{-l}h}{\partial y^{-1}}(x,y) = 0 \text{ at } y = C_u(x), l \geq 1.
\]

Using (5.2) we obtain

\[
g^0_{l,s}(x) \frac{\partial^{l+s+2}u^j}{\partial x^{l+2}\partial y^s}(x,u(x))
\]

\[
= \sum_{l+s' \leq i+2j+1, l',s' \geq 0} g^l_{l,s}(x) g^{j}_{j(i+2),s'}(x) \frac{\partial^{l'+s'}f}{\partial x^{l'}\partial y^{s'}}(x,u(x))
\]

\[
+ \sum_{k=0}^j c_k g^0_{l(i+2),m}(x) \left\{ \frac{\partial^{l-j+2k-1}}{\partial y^{l-j+2k-1}} \left[ \frac{\partial^{l+2j+2k-2}l}{\partial x^{l+2j+2k-2}} \right] \right\}(x,u(x))
\]

\[
= \sum_{l+s' \leq i+2(j+1)-1, l',s' \geq 0} g^{l+1}_{l,s}(x) \frac{\partial^{l+s'}f}{\partial x^{l}\partial y^{s'}}(x,u(x)),
\]

(5.6)

where

\[
\left| \frac{d^r}{dx^r} g^{j+1}_{l,s}(x) \right| \leq \kappa_m C_u(x)^{(-1+3(j+1)+2r+2i-2l'-s')}.
\]

(5.7)

Indeed, since \( \partial^{-l}h/\partial y^{-1}(x,y) = 0 \) at \( y = C_u(x), l \geq 1 \), we notice that the terms corresponding to \( s-j+2k-1 \leq -1 \) in (5.6) disappear and the remaining terms (for \( l'+s' = s+l+j+1 \leq i+2(j+1)-1 \)) enter in the last term in (5.6).

We can estimate the second term of the right-hand side in (5.3) in a similar way. Continuing to estimate, thanks to the assumption at order \( j \), we immediately find, using (2.13a) with \( m, i \) replaced by \( -1 \) and \( i + 2 \), that the third term is absorbed in

\[
\sum_{l+s \leq i+2(j+1)-1, l,s \geq 0} g^l_{l,s}(x,y) \frac{\partial^{l+s}f}{\partial x^{l}\partial y^{s}}(x,u(x)) + \sum_{k=0}^{j+1} c_k \left[ \frac{\partial^{l-j+2k-1}}{\partial y^{l-j+2k-1}} \left[ \frac{\partial^{l+2j+2k-2}}{\partial x^{l+2j+2k-2}} \right] \right](x,y),
\]

where \( g^l_{l,s}(x,y) \) satisfies (2.13b). The fourth term is also absorbed in (5.8); again we use (2.13a) with \( m, i \) replaced by \( 1 \) and \( i \). Finally, the fifth term is absorbed in (5.8) with \( y = C_u(x) \). Since the
terms corresponding to \(-(j+1)+2k-1 \leq -1\) in the second sum of (5.8) disappear, the fifth term is actually absorbed in the first sum of (5.8). Hence using the estimates (5.3)-(5.8) we verify that (2.13a)-(2.13b) hold at order \(j+1\) with \(i, r, q \geq 0, m = 0\).

2.b) The case for \(m \leq -1\) then easily follow. Indeed, we use (2.13a) with \(m\) replaced by 0 and \(j\) by \(j+1\), a case that we just considered. We obtain as in (5.1):

\[
\frac{\partial^{j+1} w}{\partial x^{j+1}} = \sum_{l+s \leq i+2j+1, \ l, s \geq 0} g_{i+0s}^{j}(x, y) \frac{\partial^{j+s} f}{\partial x^{l} \partial y^{s}} (x, C_{u}(x))
\]

(5.9)

\[
+ \sum_{k=0}^{j+1} c_{k+1}^{j+1} \left\{ \frac{\partial^{-j+2k-2} \left[ \frac{\partial^{j+2(j+1)-2k} f}{\partial x^{j+2(j+1)-2k}} \right]}{\partial y^{j+2k-2}} \right\} (x, y).
\]

We apply repeatedly the operator \(\partial^{-1}/\partial y^{-1}\) to (5.9) and, as for \(j = 0\), we observe that this operator is being smooth. It does not induce any additional singularity. Again in this case a polynomial dependence on \(y\) of the coefficients \(g_{i+0s}^{j+1}\) appear at this stage.

2.c) For \(m \geq 1\), using \(-u^{j+1} = \Delta u^{j}\), we write

\[
\frac{\partial^{m}}{\partial y^{m}} \left[ \frac{\partial^{j+1} w}{\partial x^{j+1}} \right] = -\frac{\partial^{m-1}}{\partial y^{m-1}} \left[ \frac{\partial^{j} \Delta w^{j}}{\partial x^{j}} \right] = -\frac{\partial^{m-1}}{\partial y^{m-1}} \left[ \frac{\partial^{j+2} w^{j}}{\partial x^{j+2}} \right] - \frac{\partial^{m+1}}{\partial y^{m+1}} \left[ \frac{\partial^{j+1} w^{j}}{\partial x^{j+1}} \right].
\]

(5.10)

We thus obtain (2.13a) for \(m \geq 1\); and for all \(i, j \geq 0\). Finally we have shown that (2.13a)-(2.13b) also hold at order \(j+1\), \(m \in \mathbb{Z}, i, r, q \geq 0\). Hence, the representation of the derivatives of \(u^{j}\) as in (2.13a)-(2.13b) are valid for all \(j, i, r, q \geq 0, m \in \mathbb{Z}\).

5.2. Proof of Lemma 3.1. To verify the lemma, we proceed by induction on \(j\). For \(j = 0\), we already found that \(\theta^{0} = \theta_{h}^{0}\) as in (3.10). We set \(v^{j}(\eta) = -u^{j}(\cos \eta, \sin \eta)\). For \(j = 1\), the solution \(\theta^{1}\) satisfies Eq. (3.9), and the right-hand side of (3.9) is then written

\[
\frac{\partial \theta^{0}}{\partial \xi} + \cos \eta \frac{\partial \theta^{0}}{\partial \eta} = -\frac{\partial \theta^{0}}{\partial \xi} + \cos \eta \frac{\partial \theta^{0}}{\partial \eta} = [-v^{0}(\eta) \sin \eta + v^{0}(\eta) \cos \eta + \bar{\xi} v^{0}(\eta) \cos^{2} \eta] \exp((\sin \eta) \bar{\xi}).
\]

(5.11)

Hence we find that

\[
\theta^{1} = \theta_{h}^{1} + \theta_{p}^{1} = \left[ v^{1}(\eta) + \bar{\xi} \left( \frac{\sin \eta}{\cos \eta} \frac{\sin \eta}{\sin \eta} v^{0}(\eta) - \frac{\cos \eta}{\sin \eta} v^{0}(\eta) \right) + \bar{\xi}^{2} \left( \frac{\cos^{2} \eta}{2 \sin \eta} v^{0}(\eta) \right) \right] \exp((\sin \eta) \bar{\xi}),
\]

(5.12)

which implies that (3.11)-(3.13) hold for \(j = 1\).

Then assuming that (3.11)-(3.13) have been proved at orders \(0, \ldots, j\), we prove it at order \(j+1\). To obtain an explicit form of \(\theta^{j+1}\), we first obtain the expression in the right-hand side of (3.9) at order \(j + 1\). It is a linear combination of \(\bar{\xi}^{j} \frac{\partial^{k} \theta^{j}}{\partial \eta^{k}}\), \(k' = 0, \ldots, j-1\), \(\bar{\xi}^{j} \frac{\partial^{k'} \theta^{j}}{\partial \xi^{k'}}\) and \((\cos \eta) \bar{\xi}^{j-k'} \frac{\partial^{k'} \theta^{j}}{\partial \eta^{k'}}\), \(k' = 0, \ldots, j\). Using the expression (3.11)-(3.13) of \(\theta^{k'}\) for \(0 \leq k' \leq j\) we find that the right-hand side of (3.9) can be written as the sum of

\[
\sum_{i=0}^{k'} \sum_{k=0}^{k'} b_{i,3k'-3i-1} \eta \bar{\xi}^{j+k-k'} \exp((\sin \eta) \bar{\xi}), \ k' = 0, 1, \ldots, j,
\]

(5.13)
and thus it can be written

\[
(5.14) \quad \left[ \sum_{i=0}^{j} \sum_{k=0}^{2j-2i} a_{i,3j-3i-k+1}(\eta)\bar{\xi}^k \right] \exp((\sin \eta)\bar{\xi}).
\]

Hence, we may rewrite (3.9) for \( j+1 \):

\[
(5.15) \begin{cases}
-\frac{\partial^2 \theta^{j+1}}{\partial \xi^2} + \sin \eta \frac{\partial \theta^{j+1}}{\partial \xi} = \left[ \sum_{i=0}^{j} \sum_{k=0}^{2j-2i} a_{i,3j-3i-k+1}(\eta)\bar{\xi}^k \right] \exp((\sin \eta)\bar{\xi}), \\
\theta^{j+1} = \nu^{j+1}(\eta) = -\nu^{j+1}(\cos \eta, \sin \eta) \text{ at } \bar{\xi} = 0, \\
\theta^{j+1} \to 0 \text{ as } \bar{\xi} \to \infty.
\end{cases}
\]

We can then explicitly find the solution \( \theta^{j+1} \) which turns out to be of the form (3.11)-(3.13) with \( j \) replaced by \( j+1 \). This proves the lemma.

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[Le50] N. Levinson, *The first boundary value problem for \( \varepsilon \Delta u + A(x,y)u_x + B(x,y)u_y + C(x,y)u = D(x,y) \) for small \( \varepsilon \). Ann. of Math. (2) 51, (1950). 428-445.


